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BAYESIAN STATISTICAL INFERENCE FOR SAMPLING A FINITE POPULATION[1]

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Bayesian statistical inference for sampling a finite population is studied by using the Dirichlet-multinomial process as prior. It is shown that if the finite population variables have a Dirichlet-multinomial prior, then the posterior distribution of the unobserved variables given a sample is also Dirichlet-multinomial. If the population size tends to infinity (the sample size is fixed), sampling without replacement from a Dirichlet multinomial process is equivalent to the iid sampling from a Dirichlet process. If both the population size and sample size tend to infinity, then given a sample, the posterior distribution of the population empirical distribution function converges in distribution to a Brownian bridge. The large-sample Bayes confidence band interval are given and shown to be equivalent to the usual ones obtained from simple random sampling.

1. Introduction. In an important article on sampling a finite population from a Bayes viewpoint, Ericson (1969) showed that if the prior distribution of the number of population variables belonging to the jth category, $j = 1, \dots, k$, is Dirichlet-multinomial (Mosimann (1962)), the posterior distribution of the number of unobserved population variables (given the observed ones) belonging to each category is also of the same type. The same result was also obtained by Hoadley (1969). Scott (1971) proved that the centered and rescaled posterior distribution converges to that of a normal random vector for arbitrary priors. In this note, we extend these results to the case of arbitrary population variables. This is accomplished by using the Dirichlet process introduced by Ferguson (1973). In Section 2 we define the Dirichlet-multinomial process, extend the Ericson–Hoadley theorem, and show that the Dirichlet process is the limit of the Dirichlet-multinomial process. In Section 3, we prove that the large-sample posterior distribution of the population empirical distribution, centered and rescaled, is a Brownian bridge with a change of time scale. As corollaries, the large-sample Bayes confidence band for the population empirical distribution function and the large-sample Bayes confidence interval for the population mean are obtained. Some difficulties in Binder (1982) are resolved.

2. The posterior distribution. Let F be a Dirichlet process on a complete and separable metric space R with index measure α (see Ferguson (1973)) denoted

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by $F \sim D(\alpha)$, and given F and any positive integer N, let $X_1, \dots, X_N$ be an iid sample from F. The marginal distribution of $X_1, \dots, X_N$ is symmetric and depends upon N and α. We denote this by $X_1, \dots, X_N \sim \overline{DM}(N; \alpha)$. Let $\{1, \dots, N\} = S + \bar{S}$ where $S \cap \bar{S} = \phi$ and let the number of elements in S be n. Denote $N - n$ by m. Let

$$(2.1) \qquad N(\cdot) = \sum_1^N \delta_{X_i}(\cdot) = m(\cdot) + n(\cdot),$$

where $m(\cdot) = \sum_{s \in \bar{S}} \delta_{X_s}(\cdot)$, $n(\cdot) = \sum_{s \in S} \delta_{X_s}(\cdot)$ and δ_x is a point mass at x. Let $\hat{F}_n(\cdot) = n^{-1} n(\cdot)$ be the empirical distribution function of $\{X_s, s \in S\}$. Let $H_N(\cdot) = N^{-1} N(\cdot)$.

DEFINITION 2.1. Suppose $X_1, \dots, X_N \sim \overline{DM}(N; \alpha)$. Denote the distribution of $N(\cdot)$ by $DM(N; \alpha)$.

The conditional distribution of $\{X_s, s \in \bar{S}\}$ given $\{X_s, s \in S\}$ is characterized by the following theorem.

THEOREM 2.1. *Suppose* $X_1, \dots, X_N \sim \overline{DM}(N; \alpha)$. *Then given* $\{X_s, s \in S\}$, $\{X_s, s \in \bar{S}\} \sim \overline{DM}(m; \alpha + n(\cdot))$. *Hence* $m(\cdot) | \{X_s, s \in S\} \sim DM(m; \alpha + n(\cdot))$.

PROOF. Since $X_1, \dots, X_N$ is exchangeable, $\{X_i, i \in \bar{S}\} | \{X_i, i \in S\} =_L \{X_i, n + 1 \le i \le N\} | \{X_i, 1 \le i \le n\}$. Hence, $F | \{X_i, 1 \le i \le n\} \sim D(\alpha + n(\cdot))$ implies $\{X_i, i \in \bar{S}\} | \{X_i, i \in S\} \sim \overline{DM}(m; \alpha + n(\cdot))$. □

The following result gives the posterior distribution of $m(\cdot)$ restricted to a subset of R and is equivalent to Theorem 5.3 of Hoadley (1969). The proof is inserted for completeness. Let B be a measurable subset of R. For any measure μ on R, let μ_B be the restriction of μ to $R - B$.

