The Annals of Statistics
1986, Vol. 14, No. 4, 1388–1398

TESTING EXPONENTIALITY VERSUS A TREND CHANGE IN MEAN RESIDUAL LIFE

BY FRANK GUESS,[1,2] MYLES HOLLANDER[1] AND FRANK PROSCHAN[1]

North Carolina State University, Florida State University and Florida State University

Given that an item is of age t, the expected value of the random remaining life is called the mean residual life (MRL) at age t. We propose two new nonparametric classes of life distributions for modeling aging based on MRL. The first class of life distributions consists of those with "increasing initially, then decreasing mean residual life" (IDMRL). The IDMRL class models aging that is initially beneficial, then adverse. The second class, "decreasing, then increasing mean residual life" (DIMRL), models aging that is initially adverse, then beneficial. We propose two testing procedures for H_0: constant MRL (i.e., exponentiality) versus H_1: IDMRL, but not constant MRL (or H_1': DIMRL, but not constant MRL). The first testing procedure assumes the turning point, τ, from IMRL to DMRL is specified by the user or is known. The second procedure assumes knowledge of the proportion, ρ, of the population that "dies" at or before the turning point (knowledge of τ itself is not assumed).

1. Introduction and summary. Let F be a life distribution (i.e., $F(t) = 0$ for $t < 0$) with a finite first moment. Let $\bar{F}(t) \equiv 1 - F(t)$. The *mean residual life function* is defined as

$$m(t) = E[X - t | X > t], \quad \text{for } \bar{F}(t) > 0, \tag{1.1}$$
$$= 0, \quad \text{for } \bar{F}(t) = 0,$$

for $t \geq 0$. When $\bar{F}(t) > 0$, $m(t)\bar{F}(t) = \int_0^\infty \bar{F}(x + t)\, dx = \int_t^\infty \bar{F}(u)\, du$. Throughout, F is assumed continuous.

We propose two new nonparametric classes of distributions relating to mean residual life. The first class of distributions, called "increasing initially then decreasing mean residual life" (IDMRL) distributions, models aging that is initially beneficial, then adverse.

DEFINITION 1.1. A life distribution with a finite first moment is called an *increasing then decreasing mean residual life* (IDMRL) distribution if there

Received September 1983; revised January 1986.

[1]Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Number AFOSR 85-C-0007. The U.S. Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereon.

[2]Research supported by the National Institute of Environmental Health Sciences under Grant 5 T32 ES 07011-05.

AMS 1980 *subject classifications*. Primary 62N05; secondary 62G10.

Key words and phrases. Nonparametric classes of distributions, mean residual life, exponentiality, increasing then decreasing mean residual life distributions, hypothesis testing, differentiable statistical functions, L-statistics.

exists a turning point $\tau \geq 0$ such that

$$\begin{aligned} &m(s) \leq m(t), \quad \text{for } 0 \leq s \leq t < \tau, \\ &m(s) \geq m(t), \quad \text{for } \tau \leq s \leq t. \end{aligned} \tag{1.2}$$

The dual class of "decreasing initially, then increasing mean residual life" (DIMRL) distributions is obtained by reversing the inequalities on the MRL function in (1.2). It is used to model aging that is initially adverse, then beneficial.

We consider tests for these two classes. Many situations arise where a model of IDMRL or DIMRL distribution can be postulated. For example, it is reasonable to consider an IDMRL model for life lengths of humans. High infant mortality explains the initial IMRL. Deterioration and aging explain the later DMRL state. [See Guess, Hollander and Proschan (1983) for additional examples.]

We develop tests of

$$H_0\text{: } F \text{ is constant MRL (equivalently, } F(x) = 1 - \exp(-x/\mu), \quad x \geq 0,\ \mu > 0,\ \mu \text{ unspecified)} \tag{1.3}$$

versus

$$H_1\text{: } F \text{ is IDMRL (and not constant MRL)}, \tag{1.4}$$

based on a random sample $X_1, \ldots, X_n$ from F. When the dual model is proposed, we test H_0 versus

$$H_1'\text{: } F \text{ is DIMRL (and not constant MRL)}. \tag{1.5}$$

Our test statistics are motivated by the Hollander and Proschan (1975) [hereafter HP (1975)] test statistic for DMRL alternatives. Its desirable efficiency properties [cf. HP (1975), Klefsjö (1983), and Hollander and Proschan (1984)] for testing exponentiality versus DMRL make it a natural choice for extension to the IDMRL and DIMRL classes.

In Section 2 we assume that the turning point τ (say) from IMRL to DMRL is known. We then derive an IDMRL test statistic for testing constant MRL versus IDMRL alternatives. By using the differentiable statistical function approach, we show asymptotic normality of the IDMRL test statistic. Knowledge of τ would be reasonable if we were working with a biological organism in a physical model of a disease process (e.g., the first two months form an incubation period). In a training program for future doctors or a recruiting program for a military service, the value of τ could be known by the length of the intensive stage designed to eliminate the weaker students or recruits.

Section 3 treats the case where we assume knowledge not of τ, but of $\rho = F(\tau)$, the proportion of the population that dies (or leaves the program, etc.) at or before the turning point τ. L-statistic theory is used to show asymptotic normality of the test statistic.

