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ON A CONVERSE TO SCHEFFÉ'S THEOREM

BY T. J. SWEETING

University of Surrey

In Boos (1985) equicontinuity conditions are given which ensure the uniform convergence of densities in $\mathbb{R}^k$, given convergence in distribution. In the present note we show that such equicontinuity conditions in fact characterize uniform local convergence with no additional assumptions on the sequence of densities, or on the limit density. Versions of these results are also given when the distributions depend on an unknown parameter; these forms will be relevant for the uniform approximation of likelihood functions.

1. Introduction. In Boos (1985) it is shown that convergence in distribution entails continuous convergence of the corresponding densities under boundedness and equicontinuity assumptions regarding the sequence of densities. This result follows from the Ascoli theorem regarding the sequential compactness of families of functions. It is further demonstrated in that paper that a strengthening of these conditions leads to uniform convergence of the densities over the entire Euclidean space. The results are applied to the problem of proving local limit theorems for translation and scale statistics.

The main purpose of the present note is to show that such equicontinuity conditions are also necessary for the stated convergences. Additionally, it is not actually necessary to assume continuity of the individual densities in these results, nor even the existence at the outset of a density for the limit distribution. When local limit results are required for the construction of likelihood functions, it is also necessary to have uniform convergence over compact subsets of the parameter space; this matter is discussed in Section 3.

2. Uniform convergence of densities. We borrow the notation in Boos (1985): Let (G_n) be a sequence of absolutely continuous (wrt Lebesgue measure μ) distributions on $\mathbb{R}^k$, and let (g_n) be a corresponding sequence of densities. We seek necessary and sufficient conditions for the sequence (g_n) to converge uniformly to some density g. Since we do not assume here that the g_n are continuous, we require the following slight modification of equicontinuity. As in Boos (1985) we let $|\cdot|$ on $\mathbb{R}^k$ be the maximum absolute coordinate. We shall say that a sequence of real-valued functions (u_n) on $\mathbb{R}^k$ is *asymptotically equicontinuous* (a.e.c.) at $x \in \mathbb{R}^k$ if given $\varepsilon > 0$, there exist $\delta(x, \varepsilon)$, $n(x, \varepsilon)$ such that whenever $|y - x| \leq \delta(x, \varepsilon)$ then $|u_n(y) - u_n(x)| < \varepsilon$ for all $n > n(x, \varepsilon)$. The sequence (u_n) is a.e.c. if it is a.e.c. at each point x of $\mathbb{R}^k$. Likewise, we say that (u_n) is *asymptotically uniformly equicontinuous* (a.u.e.c.) if it is a.e.c. and $\delta(x, \varepsilon) = \delta(\varepsilon)$, $n(x, \varepsilon) = n(\varepsilon)$ above. (These definitions actually coincide with the

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definitions of equicontinuity and uniform equicontinuity given in Boos in the case of continuous u_n; the addition of the word "asymptotic" better describes the concepts, however, especially in the case of discontinuous u_n.) Finally, the sequence (u_n) is *bounded* if $\sup_n |u_n(x)| \le M(x) < \infty$ for each $x \in \mathbb{R}^k$. The following theorems clarify the position in Boos (1985). We use the symbol $\Rightarrow$ to stand for weak convergence.

THEOREM 1. *The following two statements are equivalent.*

(1) (g_n) *is a.e.c. and bounded, and* $G_n \Rightarrow G$.

(2) $g_n \to g$ *pointwise, uniformly in compacts of* $\mathbb{R}^k$, *where* g *is the continuous density of the distribution* G.

PROOF. (1) $\Rightarrow$ (2). In the case of continuous densities (g_n) and g, the result follows from the Ascoli theorem along with Scheffé's theorem, as given in Boos (1985). However, it may readily be checked that the proof of the Ascoli result (as given in Corollary 31 of Chapter 9 in Royden (1968), for example) goes through for a general a.e.c. sequence (g_n) of (not necessarily continuous) functions with virtually no change; we omit the details. The convergence of any subsequence $(g_{n'})$ to a continuous limit g uniformly in compacts of $\mathbb{R}^k$ implies that for any compact set $K \subset \mathbb{R}^k$, $G_{n'}(K) \to \int_K g\, d\mu$. It now follows from the regularity of the measure given by $H(A) = \int_A g\, d\mu$ (Rudin (1970), Theorem 2.18) that $H = G$ and hence g is the unique continuous density of G. The result now follows from the Ascoli theorem.