THEOREM 2.2. *Suppose* $X_1, \dots, X_N \sim \overline{DM}(N; \alpha)$. *Then given* $\{X_s, s \in S\}$ *and* $m(B)$, $m_B(\cdot) \sim DM(m - m(B); \alpha_B + n_B(\cdot))$.

PROOF. Suppose $\{X_s, s \in S\}$ is given. Then $F \sim D(\alpha + n(\cdot))$, $F_B/F(R - B) \sim D(\alpha_B + n_B(\cdot))$, and $\{X_s, s \in \bar{S}\} | F$ are iid F. Hence, given F and $m(B)$, the X_s's with $s \in \bar{S}$ which do not fall in B are iid $F_B/F(R - B)$. Since $F_B/F(R - B)$ is independent of $m(B)$, $F_B/F(R - B) | m(B) \sim D(\alpha_B + n_B(\cdot))$. The last two statements imply the result. □

Theorem 2.1 specializes to the Ericson–Hoadley theorem as can be seen as follows: Let $A_1, \dots, A_k$ be a partition of R, i.e., partition R into k categories. Then $N(A_j)$, $j = 1, \dots, k$ is a Dirichlet-multinomial vector with parameters N; $\alpha(A_j)$, $j = 1, \dots, k$ in the sense of Ericson (1969) and a compound multinomial vector according to Hoadley (1969). Since given $\{X_s, s \in S\}$, $\{X_s, s \in \bar{S}\} \sim \overline{DM}(m; \alpha + n(\cdot))$ by Theorem 2.1, the posterior distribution of $m(A_j)$, $j = 1, \dots, k$, given $\{X_s, s \in S\}$ is a Dirichlet-multinomial vector with parameters m; $\alpha(A_j) + n(A_j)$, $j = 1, \dots, k$. The last statement with $\alpha = \sum_{j=1}^{k} \alpha_j \delta_{y_j}$ where y_j, $j = 1, \dots, k$ are the (distinct) categorical values of the population variables and $\alpha_j \geq 0$ implies the result of Ericson and Hoadley.

The next result states that sampling without replacement from a finite population with a Dirichlet-multinomial prior is equivalent to the iid sampling from a Dirichlet process if the population size is large. Denote convergence in distribution of random variables by $\to_L$.

PROPOSITION 2.1. *Let $M_n \sim DM(n; \alpha)$. Then $n^{-1} M_n \to_L F$ where $F \sim D(\alpha)$.*

PROOF. Let $F \sim D(\alpha)$ and given F, $Y_1, \dots, Y_n$ are iid F. Then $\sum_1^n \delta_{Y_i} \sim DM(n; \alpha)$. Let $H_n = n^{-1} \sum_1^n \delta_{Y_i}$. For each (measurable) subset A of R and each F $P\{H_n(A) \to F(A) | F\} = 1$. By Fubini's theorem $H_n(A) \to F(A)$ a.s. and similarly, $H_n(A_j)$, $j = 1, \dots, k \to F(A_j)$, $j = 1, \dots, k$ a.s. for each collection $\{A_j, j = 1, \dots, k\}$. It follows that $H_n(A_j)$, $j = 1, \dots, k \to_L F(A_j)$, $j = 1, \dots, k$. Since R is a complete and separable metric space, Theorem 3.17 in Matthes, Kerstan, and Mecke (1978) implies the distribution of H_n converges to that of F. □

The above proposition together with Theorem 2.1 can be used to obtain some Bayes estimates of Ferguson (1973). For example, since $m(\cdot) | \{X_s, s \in S\} \sim DM(m; \alpha + n(\cdot))$, by Proposition 2.1 and for a fixed sample size n,

$$E[H_N(A) | X_s, s \in S] \to E[F(A) | X_s, s \in S], \tag{2.2}$$

and if $\int |x| \alpha(dx) < \infty$,

$$E\left[\int x H_N(dx) | X_s, s \in S\right] \to E\left[\int x F(dx) | X_s, s \in S\right], \tag{2.3}$$

where F given $\{X_s, s \in S\}$ is a Dirichlet process with index measure $\alpha + n(\cdot)$. Hence,

$$E[F(A) | X_s, s \in S] = \frac{\alpha(A)}{\alpha(R) + n} + \frac{n}{\alpha(R) + n} \hat{F}_n(A), \tag{2.4}$$

and if $\int |x| \alpha(dx) < \infty$,

$$E\left[\int x F(dx) | X_2, s \in S\right] = \frac{1}{\alpha(R) + n} \int x \alpha(dx) + \frac{n}{\alpha(R) + n} \overline{X}, \tag{2.5}$$

where $\overline{X}$ is the sample average. If the X's can only take values in a finite set, Ericson (1969, pages 212 and 213) established the above results by computing the posterior means of functionals of H_N first and then letting $N \to \infty$ for a fixed n. His expressions for the posterior means are applicable in our case as well (with obvious change of notation) and will not be reproduced.