Section 4 contains two examples which illustrate the two tests.

2. The IDMRL test when the turning point τ is known. In this section we assume that the turning point τ is known or has been specified by the user. Motivated by HP (1975), we consider the parameter

$$T(F) = \int_0^\tau \int_0^t \bar{F}(s)\bar{F}(t)[m(t) - m(s)]\, dF(s)\, dF(t) + \int_\tau^\infty \int_\tau^t \bar{F}(s)\bar{F}(t)[m(s) - m(t)]\, dF(s)\, dF(t). \tag{2.1}$$

From (2.1) we see that $T(F)$ is a weighted measure of the degree to which F satisfies the IDMRL property. If the MRL is constant, then $T(F) = 0$. Define $D(s, t) = \bar{F}(s)\bar{F}(t)\{m(t) - m(s)\}$. Then

$$T(F) = E\{I(S < T < \tau)D(S, T) - I(\tau < S < T)D(S, T)\},$$

when S, T are independent random variables, each with distribution F. The weights $\bar{F}(s)$ and $\bar{F}(t)$ represent the proportions of the population still alive at s and t, respectively, and thus furnish comparisons concerning the mean residual lifelengths from s and t, respectively.

Using integration by parts, we can rewrite $T(F)$ as

$$\begin{aligned} T(F) = \int_0^\tau \{ &\left[\tfrac{2}{3} - F(\tau) + \tfrac{1}{2}F^2(\tau)\right]\bar{F}(x) \\ &+ \left[-1 + F(\tau) - \tfrac{1}{2}F^2(\tau)\right]\bar{F}^2(x) + \tfrac{1}{3}\bar{F}^4(x)\}\, dx \\ + \int_\tau^\infty \{ &\left[-\tfrac{1}{6} + \tfrac{1}{2}F(\tau) - \tfrac{1}{2}F^2(\tau) + \tfrac{1}{3}F^3(\tau)\right]\bar{F}(x) \\ &+ \left[\tfrac{1}{2} - F(\tau) + \tfrac{1}{2}F^2(\tau)\right]\bar{F}^2(x) - \tfrac{1}{3}\bar{F}^4(x)\}\, dx, \end{aligned} \tag{2.2}$$

a form which we find convenient.

Let F_n be the empirical distribution formed by a random sample $X_1, \ldots, X_n$ from F. $T_n =_{\text{def.}} T(F_n)$ is a natural statistic for testing H_0 versus H_1. Integrating, we find the computationally simpler expression

$$\begin{aligned} T_n = &\sum_{i=1}^{i^*} B_1\left(\frac{n - i + 1}{n}\right)(X_{in} - X_{(i-1)n}) + B_1\left(\frac{n - i^*}{n}\right)(\tau - X_{i^*n}) \\ &+ B_2\left(\frac{n - i^*}{n}\right)(X_{(i^*+1)n} - \tau) + \sum_{i=i^*+2}^{n} B_2\left(\frac{n - i + 1}{n}\right)(X_{in} - X_{(i-1)n}), \end{aligned}$$

where $0 = X_{0n} < X_{1n} < \cdots < X_{i^*n} \le \tau < X_{(i^*+1)n} < \cdots < X_{nn}$, and

$$\begin{aligned} B_1(u) &=_{\text{def.}} \left[\tfrac{2}{3} - F_n(\tau) + \tfrac{1}{2}F_n^2(\tau)\right]u + \left[-1 + F_n(\tau) - \tfrac{1}{2}F_n^2(\tau)\right]u^2 + \tfrac{1}{3}u^4, \\ B_2(u) &=_{\text{def.}} \left[-\tfrac{1}{6} + \tfrac{1}{2}F_n(\tau) - \tfrac{1}{2}F_n^2(\tau) + \tfrac{1}{3}F_n^3(\tau)\right]u \\ &\quad + \left[\tfrac{1}{2} - F_n(\tau) + \tfrac{1}{2}F_n^2(\tau)\right]u^2 - \tfrac{1}{3}u^4. \end{aligned}$$

[The dependence of $B_i(u)$, $i = 1, 2$, on $F_n(\tau)$ is suppressed for convenience.] Under our continuity assumption on F, with probability one ties will not occur. However, in industrial, medical, and other settings ties may occur due to

grouping of the data. For the case of ties, use

$$T_n = \sum_{i=1}^{i^*} B_1\left(\frac{s_{i-1}}{n}\right)\left(\tilde{X}_{ik} - \tilde{X}_{(i-1)k}\right) + B_1\left(\frac{s_{i^*}}{n}\right)\left(\tau - \tilde{X}_{i^*k}\right)$$

$$+ B_2\left(\frac{s_{i^*}}{n}\right)\left(\tilde{X}_{(i^*+1)k} - \tau\right) + \sum_{i=i^*+2}^{k} B_2\left(\frac{s_{i-1}}{n}\right)\left(\tilde{X}_{ik} - \tilde{X}_{(i-1)k}\right),$$

where $0 = \tilde{X}_{0k} < \tilde{X}_{1k} < \cdots < \tilde{X}_{i^*k} \le \tau < \tilde{X}_{(i^*+1)k} < \cdots < \tilde{X}_{kk}$ are the distinct ordered observations from the random sample and

$$n_i = \text{number of observed deaths at time } \tilde{X}_{ik},$$

$$s_i = n - \sum_{l=0}^{i} n_l, \qquad \text{for } i = 0,1,\ldots,k < n.$$

Note that $n_i \neq 0$, $i = 1,\ldots,k$, while $n_0 = 0$ is allowed.

