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OPTIMAL PROPERTIES OF THE BECHHOFER–KULKARNI BERNOULLI SELECTION PROCEDURE[1]

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In a recent article Bechhofer and Kulkarni proposed a class of closed adaptive sequential procedures for selecting that one of $k \geq 2$ Bernoulli populations with the largest single-trial success probability. These sequential procedures which take no more than n observations from any one of the k populations achieve the same probability of a correct selection as does a single-stage procedure which takes exactly n observations from every one of the k populations. In addition, they often require substantially less than a total of kn observations to terminate sampling. Amongst other problems, Bechhofer and Kulkarni considered the problem of devising a procedure within this class which minimizes the expected total number of observations to terminate sampling. For their proposed procedure they cited several optimality properties for the case $k = 2$ and conjectured additional optimality properties for the case $k \geq 3$.

In this article we use a new method of proof to establish stronger results than those cited by Bechhofer and Kulkarni for the case $k = 2$, and prove stronger results than those conjectured for $k \geq 3$. We also describe a new procedure for $k \geq 3$ and prove that it minimizes the expected total number of observations to terminate sampling when all of the success probabilities are small.

1. Introduction. Let Π_i $(1 \leq i \leq k)$ denote $k \geq 2$ Bernoulli populations with corresponding single-trial "success" probabilities p_i. We denote the ordered values of the p_i by $p_{[1]} \leq \cdots \leq p_{[k]}$. Let $\mathbf{p} = (p_1, \ldots, p_k)$ and $\bar{\mathbf{p}} = (p_{[1]}, \ldots, p_{[k]})$. The pairing of the Π_i with the $p_{[j]}$ $(1 \leq i, j \leq k)$ is assumed to be completely unknown. Thus for given $\bar{\mathbf{p}}$, $\mathbf{p}$ has probability $1/k!$ of being any *particular* permutation of $\bar{\mathbf{p}}$. The goal of the experimenter is to select as "best" a population with success probability $p_{[k]}$; when such a population is selected we say that a *correct selection* (CS) has been made.

Define $N_{(i)}$ to be the number of observations taken from the population associated with $p_{[i]}$ at the termination of sampling and $N = \sum_{i=1}^{k} N_{(i)}$ to be the total number of observations taken. For a given value of $\bar{\mathbf{p}}$ we denote the expected value of N using procedure $\mathscr{P}$ by $E_{\mathscr{P}}\{N|\bar{\mathbf{p}}\}$.

We shall define a class $\mathscr{C}$ of procedures, all of which achieve the same $P\{\text{CS}\}$, uniformly in $\bar{\mathbf{p}}$, as does the single-stage procedure which takes n observations from each population and selects the population with the largest number of successes, breaking ties at random. The procedures in $\mathscr{C}$ share a common stopping rule and terminal decision rule but use different sampling rules; they

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take no more than n observations from any one population and curtail sampling as soon as is possible without decreasing the $P\{\mathrm{CS}\}$.

We are concerned with finding procedures in $\mathscr{C}$ which minimize $E\{N|\bar{\mathbf{p}}\}$ and/or minimize $E\{\sum_{i=1}^{k}\lambda_i N_{(i)}|\bar{\mathbf{p}}\}$ where $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$. (This latter goal generalizes to $k \geq 3$ the goal of minimizing $E\{N_{(1)}|\bar{\mathbf{p}}\}$, the expected number of observations from the "inferior" population, when $k = 2$.) This approach contrasts with that of comparing the performance of procedures which guarantee a certain $P\{\mathrm{CS}\}$ requirement in, say, an indifference-zone formulation, in that by restricting attention to procedures in $\mathscr{C}$ we compare procedures which guarantee exactly the *same* $P\{\mathrm{CS}\}$ *uniformly* in $\bar{p}$.