3. The large-sample posterior distribution. Our next result deals with the large-sample posterior distribution of H_N. In the following, we assume that R is a finite q-dimensional cube in a Euclidean space, say $R = [0, 1]^q$. For brevity we denote $\mu((0, x])$ by $\mu(x)$ for any measure μ on $[0,1]^q$ and for each $x \in [0,1]^q$. We first compute the posterior mean and variance of $H_N(x) = N^{-1}\Sigma_1^N I_{[X_i \le x]}$ where $\le$ is the coordinatewise inequality.

Since $N(\cdot) \sim DM(N; \alpha)$ and $H_N(\cdot) = N^{-1}N(\cdot)$, from Mosimann (1962) or Ericson (1969) we find

$$E[H_N(x)|X_s, s \in S] = \frac{1}{N(\alpha(R) + n)} \times \left[(\alpha(R) + N) \sum_{s \in S} \delta_{X_s}(x) + m\alpha(x)\right] \tag{3.1}$$

and

$$\begin{aligned}\operatorname{var}[H_N(x)|X_s, s \in S] &= \left(\frac{m}{N^2}\right)\left(\frac{\alpha(x) + \Sigma_{s \in S}\delta_{X_s}(x)}{\alpha(R) + n}\right)\\ &\quad \times \left(\frac{\alpha(R) - \alpha(x) + n - \Sigma_{s \in S}\delta_{X_s}(x)}{\alpha(R) + n}\right)\left(\frac{N + \alpha(R)}{1 + n + \alpha(R)}\right).\end{aligned} \tag{3.2}$$

Let us denote the centered (at $\hat{F}_n$) and rescaled H_N by Y_{mn}, i.e.,

$$\begin{aligned} Y_{mn}(x) &= \left(\frac{nN}{m}\right)^{1/2}\{H_N(x) - \hat{F}_n(x)\}\\ &= \left(\frac{mn}{N}\right)^{1/2}\{m^{-1}m(x) - \hat{F}_n(x)\}\\ &= \tilde{Y}_{mn}(x) + \varepsilon_{mn}(x),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\tilde{Y}_{mn}(x) &= \left(\frac{mn}{N}\right)^{1/2}\{m^{-1}m(x) - \tilde{F}_n(x)\},\\ \varepsilon_{mn}(x) &= \left(\frac{mn}{N}\right)^{1/2}\{\tilde{F}_n(x) - \hat{F}_n(x)\},\\ \tilde{F}_n(x) &= \frac{\alpha(x)}{\alpha(R) + n} + \frac{n}{\alpha(R) + n}\hat{F}_n(x).\end{aligned} \tag{3.4}$$

The main result of this section is that $Y_{mn}(\cdot)$ coverges in distribution to a Brownian bridge on $[0,1]^q$. This convergence can be established by showing that $\tilde{Y}_{mn}(\cdot)$ converges in distribution to the same limit since $\sup_x|\varepsilon_{mn}(x)| \le 2\alpha(R)n^{-1/2}$. We first establish the tightness of $\tilde{Y}_{mn}(\cdot)$. Denote weak convergence of measure by $\Rightarrow$.

LEMMA 3.1. *Given $\{X_s, s \in S\}$, the sequence $\{\tilde{Y}_{mn}\}$ is tight in $D[0,1]^q$ if F is continuous and $\hat{F}_n \Rightarrow F$.*

PROOF. Denote $\alpha(R) + N$ by α_N and $\alpha(R) + n$ by α_n. We use the fluctuation inequality of Bickel and Wichura (1971) to show tightness. Let B, C be two neighboring blocks; then

$$E\tilde{Y}^2_{mn}(B)\tilde{Y}^2_{mn}(C) = \left(\frac{n}{Nm}\right)^2 E\Big\{[m(B) - m\tilde{F}_n(B)]^2 \times E\Big[(m(C) - m\tilde{F}_n(C))^2 | m(B)\Big]\Big\}, \tag{3.5}$$

where E denotes conditional expectation given $\{X_s, s \in S\}$. Let $A = (B + C)^c$. According to Theorem 2.2 (or Theorem 5.3 of Hoadley (1969)), given $m(B)$, the joint distribution of $m(C)$ and $m(A)$ is a Dirichlet-multinomial vector with parameters $m - m(B)$, $n(C)$, and $n(A)$. Therefore the moment expression in Mosimann (1982, equation (15)) can be applied to obtain