To establish asymptotic normality of T_n, we use the differentiable statistical function (DSF) approach of von Mises (1947) [cf. Boos and Serfling (1980) and Serfling (1980)]. The approach uses a Taylor expansion for functionals.

The first-order Gâteaux differential of a functional T at the point F in the direction G is defined as

$$d_1T(F, G - F) = \lim_{\lambda \to 0^+} \frac{T(F_\lambda) - T(F)}{\lambda},$$

where $F_\lambda = F + \lambda(G - F)$, F and G are distributions in the domain of $T(\cdot)$, and $0 < \lambda \le 1$. Notice that the differential is a function of two arguments: the distribution F, and the increment $D =_{\text{def.}} G - F$. Let $\mu(T, F) =_{\text{def.}} E_F[d_1T(F, \delta_{X_1} - F)]$ and $\sigma^2(T, F) =_{\text{def.}} \text{Var}_F[d_1T(F, \delta_{X_1} - F)]$.

For the IDMRL functional T, the Gâteaux differential is

$$d_1T(F, D) = \left[\int_0^\tau \left\{-\bar{F}(\tau)\bar{F}(x) + \bar{F}(\tau)\bar{F}^2(x)\right\} dx\right] D(\tau)$$

$$- \int_0^\tau \left\{\left[\tfrac{2}{3} - F(\tau) + \tfrac{1}{2}F^2(\tau)\right]\right.$$

$$\left. + 2\left[-1 + F(\tau) - \tfrac{1}{2}F^2(\tau)\right]\bar{F}(x) + \tfrac{4}{3}\bar{F}^3(x)\right\} D(x)\, dx$$

$$+ \left[\int_\tau^\infty \left\{\left[\tfrac{1}{2} - F(\tau) + F^2(\tau)\right]\bar{F}(x) - \bar{F}(\tau)\bar{F}^2(x)\right\} dx\right] D(\tau)$$

$$- \int_\tau^\infty \left\{\left[-\tfrac{1}{6} + \tfrac{1}{2}F(\tau) - \tfrac{1}{2}F^2(\tau) + \tfrac{1}{3}F^3(\tau)\right]\right.$$

$$\left. + 2\left[\tfrac{1}{2} - F(\tau) + \tfrac{1}{2}F^2(\tau)\right]\bar{F}(x) - \tfrac{4}{3}\bar{F}^3(x)\right\} D(x)\, dx.$$

Set $D_n = F_n - F$. Note $F_n = n^{-1}\sum_{i=1}^n \delta_{X_i}$, where $\delta_{X_i}(x) = 0$ if $x < X_i$ and $= 1$ if $x \ge X_i$. The differential is linear in the increment argument; and thus,

$$d_1T(F, D_n) = \frac{1}{n}\sum_{i=1}^{n} d_1T(F, \delta_{X_i} - F). \tag{2.3}$$

Our proof of asymptotic normality approximates $T(F_n) - T(F)$ by $d_1T(F, D_n)$

and shows the remainder term R_n converges in probability to 0. Our proof uses the fact that $T(F)$ can be represented as

$$T(F) = T_{11}(F)T_{12}(F) + T_{21}(F)T_{22}(F) + T_{3*}(F) + T_{41}(F)T_{42}(F) + T_{51}(F)T_{52}(F) + T_{6*}(F). \tag{2.4}$$

The functionals $T_{ij}(\cdot)$, $i = 1,2,3$, $j = 1,2,*$ are defined in the obvious way from (2.2), e.g.,

$$T_{11}(F) = \left[\tfrac{2}{3} - F(\tau) + \tfrac{1}{2}F^2(\tau)\right],$$

$$T_{12}(F) = \int_0^\tau \bar{F}(x)\,dx, \qquad T_{3*}(F) = \int_0^\tau \tfrac{1}{3}\bar{F}^4(x)\,dx,$$

etc.

THEOREM 2.1. *Let F be a life distribution such that $0 < F(\tau) < 1$, $0 < \sigma^2(T, F) < \infty$, and $0 < \sigma^2(T_{ij}, F) < \infty$ for $i = 1,2,3$, $j = 1,2,*$. Then*

$$n^{1/2}\left[T(F_n) - T(F)\right] \to_d N(0, \sigma^2(T, F)). \tag{2.5}$$

PROOF. Note that $\mu(T, F) = 0$. Applying the classical Lindeberg–Lévy central limit theorem to (2.3), we have

$$n^{1/2}\left[d_1T(F, D_n)\right] \to_d N(0, \sigma^2(T, F)).$$

Next, we show that $R_n = n^{1/2}[T(F_n) - T(F) - d_1T(F, D_n)]$ converges in probability to zero. Equation (2.4) allows us to express R_n as a sum of simpler products. Rewrite R_n as