If $\bar{\mathbf{p}}$ is known or if a prior distribution for $\bar{\mathbf{p}}$ is specified, an optimal procedure within $\mathscr{C}$ for a particular goal exists and can be found by backwards induction. However, calculation of such a procedure becomes prohibitively expensive as n or k increases. We shall show that certain simply-described procedures are optimal for particular regions of $\bar{\mathbf{p}}$ space. This allows an experimenter to use the procedure which is optimal for an initial estimate of $\bar{\mathbf{p}}$, with the option of switching to a different procedure as further information on $\bar{\mathbf{p}}$ is obtained.

Bechhofer and Kulkarni (1982a) state several theorems regarding the optimality of one procedure in $\mathscr{C}$ for $k = 2$. These theorems are proved in Kulkarni (1981) and the performance characteristics of this procedure and of a generalized procedure for $k \geq 3$ are described in Bechhofer and Kulkarni (1982b), Bechhofer and Frisardi (1983), Percus and Percus (1984), Jennison (1984) and Kulkarni and Kulkarni (1985). In the present paper we use a new method of proof to strengthen results for the case $k = 2$ and obtain new results for $k \geq 3$. The same method is used in Jennison and Kulkarni (1984) to derive optimal procedures for the problem of selecting the s "best" populations, where $1 \leq s \leq k - 1$.

In Section 2 we define the class of procedures $\mathscr{C}$. Sections 3 and 4 are concerned with the case $k = 2$. In Section 3 we define the procedure $\mathscr{P}^*$ and prove that it minimizes $E\{N|\bar{\mathbf{p}}\}$ among procedures in $\mathscr{C}$ whenever $p_{[1]} + p_{[2]} \geq 1$; a conjugate procedure $\bar{\mathscr{P}}^*$ minimizes $E\{N|\bar{\mathbf{p}}\}$ whenever $p_{[1]} + p_{[2]} \leq 1$. In Section 4 we show that $\mathscr{P}^*$ also minimizes $E\{N_{(1)}|\bar{\mathbf{p}}\}$ in a specific region of the parameter space and give conditions for $\mathscr{P}^*$ and $\bar{\mathscr{P}}^*$ to minimize the expected total number of failures.

Section 5 extends the above results to the case $k \geq 3$. We prove that the generalization of $\mathscr{P}^*$ proposed by Bechhofer and Kulkarni (1982a) minimizes $E\{N|\bar{\mathbf{p}}\}$ in a specific region of $\bar{\mathbf{p}}$-space where all the $p_{[i]}$ $(1 \leq i \leq k)$ are sufficiently large. We define a generalization of $\bar{\mathscr{P}}^*$ and prove that it minimizes $E\{N|\bar{\mathbf{p}}\}$ in another specific region of $\bar{\mathbf{p}}$-space where the $p_{[i]}$ $(1 \leq i \leq k)$ are sufficiently small. The goal of minimizing the expected number of observations from an inferior population is generalized to the case $k \geq 3$ and we prove that $\mathscr{P}^*$ is optimal in this respect over a specific region of $\bar{\mathbf{p}}$-space; in particular, $\mathscr{P}^*$ minimizes $E\{\sum_{i=1}^{k}\lambda_i N_{(i)}|\bar{\mathbf{p}}\}$ for any $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ whenever $p_{[1]} + p_{[2]} \geq 1$.

2. A class of closed sequential selection procedures. A selection procedure is characterized by a sampling rule, a stopping rule, and a terminal decision rule. We shall use the following notation in specifying such rules: when a total of

m observations have been taken the experiment is said to be at stage m, and the state of the experiment at this point is the complete history of the first m observations. This is denoted by $y_m = \{(i_1, v_1), (i_2, v_2), \ldots, (i_m, v_m)\}$, where i_r is the index of the population from which the rth observation is taken, and v_r is the outcome of the rth observation (either a success or a failure); the notation y without a subscript will be used to denote a general state of an experiment. The following variables are functions of y_m, although the dependence on m is suppressed in the notation:

n_i = number of observations taken from Π_i in the first m stages,

s_i = number of "successes" yielded by Π_i in the first m stages,

f_i = number of "failures" yielded by Π_i in the first m stages.