(2) $\Rightarrow$ (1). The following form of converse uses the local compactness of $\mathbb{R}^k$. Let $x \in \mathbb{R}^k$ and $\varepsilon > 0$. Choose $\delta(x, \varepsilon)$, $n(x, \varepsilon)$ such that $|g(y) - g(x)| < \varepsilon$ and $|g_n(y) - g(y)| < \varepsilon$ whenever $|y - x| < \delta(x, \varepsilon)$ and $n > n(x, \varepsilon)$ (from the continuity of g and uniform convergence of (g_n) in compacts). Then if $|y - x| < \delta(x, \varepsilon)$ and $n > n(x, \varepsilon)$ we have

$$
\begin{aligned}
|g_n(y) - g_n(x)| &\le |g_n(y) - g(y)| + |g_n(x) - g(x)| + |g(y) - g(x)| \\
&< 3\varepsilon
\end{aligned}
\tag{3}
$$

and, hence, (g_n) is a.e.c. Also, (g_n) is trivially bounded as (g_n) converges pointwise. The weak convergence $G_n \Rightarrow G$ follows from Scheffé's theorem. □

Thus, apart from allowing arbitrary densities, the sufficient conditions in Boos (1985) for density convergence cannot further be weakened if uniformity in compacts of $\mathbb{R}^k$ is demanded. We remark that it is not necessary to assume at the outset the existence of a density for G in the implication (1) $\Rightarrow$ (2). When the sequence (g_n) is continuous, then of course asymptotic equicontinuity is equivalent to equicontinuity. Finally, we note that when (g_n) is continuous the uniform convergence of (g_n) in (2) itself entails the continuity of the limit g.

The next result treats uniform convergence over $\mathbb{R}^k$.

THEOREM 2. *The following two statements are equivalent.*

(4) (g_n) *is a.u.e.c. and bounded, and* $G_n \Rightarrow G$.

(5) $g_n \to g$ *pointwise, uniformly in* $\mathbb{R}^k$, *where* g *is the uniformly continuous density of the distribution* G.

PROOF. (4) ⇒ (5). Since (g_n) is a.e.c. and bounded we know that $g_n \to g$ pointwise from Theorem 1 where g is the continuous density of G. But (g_n) a.u.e.c. implies that given $\varepsilon > 0$ there exist $\delta(\varepsilon)$, $n(\varepsilon)$ such that $|g_n(y) - g_n(x)| < \varepsilon$ whenever $|y - x| < \delta(\varepsilon)$ and $n > n(\varepsilon)$. Then if $|y - x| < \delta(\varepsilon)$

$$|g(y) - g(x)| = \lim_{n \to \infty} |g_n(y) - g_n(x)| < \varepsilon,$$

and hence g is uniformly continuous in $\mathbb{R}^k$.

We give a slightly different proof of the uniform convergence of (g_n) to that in Boos (1985). In view of Theorem 1, it suffices to prove that (i) $g_n(x_n) \to 0$ and (ii) $g(x_n) \to 0$ whenever $|x_n| \to \infty$. Consider first (i) and suppose to the contrary that there exists $\varepsilon > 0$ and a sequence (x_n) with $|x_n| \to \infty$ such that $g_{n'}(x_{n'}) > 2\varepsilon$ along some subsequence (n'). From asymptotic uniform equicontinuity it follows that there exist $\delta(\varepsilon) > 0$, $n(\varepsilon)$ such that $g_{n'}(y) > \varepsilon$ whenever $|y - x_{n'}| < \delta(\varepsilon)$ and $n' > n(\varepsilon)$. Hence $\int_{|y - x_{n'}| < \delta(\varepsilon)} g_{n'}(y)\, d\mu > \varepsilon[2\delta(\varepsilon)]^k$ for each n'. Thus $G_{n'}(\{y\colon |y| > |x_{n'}| - \delta(\varepsilon)\}) > \varepsilon[2\delta(\varepsilon)]^k$ and since $|x_{n'}| \to \infty$ the sequence (G_n) is not tight and hence cannot converge weakly. A similar argument using the uniform continuity of g shows that if (ii) fails then G is not tight and hence is not a probability measure.

(5) ⇒ (4). We can choose $\delta(\varepsilon)$ and $n(\varepsilon)$ such that $|g(y) - g(x)| < \varepsilon$ whenever $|y - x| < \delta(\varepsilon)$ and $|g_n(y) - g(y)| < \varepsilon$ for all y. Then if $|y - x| < \delta(\varepsilon)$ and $n > n(\varepsilon)$ (3) holds and (g_n) is a.u.e.c as required. The boundedness of (g_n) and weak convergence of (G_n) follow as before. □

Note that the conditions that g be continuous and $g(x_n) \to 0$ as $|x_n| \to \infty$ stipulated by Boos (1985) are redundant, since the asymptotic uniform equicontinuity and boundedness of (g_n) entail the existence and uniform continuity of the limit, which in turn imply that $g(x_n) \to 0$ as $|x_n| \to \infty$.