$$\begin{aligned} R_n = &\sum_{i\in\{1,2,4,5\}} \{T_{i2}(F)n^{1/2}\left[T_{i1}(F_n) - T_{i1}(F) - d_1T_{i1}(F, F_n - F)\right] \\ &+ T_{i1}(F)n^{1/2}\left[T_{i2}(F_n) - T_{i2}(F) - d_1T_{i2}(F, F_n - F)\right] \\ &+ \left[T_{i1}(F_n) - T_{i1}(F)\right]n^{1/2}\left[T_{i2}(F_n) - T_{i2}(F)\right]\} \\ &+ \sum_{i\in\{3,6\}} n^{1/2}\left[T_{i*}(F_n) - T_{i*}(F) - d_1T_{i*}(F, F_n - F)\right] \\ =_{\text{def.}} &\sum_{i\in\{1,2,4,5\}} \{R_{i1n} + R_{i2n} + R_{i3n}\} + \sum_{i\in\{3,6\}} R_{in}. \end{aligned}$$

Consider the $i = 1$ terms. R_{11n} reduces to

$$R_{11n} = \tfrac{1}{2}T_{12}(F)\left[F_n(\tau) - F(\tau)\right]n^{1/2}\left[F_n(\tau) - F(\tau)\right],$$

which converges in probability to 0.

$$R_{12n} = T_{11}(F)n^{1/2}[0] = 0,$$

$$R_{13n} = \left[T_{11}(F_n) - T_{11}(F)\right]n^{1/2}\left[T_{12}(F_n) - T_{12}(F)\right] \to_p 0,$$

since

$$T_{11}(F_n) - T_{11}(F) \to_p 0$$

and

$$n^{1/2}[T_{12}(F_n) - T_{12}(F)] \to_d N(0, \sigma^2(T_{12}, F)).$$

[If $\sigma^2(T_{12}, F) = 0$, we follow Serfling's (1980) convention that $N(0,0)$ is degenerate at 0.] Hence, $R_{11n} + R_{12n} + R_{13n} \to_p 0$. Similar straightforward arguments for $i = 2$, 4, and 5 yield $R_{i1n} + R_{i2n} + R_{i3n} \to_p 0$. For the $i = 3$ term,

$$\begin{aligned} R_{3n} &= \tfrac{1}{3}n^{1/2}\int_0^\tau [\bar{F}_n^4(x) - 4\bar{F}^3(x)\bar{F}_n(x) + 3\bar{F}^4(x)]\,dx \\ &= \tfrac{1}{3}n^{1/2}\int_0^\tau [\bar{F}_n(x) - \bar{F}(x)][\bar{F}_n^3(x) - \bar{F}^3(x) + \bar{F}(x)(\bar{F}_n^2(x) - \bar{F}^2(x)) \\ &\qquad + \bar{F}^2(x)(\bar{F}_n(x) - \bar{F}(x))]\,dx. \end{aligned}$$

We have,

$$\begin{aligned} |R_{3n}| \le \tfrac{1}{3}n^{1/2}\Big[\sup_{0\le x\le\tau}|\bar{F}_n(x) - \bar{F}(x)|\Big]\int_0^\tau \big|\bar{F}_n^3(x) - \bar{F}^3(x) + \bar{F}(x)(\bar{F}_n^2(x) \\ - \bar{F}^2(x)) + \bar{F}^2(x)(\bar{F}_n(x) - \bar{F}(x))\big|\,dx. \end{aligned}$$

Now,

$$n^{1/2}\Big[\sup_{0\le x\le\tau}|\bar{F}_n(x) - \bar{F}(x)|\Big] \le n^{1/2}\Big[\sup_{0\le x\le\infty}|F_n(x) - F(x)|\Big] \to_d \mathcal{K},$$

the Kolmogorov–Smirnov limiting distribution [cf. Serfling (1980)]. Also,

$$\begin{aligned} \int_0^\tau \big|\bar{F}_n^3(x) - \bar{F}^3(x) + \bar{F}(x)(\bar{F}_n^2(x) - \bar{F}^2(x)) \\ + \bar{F}^2(x)(\bar{F}_n(x) - \bar{F}(x))\big|\,dx \to_p 0. \end{aligned}$$

Thus, $|R_{3n}| \to_p 0$. A similar argument holds for the $i = 6$ term. Thus $R_n \to_p 0$, and Slutsky's theorem yields (2.5). □

If $\int_0^\infty x^2\,dF(x) < \infty$, then all the variance terms in the assumptions of Theorem 2.1 will be finite.

We formally develop the IDMRL(τ) test procedure. Under H_0, $T(F) = 0$; and thus, $n^{1/2}[T(F_n) - T(F)] = n^{1/2}[T(F_n)]$, and the appropriate variance conditions hold. [Assume $\tau \ne 0$ to avoid $F(\tau) = 0$. If $\tau = 0$, the test reduces to testing for DMRL (IMRL); see HP (1975).] From (2.5), we have $n^{1/2}[T(F_n)] \to_d N(0, \sigma^2(T, F))$. Straightforward calculations show

$$\sigma^2(T, F) = \mu^2\big[-\tfrac{1}{15}F^5(\tau) + \tfrac{1}{6}F^4(\tau) - \tfrac{1}{6}F^3(\tau) + \tfrac{1}{10}F^2(\tau) - \tfrac{1}{30}F(\tau) + \tfrac{1}{210}\big],$$

where $\mu = \int_0^\infty \bar{F}(x)\,dx$. $\hat{\sigma}_n^2 =_{\text{def.}} \sigma^2(T, F_n)$ is a consistent estimator of $\sigma^2(T, F)$; thus, $\{n^{1/2}[T(F_n) - T(F)]\}/\hat{\sigma}_n \to_d N(0,1)$.