$$\begin{aligned} &E\Big[(m(C) - m\tilde{F}_n(C))^2 | m(B)\Big] \\ &= E\left[\left(m(C) - \frac{(m - m(B))\tilde{F}_n(C)}{\tilde{F}_n(A + C)}\right)^2 | m(B)\right] \\ &\quad + \left\{[m - m(B)]\frac{\tilde{F}_n(C)}{\tilde{F}_n(A + C)} - m\tilde{F}_n(C)\right\}^2 \\ &= [m\tilde{F}_n(B) - m(B)]^2 \frac{\tilde{F}_n(C)}{\tilde{F}_n(A + C)}\left[\frac{1 + \alpha_n\tilde{F}_n(C)}{1 + \alpha_n\tilde{F}_n(A + C)}\right] \\ &\quad + [m\tilde{F}_n(B) - m(B)](\alpha_N + m)\frac{\tilde{F}_n(C)\tilde{F}_n(A)}{\tilde{F}_n(A + C)(1 + \alpha_n\tilde{F}_n(A + C))} \\ &\quad + \frac{\alpha_N m\tilde{F}_n(C)\tilde{F}_n(A)}{1 + \alpha_n\tilde{F}_n(A + C)}. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned} &E\Big\{[m(B) - m\tilde{F}_n(B)]^2[m(C) - m\tilde{F}_n(C)]^2\Big\} \\ &= E[m(B) - m\tilde{F}_n(B)]^4 \frac{\tilde{F}_n(C)}{\tilde{F}_n(A + C)}\frac{1 + \alpha_n\tilde{F}_n(C)}{1 + \alpha_n\tilde{F}_n(A + C)} \\ &\quad - E[m(B) - m\tilde{F}_n(B)]^3(\alpha_N + m)\frac{\tilde{F}_n(C)\tilde{F}_n(A)}{\tilde{F}_n(A + C)(1 + \alpha_n\tilde{F}_n(A + C))} \\ &\quad + E[m(B) - m\tilde{F}_n(B)]^2 \frac{\alpha_N m\tilde{F}_n(C)\tilde{F}_n(A)}{1 + \alpha_n\tilde{F}_n(A + C)} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{3.7}$$

From Johnson and Kotz (1969, page 231), we find

$$E[m(B) - m\tilde{F}_n(B)]^4 = m\tilde{F}_n(B)\tilde{F}_n(A+C)\left(\frac{\alpha_N}{\alpha_n+1}\right)\left(\frac{1}{\alpha_n+2}\right)\left(\frac{1}{\alpha_n+3}\right)$$
$$\times\{(\alpha_N+m)(\alpha_N+2m)[1-3\tilde{F}_n(B)\tilde{F}_n(A+C)] + \alpha_n(m-1)[1+3\alpha_N\tilde{F}_n(B)\tilde{F}_n(A+C)]\}, \tag{3.8}$$

$$E[m(B) - m\tilde{F}_n(B)]^3 = m\tilde{F}_n(B)\tilde{F}_n(A+C)\left(\frac{\alpha_N}{\alpha_n+1}\right)\times\left(\frac{\alpha_N+m}{\alpha_n+2}\right)[\tilde{F}_n(B) - \tilde{F}_n(A+C)], \tag{3.9}$$

and

$$E[m(B) - m\tilde{F}_n(B)]^2 = m\tilde{F}_n(B)\tilde{F}_n(A+C)\left(\frac{\alpha_N}{\alpha_n+1}\right). \tag{3.10}$$

Substitute (3.8) into I to obtain

$$\mathrm{I} \le 8\left(\frac{\alpha_N m}{\alpha_n}\right)^2 \tilde{F}_n(B)\tilde{F}_n(C) \quad \text{if } m \ge 2 \text{ and } n \ge 2.$$

Similar substitutions for II and III yield

$$\mathrm{II} \le 4\left(\frac{\alpha_N m}{\alpha_n}\right)^2 \tilde{F}_n(B)\tilde{F}_n(C)$$

and

$$\mathrm{III} \le \left(\frac{\alpha_N m}{\alpha_n}\right)^2 \tilde{F}_n(B)\tilde{F}_n(C).$$

Thus

$$\mathrm{I} + \mathrm{II} + \mathrm{III} \le 13\left(\frac{\alpha_N m}{\alpha_n}\right)^2 \tilde{F}_n(B)\tilde{F}_n(C),$$

implying $E\tilde{Y}_{mn}^2(B)\tilde{Y}_{mn}^2(C) \le 13\ (1+\alpha(R))^2\tilde{F}_n(B)\tilde{F}_n(C)$, provided $m \ge 2$ and $n \ge 2$. By the extended version of Theorem 3 in Bickel and Wichura (1971), $\{\tilde{Y}_{mn}\}$ is tight. □

Let $B(x)$ be a Gaussian process with zero means and covariance $\operatorname{cov}(B(t), B(s)) = F(\min(t,s)) - F(t)F(s)$ where the minimum is computed coordinatewise.