As remarked by Boos, it is more convenient to consider uniform convergence on compacts than pointwise convergence. In the latter case, it is possible to give a version of Theorem 1 by simply replacing the a.e.c and boundedness conditions on (g_n) in (1) by an appropriate sequential compactness condition. Such a result however is of little use, as it seems difficult to characterize compactness in the topology of pointwise convergence.

3. Uniform approximation of likelihood functions. As remarked in Boos (1985), local limit results are especially important in statistics whenever one wishes to construct an approximate likelihood function based on some appropriate statistic. It would also appear to be essential that such an approximation can be made uniformly in compact subsets of the parameter space Ω.

Suppose then that the distributions $G_n(x; \theta)$ depend on an unknown parameter $\theta \in \Omega$, which we assume to be a subset of $\mathbb{R}^l$ (although the results below may be stated for an arbitrary separable locally compact metric space). Again, for each θ we assume $G_n(x; \theta)$ is absolutely continuous with density $g_n(x; \theta)$ (not necessarily continuous in either x or θ). The following version of Theorem 1 follows easily on replacing x by (x, θ) throughout in the proof of Theorem 1; otherwise the proof requires no change.

THEOREM 3. *The following two statements are equivalent.*

(6) (g_n) *is a.e.c. and bounded in* $\mathbb{R}^k \times \Omega$, *and* $G_n(\ ;\theta) \Rightarrow G(\ ;\theta)$ *for each* $\theta \in \Omega$.

(7) $g_n \to g$ *pointwise, uniformly in compacts of* $\mathbb{R}^k \times \Omega$, *where* $g(\ ;\theta)$ *is the continuous (in* $\mathbb{R}^k \times \Omega$*) density of* $g(\ ;\theta)$.

Note that since (6) $\Rightarrow$ (7), the conditions in (6) also ensure that $G_n(\ ;\theta) \Rightarrow G(\ ;\theta)$ uniformly in compacts of $\mathbb{R}^k \times \Omega$, and that $G(\ ;\theta)$ is continuous in $\mathbb{R}^k \times \Omega$.

In practice, uniform convergence in $\mathbb{R}^k$ will often be necessary for likelihood approximations, as $G_n(x; \theta)$ will usually be the distribution of some quantity $u_n(T_n, \theta)$ where T_n is a statistic. Then the density of T_n is given by

$$f_n(t; \theta) = g_n(u_n(t; \theta); \theta)|(\partial/\partial t)u_n(t, \theta)|,$$

where g_n is the density of $u_n(T_n, \theta)$, and as θ ranges over compacts of Ω, the values of $u_n(t, \theta)$ for given t and all n will usually not be confined to some compact set. It will therefore be necessary to show that $g_n \to g$ uniformly in $\mathbb{R}^k \times K$ for every compact $K \in \Omega$.

THEOREM 4. *Let K be any compact subset of Ω. Then the following two statements are equivalent.*

(8) (g_n) *is a.u.e.c. and bounded in* $\mathbb{R}^k \times K$ *and* $G_n(\ ;\theta) \Rightarrow G(\ ;\theta)$ *uniformly in* $\mathbb{R}^k \times K$ *where G is continuous in* θ.

(9) $g_n \to g$ *pointwise, uniformly in* $\mathbb{R}^k \times K$, *where* $g(\ ;\theta)$ *is the density of G, uniformly continuous in* $\mathbb{R}^k \times K$.

Note that (g_n) a.e.c. in $\mathbb{R}^k \times \Omega$ and a.u.e.c. in $\mathbb{R}^k$ for each θ is equivalent to (g_n) a.u.e.c. in $\mathbb{R}^k \times K$ for every compact $K \in \Omega$. (This follows from the separability of Ω.) Theorem 4 follows as a corollary to Theorem 2, given the following basic lemma. Let X and Y be two metric spaces. The sequence (u_n) of functions u_n: $X \to Y$ is said to converge continuously to u if $u_n(x_n) \to u(x)$ for every sequence (x_n) with $x_n \to x$ for all $x \in X$.

LEMMA. *Let (u_n) be an arbitrary sequence of functions u_n: $X \to Y$ where X is locally compact. Then $u_n \to u$, a continuous limit, uniformly in compacts of X, if and only if $u_n \to u$ continuously.*

The lemma follows from standard arguments, and the proof is omitted. Theorem 4 now follows from Theorem 2 on taking convergent sequences (θ_n) and writing $g_n^*(x) = g_n(x; \theta_n)$. As an example, consider a translation statistic T_n satisfying the conditions of Theorem 1 in Boos (1985). Then if T_n is also a *scale* statistic, it is easily shown, following the proof in Boos, that (8) holds with $G_n(\ ; \theta)$ the distribution of $n^{1/2}(T_n - \mu)$ and $\theta = (\mu, \sigma)$, where μ is the target parameter of T_n, giving the desired uniform density convergence in (9).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SURREY
GUILDFORD, SURREY GU2 5XH
ENGLAND