The IDMRL test procedure rejects H_0 in favor of H_1 at the approximate level α if $\tilde{T}_n =_{\text{def.}} n^{1/2}[T(F_n)]/\hat{\sigma}_n \ge z_\alpha$, where z_α is the upper α-quantile of the standard normal distribution. If $\tilde{T}_n < z_\alpha$, H_0 is accepted. The DIMRL test rejects H_0 in favor of H_1' at the approximate α level if $\tilde{T}_n \le -z_\alpha$. If $\tilde{T}_n > -z_\alpha$, H_0 is accepted.

3. The IDMRL test when the proportion ρ is known. In this section we do not assume knowledge of τ, the turning point. Instead, we assume knowledge of the proportion ρ of the population that "dies" at or before the turning point (e.g., in a training or recruiting program of students or military personnel, knowledge of ρ would be reasonable). Let $F^{-1}(\rho) = \inf\{x|F(x) \geq \rho\}$ for $0 < \rho < 1$. Note that $\tau = F^{-1}(\rho)$.

Recall that we want to test H_0 (1.3) versus H_1 (1.4). In this section we find form (2.1) of $T(F)$ convenient. A natural statistic to consider first is $U_n = T(F_n)$.

Let $X_{1n} \leq \cdots \leq X_{nn}$ denote the order statistics from a random sample of F. Let $[x]$ = greatest integer less than or equal to x. Let $m_n(t)$ denote the empirical mean residual life function [i.e., $m_n(t) = (\sum_{k=i+1}^{n}(X_{kn} - t))/(n-i)$ for $t \in [X_{in}, X_{(i+1)n})$, $i = 0, 1, \ldots, n-1$, where $X_{0n} = 0$. $m_n(t) = 0$, for $t \geq X_{nn}$]. Simplifying U_n by integrating, we get

$$
\begin{aligned}
U_n = &\sum_{i=1}^{j^*-1}\sum_{j=i+1}^{j^*}\frac{1}{n^2}\bar{F}_n(X_{in})\bar{F}_n(X_{jn})\left[m_n(X_{jn}) - m_n(X_{in})\right] \\
&+\sum_{i=j^*}^{n-1}\sum_{j=i+1}^{n}\frac{1}{n^2}\bar{F}_n(X_{in})\bar{F}_n(X_{jn})\left[m_n(X_{in}) - m_n(X_{jn})\right],
\end{aligned} \tag{3.1}
$$

where $j^* = n\rho$ if $n\rho$ is an integer, $= [n\rho] + 1$ if $n\rho$ is not an integer. Note that $\hat{\tau} = F_n^{-1}(\rho) = X_{j^*n}$.

Now we modify U_n to get another statistic V_n which uses information on the total lifelengths of the items:

$$
V_n = U_n + \sum_{j=1}^{j^*}\frac{1}{n^2}\bar{F}_n(X_{jn})\left[m_n(X_{jn}) - m_n(X_{0n})\right]. \tag{3.2}
$$

The statistic V_n (U_n) as expressed in (3.2) [(3.1)] implicitly uses the fact that F is continuous. However, in actual practice ties may occur due to grouping of the data, even though the underlying distribution is continuous. To accommodate ties, use the following computational expression:

$$
\begin{aligned}
V_n = &\frac{1}{n^2}\sum_{i=0}^{j^*-1}\sum_{j=i+1}^{j^*}\bar{F}_n(\tilde{X}_{in})\bar{F}_n(\tilde{X}_{jn})\left[m_n(\tilde{X}_{jn}) - m_n(\tilde{X}_{in})\right]n_i n_j \\
&+\frac{1}{n^2}\sum_{i=j^*}^{k-1}\sum_{j=i+1}^{k}\bar{F}_n(\tilde{X}_{in})\bar{F}_n(\tilde{X}_{jn})\left[m_n(\tilde{X}_{in}) - m_n(\tilde{X}_{jn})\right]n_i n_j,
\end{aligned}
$$

where k, n_i, and $\tilde{X}_{ik}$, $i = 1, \ldots, k$, are defined as in Section 2 and $n_0 =_{\text{def.}} 1 +$ number of observed deaths at time $\tilde{X}_{0k} = 0$. (Note that $j^* \neq i^*$ can happen.)