THEOREM 3.1. *Suppose F is continuous and $\hat{F}_n \Rightarrow F$; then given $\{X_s, s \in S\}$, $Y_{mn}(\cdot) \to_L B(\cdot)$ in $D[0,1]^q$ as $m \to \infty$ and $n \to \infty$.*

PROOF. According to Scott (1971), the finite dimensional conditional distribution of $Y_{mn}(\cdot)$ converges to that of $B(\cdot)$. Hence, the finite dimensional conditional distribution of $\tilde{Y}_{mn}(\cdot)$ also converges to that of $B(\cdot)$. An application of Lemma 3.1 entails $\tilde{Y}_{mn}(\cdot) \to_L B(\cdot)$, implying $Y_{mn}(\cdot) \to_L B(\cdot)$. □

If $q = 1$, Theorem 3.1 and the continuous mapping theorem can be applied to give the following corollaries:

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1,

$$\lim_{m,n\to\infty} P\Big\{\sup_{0\le t\le 1} |Y_{mn}(t)| > \lambda | X_s, s \in S\Big\} = 2\sum_{j=1}^{\infty} (-1)^{j+1} e^{-2j^2\lambda^2}. \tag{3.11}$$

According to Corollary 3.1, a $(1-\alpha)$ large-sample Bayes confidence band for H_N is given by

$$\hat{F}_n \pm \lambda\left(\frac{1-f}{n}\right)^{1/2}, \tag{3.12}$$

where $1 - f = 1 - n/N$ is the finite population correction factor (fpc) and λ is the $(1-\alpha)$-percentile point of $\sup_s|B(s)|$ defined by $2\sum_{j=1}^{\infty}(-1)^{j+1}\exp^{(-2j^2\lambda^2)} = \alpha$. Deleting the fpc, the last band becomes the Kolmogorov–Smirnov band obtained by Lo (1983) using a Dirichlet prior and iid sampling.

COROLLARY 3.2. *Suppose* $\lim_{n\to\infty}(n-1)^{-1}\sum_{s\in S}(X_s - \overline{X})^2 = \sigma^2$. *Under the assumptions of Theorem* 3.1,

$$\begin{aligned} &\lim_{m,n\to\infty} P\left\{\left(\frac{Nn}{m}\right)^{1/2} \left|\int sH_N(ds) - \overline{X}\right| \le \lambda | X_s, s \in S\right\} \\ &\quad = (2\pi\sigma^2)^{-1/2}\int_{-\lambda}^{\lambda} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right) dx. \end{aligned} \tag{3.13}$$

Corollary 3.2 partly extends the corollary of Scott (1971), who established the result for more general priors, but under the restriction that the X's can only take values in a finite set.

REMARK 3.1. The conclusion of Corollary 3.2 also appears in Binder (1982, Section 2). However, Binder's argument is based on a result of Scott (1971) and does not work: The definition of A_k there implies that k grows with the sample size whereas Scott's result is applicable for a fixed k only.

An immediate consequence of Corollary 3.2 is that a $(1-\alpha)$ large-sample confidence interval for the population mean $\int sH_N(ds)$ is given by

$$\overline{X} \pm \lambda S_n\left(\frac{1-f}{n}\right)^{1/2}, \tag{3.14}$$

where $S_n^2 = (n-1)^{-1}\sum_{s\in S}(X_s - \overline{X})^2$ is the sample variance and λ is the $(1-\alpha/2)$-percentile point of a standard normal random variable. This large-sample Bayes confidence interval coincides with the well known large-sample confidence interval for the population mean from simple random sampling (Cochran (1977)). Deleting the fpc, the last interval coincides with the usual large-sample sample theorist confidence interval for a population mean obtained from iid sampling.

4. Concluding remarks. The results in the previous sections can be applied to stratified population models. The idea is that each stratum can be treated as an independent population model and a Dirichlet-multinomial prior can be assigned to the population variables of the stratum in question. If the priors for different strata are assumed to be independent, the results in the previous sections hold for each stratum. These can be applied to justify the large-sample result in Binder (1982, Section 2.3).

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