We restrict attention to procedures which take no more than n observations from any one of the k populations where $n \geq 1$ is a prespecified integer; thus $0 \leq m \leq kn$ and $n_i \leq n$ $(1 \leq i \leq k)$. Let the set of all possible states be

$$\Omega = \left\{ y_m : 0 \leq m \leq kn;\ n_i \leq n, 1 \leq i \leq k;\ \sum_{i=1}^{k} n_i = m \right\}.$$

A sampling rule, $\mathscr{R}$, is a (possibly random) function from Ω tc $\{1, 2, \ldots, k\}$ which specifies the index of the population from which the next observation is to be taken.

The stopping rule $\mathscr{S}^*$ is: "Stop at the first stage m at which there exists at least one population Π_i such that

$$s_i \geq s_j + n - n_j \quad \text{for all } j \neq i." \tag{2.1}$$

The terminal decision rule $\mathscr{T}^*$ is: "Select as best the population Π_i which satisfies (2.1). If more than one population satisfies (2.1) then select one of these populations at random." Since we shall restrict attention to procedures using the stopping rule $\mathscr{S}^*$, the notation y_m will be used to refer only to states which could arise under the stopping rule $\mathscr{S}^*$.

We shall consider the class $\mathscr{C}$ of sequential procedures which use the stopping rule $\mathscr{S}^*$, terminal decision rule $\mathscr{T}^*$, and arbitrary sampling rule $\mathscr{R}$, which takes no more than n observations from any one population; we write $\mathscr{P} = (\mathscr{R}, \mathscr{S}^*, \mathscr{T}^*)$ to denote the procedure in $\mathscr{C}$ using the sampling rule $\mathscr{R}$.

THEOREM 2.1. *If $\mathscr{P}_1$ and $\mathscr{P}_2$ are in $\mathscr{C}$ then $P\{\mathrm{CS}|\bar{\mathbf{p}}\}$ is equal for $\mathscr{P}_1$ and $\mathscr{P}_2$, uniformly in $\bar{\mathbf{p}}$.*

This theorem is proved in Bechhofer and Kulkarni (1982a) and is a special case of a more general theorem proved in Jennison (1983). A consequence of the general theorem is that the $P\{\mathrm{CS}\}$ for the single-stage procedure of Sobel and Huyett (1957) equals the $P\{\mathrm{CS}\}$ for any $\mathscr{P} \in \mathscr{C}$, uniformly in $\bar{\mathbf{p}}$.

REMARK 2.1. The stopping rule $\mathscr{S}^*$ terminates the experiment as soon as one or more populations which will have the most successes in their first n

observations can be identified. To stop sampling any earlier would decrease the $P\{\mathrm{CS}\}$. Hence, a procedure which is optimal within $\mathscr{C}$ for some objective function is also optimal within the class of all procedures taking at most n observations from any one population, that achieve the same $P\{\mathrm{CS}\}$ as the single-stage procedure, uniformly in $\bar{\mathbf{p}}$.

REMARK 2.2. For $k = 2$, if $\mathscr{R}$ is play-the-winner sampling the procedure $(\mathscr{R}, \mathscr{S}^*, \mathscr{T}^*)$ is the same as that proposed by Hoel (1972); however, this procedure is *not* optimal for our problem.

3. Minimizing the expected total number of observations: $k = 2$. We seek a procedure which is optimal within the class $\mathscr{C}$, in the sense that it minimizes $E\{N|(p_{[1]}, p_{[2]})\}$. We first consider the case in which $p_{[1]} < p_{[2]}$ are *known* but the pairing of Π_1, Π_2 with $p_{[1]}, p_{[2]}$ is unknown. Let $\omega \in \{1,2\}$ denote the state of nature, where $\omega = 1$ if $(p_1, p_2) = (p_{[2]}, p_{[1]})$, i.e., Π_1 is "best", and $\omega = 2$ if $(p_1, p_2) = (p_{[1]}, p_{[2]})$, i.e., Π_2 is "best"; the prior distribution for ω is given by $P\{\omega = 1\} = P\{\omega = 2\} = \frac{1}{2}$.