In establishing asymptotic normality, we find it useful to express V_n as a linear combination of order statistics (i.e., as an L-statistic). To represent V_n as an L-statistic, we reverse the triple summation implicit in (3.2); thus, $V_n = n^{-4}\sum_{k=1}^{n}c_{kn}X_{kn}$, where:

CASE 1. $k < j^*$:

$$
\begin{aligned}
c_{kn} = &-\tfrac{4}{3}k^3 + \tfrac{1}{2}k^2 - \tfrac{1}{6}k + 4nk^2 - 2n^2k - 2nkj^* \\
&+kj^{*2} + kj^* - nk + n^2j^* - \tfrac{1}{2}nj^{*2} - \tfrac{1}{2}nj^*.
\end{aligned}
$$

CASE 2. $k = j^*$:

$$c_{j^*n} = -\tfrac{1}{2}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}j^{*3} + 2j^{*2} - \tfrac{1}{6}j^* + \tfrac{1}{2}n^2j^* - \tfrac{5}{2}nj^*.$$

CASE 3. $k > j^*$:

$$c_{kn} = -\tfrac{1}{2}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{3}j^{*3} + \tfrac{1}{2}j^{*2} + \tfrac{1}{6}j^* + \tfrac{4}{3}k^3 - \tfrac{1}{2}k^2 + \tfrac{1}{6}k - kj^{*2} + 3n^2k + 2nkj^* - 4nk^2 + \tfrac{1}{2}nj^{*2} - nj^* - \tfrac{3}{2}n^2j^* + kj^*.$$

Note that c_{kn} depends on ρ, as well as on k and n.

From cases 1 and 3, we are led to the following weight function:

$$J(x) = \begin{cases} J_1(x) = -\tfrac{4}{3}x^3 + 4x^2 + (\rho^2 - 2\rho - 2)x + (\rho - \tfrac{1}{2}\rho^2), & \text{if } 0 \le x < \rho, \\ J_2(x) = \tfrac{4}{3}x^3 - 4x^2 + (3 + 2\rho - \rho^2)x + (\tfrac{1}{3}\rho^3 + \tfrac{1}{2}\rho^2 - \tfrac{3}{2}\rho - \tfrac{1}{2}), & \text{if } \rho \le x \le 1. \end{cases}$$

(This comes from dividing by n^3 in cases 1 and 3, equating j^*/n with ρ and k/n with x, and ignoring terms that are not cubic.) Let $S_n = (1/n)\sum_{k=1}^{n} J(k/n)X_{kn}$. Note that $n^{1/2}(V_n - S_n)$ converges in probability to zero. To form our IDMRL test statistic, we modify V_n to get the scale invariant statistic $V_n^* = V_n/\bar{X}_n$, where $\bar{X}_n \equiv m_n(0) = (1/n)\sum_{i=1}^{n} X_{in}$.

THEOREM 3.1. *Let F be a life distribution such that A_1: $\int_0^\infty x^2\,dF(x) < \infty$, A_2: $\int_0^\infty (F(x)\bar{F}(x))^{1/2}\,dx < \infty$, A_3: $\sigma^2(J^*, F) > 0$, and A_4: F has a unique ρ-quantile. Then*

$$V_n' =_{\text{def.}} n^{1/2}[V_n^* - \mu(J, F)/\mu] \to_d N(0, \sigma^2(J^*, F)/\mu^2), \tag{3.3}$$

where

$$\mu(J, F) = \int_0^\infty xJ(F(x))\,dF(x),$$

$$\sigma^2(J, F) = \int_0^\infty \int_0^\infty J(F(x))J(F(y))[F(\min(x, y)) - F(x)F(y)]\,dx\,dy,$$

and

$$J^*(x) = J(x) - \mu(J, F)/\mu, \quad \text{for } 0 \le x \le 1.$$

PROOF. We establish (3.3) using results of Stigler (1974, 1979) and Mason (1981). With $S_n^* = S_n/\bar{X}_n$, we first note $n^{1/2}(V_n^* - S_n^*)$ converges in probability to zero. We will show

$$S_n' =_{\text{def.}} n^{1/2}[S_n^* - \mu(J, F)/\mu] \to_d N(0, \sigma^2(J^*, F)/\mu^2), \tag{3.4}$$

which, along with Slutsky's theorem, yields (3.3). Note that S_n' can be rewritten as $S_n' = n^{1/2}[(1/n)\sum_{i=1}^{n} J^*(i/n)X_{in}]/\bar{X}_n$.

To apply the results of Stigler (1974) and Mason (1981), we observe the following conditions hold. J^* is bounded on $[0, 1]$. By A_4, J^* is continuous a.e. F^{-1}. J^* also satisfies a Hölder condition for $\alpha > \frac{1}{2}$ (e.g., $\alpha = 1$, which is simply a Lipschitz condition) except at the one point ρ. Under A_4, however, ρ is a continuity point of F^{-1}. Using Theorem 2 of Stigler (1974), then Theorem 1 of

Stigler (1974) and Theorem 2 of Mason (1981), we have under A_1, A_2, A_3, and A_4 that

$$(3.5) \qquad n^{1/2}\left[\frac{1}{n}\sum_{i=1}^{n} J^*\left(\frac{i}{n}\right)X_{in} - \mu(J^*, F)\right] \to_d N(0, \sigma^2(J^*, F)).$$

[See Remark 2 of Stigler (1974) and consider $J_n^*(i/(n+1)) =_{\text{def.}} J^*(i/n)$ as it applies to Theorems 1 and 2. The proof of Theorem 2 of Mason (1981) can be modified in a straightforward fashion to handle our weight function, J^*, using i/n instead of $i/(n+1)$.) Since $\mu(J^*, F) = 0$, we have from (3.5) that $n^{1/2}[(1/n)\Sigma_{i=1}^{n} J^*(i/n)X_{in}] \to_d N(0, \sigma^2(J^*, F))$. Applying Slutsky's theorem, we get (3.4). □

We now formally develop the IDMRL(ρ) test procedure. Note that $\mu(J, F) = T(F)$. From this, we have $\mu(J, F) = 0$ when H_0 holds. Also, $\sigma^2(J^*, F) = \sigma^2(J, F)$ under H_0. [Recall $J^*(x) = J(x) - \mu(J, F)/\mu$.] Since V_n^* is a scale invariant statistic, the calculation of the asymptotic variance under H_0 can be made with scale parameter λ taken to be 1. Note that $\mu = 1$.

Set $F_0(x) = 1 - \exp(-x)$, $x \geq 0$. Calculations show that

$$\sigma^2(\rho) =_{\text{def.}} \sigma^2(J, F_0) = -\tfrac{1}{15}\rho^5 + \tfrac{1}{6}\rho^4 - \tfrac{1}{6}\rho^3 + \tfrac{1}{10}\rho^2 - \tfrac{1}{30}\rho + \tfrac{1}{210}.$$

[When $\rho = 0$ and $\rho = 1$, $\sigma^2(\rho) = \frac{1}{210}$, which is the asymptotic variance of the HP (1975) DMRL test statistic.] Since A_1–A_4 are satisfied under H_0, we have

$$\tilde{V}_n =_{\text{def.}} n^{1/2}[V_n^*]/\sigma(\rho) \to_d N(0, 1).$$

The IDMRL(ρ) test procedure rejects H_0 in favor of H_1 at the approximate α-level if $\tilde{V}_n \geq z_\alpha$, where z_α is the upper α-quantile of the standard normal distribution. If $\tilde{V}_n < z_\alpha$, H_0 is accepted. The DIMRL(ρ) test rejects H_0 in favor of H_1' at the approximate α-level if $\tilde{V}_n \leq -z_\alpha$. If $\tilde{V}_n > -z_\alpha$, H_0 is accepted.

By rewriting V_n as a sum of weighted normalized spacings, we can use the approach of Langenberg and Srinivasan (1979) to find the exact distribution of $V_n^* = V_n/\bar{X}_n$. See Guess, Hollander, and Proschan (1983) for details. Table 1 contains critical values of $\tilde{V}_n$ for $\rho = 0.25$ for the sample sizes $n = 2, \ldots, 30$ in the lower and upper $\alpha = 0.01, 0.05, 0.10$ regions. Tables for $\rho = 0(0.1)1, 0.75, \frac{1}{3}$, and $\frac{2}{3}$ are available from Frank Guess on request.

In this section we considered the case where ρ is known. Section 2 treated the situation where τ is known. Open problems suggested by this paper include: What procedures could be used to test exponentiality against IDMRL(τ) [IDMRL(ρ)] when $\tau[\rho]$ is unknown? What are the properties of T_n when τ is estimated by a value which maximizes the empirical mean residual life function? What are optimal estimators of F when it is known that F is IDMRL(τ) [DIMRL(τ)]? How should these procedures be extended to accommodate censored data?

4. An example. We illustrate the use of the two tests on a data set from Bjerkedal (1960). We give a brief description of the data and then the results of the two sets. Bjerkedal (1960) studies the lifelengths of guinea pigs injected with

TABLE 1
Exact critical values of the IDMRL(ρ) *test statistic* $\tilde{V}_n$, $\rho = 0.25$.

	Lower tail			Upper tail		
n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
2	−6.74	−6.19	−5.50	5.50	6.19	6.74
3	−4.29	−3.41	−2.76	2.76	3.41	4.29
4	−3.22	−2.67	−2.24	2.26	2.80	3.66
5	−3.34	−2.55	−2.05	2.05	2.55	3.34
6	−2.64	−2.06	−1.68	1.72	2.17	2.89
7	−2.71	−2.03	−1.62	1.65	2.07	2.78
8	−3.09	−2.16	−1.68	1.68	2.10	2.81
9	−2.33	−1.75	−1.40	1.43	1.81	2.47
10	−2.45	−1.78	−1.41	1.42	1.80	2.45
11	−2.68	−1.88	−1.46	1.45	1.83	2.49
12	−2.93	−2.01	−1.54	1.51	1.90	2.57
13	−2.29	−1.65	−1.30	1.30	1.65	2.28
14	−2.46	−1.73	−1.34	1.34	1.69	2.32
15	−2.65	−1.83	−1.41	1.38	1.75	2.39
16	−2.83	−1.93	−1.48	1.44	1.81	2.47
17	−2.33	−1.63	−1.27	1.26	1.60	2.21
18	−2.47	−1.71	−1.32	1.30	1.65	2.27
19	−2.62	−1.80	−1.38	1.35	1.70	2.34
20	−2.76	−1.89	−1.44	1.40	1.76	2.41
21	−2.35	−1.64	−1.26	1.25	1.58	2.19
22	−2.48	−1.71	−1.31	1.29	1.63	2.24
23	−2.60	−1.78	−1.36	1.33	1.68	2.31
24	−2.71	−1.85	−1.42	1.37	1.73	2.38
25	−2.37	−1.64	−1.26	1.24	1.57	2.18
26	−2.48	−1.70	−1.31	1.28	1.62	2.23
27	−2.58	−1.77	−1.35	1.32	1.66	2.29
28	−2.68	−1.83	−1.40	1.36	1.71	2.35
29	−2.38	−1.65	−1.27	1.24	1.57	2.17
30	−2.47	−1.70	−1.30	1.27	1.61	2.23

different amounts of tubercle bacilli. (Guinea pigs are known to have a high susceptibility to human tuberculosis, which is one reason for choosing this species.) We describe the only study (M) in which animals in a single cage are under the same regimen. The regimen number is the common log of the number of bacillary units in 0.5 ml of the challenge solution, e.g., regimen 4.3 corresponds to 2.2×10^4 bacillary units per 0.5 ml ($\log_{10}(2.2 \times 10^4) = 4.342$).

Before conducting such an experiment, it is reasonable to conjecture that the injection of tubercle bacilli causes an adverse stage of aging (DMRL). After the guinea pigs have survived this adverse stage, the guinea pigs' natural systems recoup to yield a beneficial stage (IMRL).

Hall and Wellner (1981) examine regimen 4.3 and fit a parametric distribution that is in the DIMRL class. They estimate the point at which the MRL changes trend as $\hat{\tau}_{4.3} = 91.9$ (τ corresponds to "α" in the notation of their parametric model). We use $\hat{\rho}_{4.3} = F_n(\hat{\tau}_{4.3}) = \frac{1}{9}$ as a natural estimator of $\rho_{4.3}$.

Using the information gained from regimen 4.3, we apply the DIMRL tests to regimen 5.5. (The sample sizes for regimen 4.3 and 5.5 are both 72.) For regimen 5.5, we use $\tau_{5.5} = 91.9$ and $\rho_{5.5} = \frac{1}{9}$. Note that this is a reasonable a priori assertion concerning these two population parameters under regimen 5.5 since it is based on data from a closely related population.

For the DIMRL test with τ known, we get $T_n = -0.6419$, $\hat{\sigma}_n^2 = 7.1072$, and $\tilde{T}_n = -2.04$, yielding a P-value of 0.0207 in the normal approximation. For the DIMRL test with ρ known, we get $V_n^* = -0.01106$, $\sigma^2(\rho) = 0.00209$, and $\tilde{V}_n = -2.05$, yielding a P-value of 0.0202 in the normal approximation. Both the DIMRL(τ) and the DIMRL(ρ) test procedures suggest significant evidence to reject H_0 in favor of the alternative H_1'.

Acknowledgments. We thank Dr. Dennis Boos, Department of Statistics, North Carolina State University, for valuable comments. We are grateful to Mr. Kermit Rose of the Florida State University Computer Center for his suggestions concerning the computer program that produced the exact critical values for the IDMRL(ρ) test. We also thank the Associate Editor and referees for their comments and suggestions.

REFERENCES

BJERKEDAL, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *Amer. J. Hygiene* **72** 130–148.

BOOS, D. D. and SERFLING, R. J. (1980). A note on differentials and the CLT and LIL for statistical functions, with application to M-estimates. *Ann. Statist.* **8** 618–624.

GUESS, F., HOLLANDER, M. and PROSCHAN, F. (1983). Testing whether mean residual life changes trend. Report M665, Dept. of Statistics, Florida State Univ.

HALL, W. J. and WELLNER, J. A. (1981). Mean residual life. In *Statistics and Related Topics* (M. Csörgő, D. A. Dawson, J. N. K. Rao and A. K. Md. E. Saleh, eds.) 169–184. North-Holland, Amsterdam.

HOLLANDER, M. and PROSCHAN, F. (1975). Tests for the mean residual life. *Biometrika* **62** 585–593.

HOLLANDER, M. and PROSCHAN, F. (1984). Nonparametric concepts and methods in reliability. In *Handbook of Statistics, Nonparametric Methods* (P. R. Krishnaiah and P. K. Sen, eds.) 4 613–655. North-Holland, Amsterdam.

KLEFSJÖ, B. (1983). Some tests against aging based on the total time on test transform. *Comm. Statist. A—Theory Methods* **12** 907–927.

LANGENBERG, P. and SRINIVASAN, R. (1979). Null distribution of the Hollander–Proschan statistic for decreasing mean residual life. *Biometrika* **66** 679–680.

MASON, D. M. (1981). Asymptotic normality of linear combinations of order statistics with a smooth score function. *Ann. Statist.* **9** 899–908.

SERFLING, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley, New York.

STIGLER, S. M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2** 676–693. Correction (1979). *Ann. Statist.* **7** 466.

VON MISES, R. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18** 309–348.

FRANK GUESS
DEPARTMENT OF STATISTICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA 27695

MYLES HOLLANDER
FRANK PROSCHAN
DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306