For $k = 2$, the complete history, y_m, at stage m can be summarized by $x(y_m) = (s_1, f_1, n_1; s_2, f_2, n_2)$. Here, $s_i + f_i = n_i$ $(i = 1,2)$ and the stopping rule $\mathscr{S}^*$ given in (2.1) can be written as: "Stop as soon as

$$s_1 + f_2 = n \quad (\text{select } \Pi_1) \tag{3.1}$$

or

$$s_2 + f_1 = n \quad (\text{select } \Pi_2)." \tag{3.2}$$

For given $\bar{\mathbf{p}} = (p_{[1]}, p_{[2]})$, Bayes optimal procedures which minimize $E\{N|\bar{\mathbf{p}}\}$ can be found by backwards induction; Kulkarni (1981) used dynamic programming to construct such procedures. It can be seen from the backwards induction argument that there are Bayes optimal procedures with nonrandomized sampling rules. In fact, there are Bayes optimal procedures whose sampling rules depend on $x(y_m)$ only, but we shall make use of the more general form of procedure in proving Theorem 3.1 below. For a nonrandomized procedure $\mathscr{P} = (\mathscr{R}, \mathscr{S}^*, \mathscr{T}^*) \in \mathscr{C}$ we denote by $d(y_m, \mathscr{R}) \in \{1,2\}$ the index of the population to be sampled next when in state y_m, if the experiment has not yet stopped; since a procedure in $\mathscr{C}$ is determined by its sampling rule we use the notation $d(y, \mathscr{R})$ and $d(y, \mathscr{P})$ interchangeably.

The procedure $\mathscr{P}^* = (\mathscr{R}^*, \mathscr{S}^*, \mathscr{T}^*)$ defined in Kulkarni (1981) and Bechhofer and Kulkarni (1982a) uses the following sampling rule which is also the Least Failures Rule defined by Kelly (1981).

DEFINITION OF SAMPLING RULE $\mathscr{R}^*$. "Sample Π_1 next when $f_1 < f_2$ or $(f_1 = f_2$ and $s_1 > s_2)$. Sample Π_2 next when $f_2 < f_1$ or $(f_1 = f_2$ and $s_2 > s_1)$. Sample Π_1 or Π_2 next with probability $\frac{1}{2}$ each if $s_1 = s_2$ and $f_1 = f_2$."

We now define a subclass $\mathscr{C}^*$ of the nonrandomized procedures in $\mathscr{C}$. Let $\Omega_B = \{y\colon s_1 = s_2, f_1 = f_2)$ and let $\Omega_A = \Omega \setminus \Omega_B$. Procedure $\mathscr{P}$ is in $\mathscr{C}^*$ if it uses the stopping rule $\mathscr{S}^*$, the terminal decision rule $\mathscr{T}^*$ and a nonrandomized sampling rule $\mathscr{R}$ for which $d(y; \mathscr{R}) = 1$ when $f_1 < f_2$ or ($f_1 = f_2$ and $s_1 > s_2$) and $d(y; \mathscr{R}) = 2$ when $f_2 < f_1$ or ($f_1 = f_2$ and $s_2 > s_1$); there is no restriction on $d(y; \mathscr{R})$ when $y \in \Omega_B$. The notation $d(y; \mathscr{C}^*)$ will be used to denote the above function for $y \in \Omega_A$. Note that $\mathscr{P}^*$ differs from the procedures in $\mathscr{C}^*$ only in that it randomizes between Π_1 and Π_2 for $y \in \Omega_B$.

Theorem 3.1. Amongst all procedures in $\mathscr{C}$, those in $\mathscr{C}^*$ are optimal, in the sense that they minimize $E\{N|(p_{[1]}, p_{[2]})\}$ whenever $p_{[1]} + p_{[2]} \geq 1$.

Method of proof. For a particular pair $\bar{\mathbf{p}} = (p_{[1]}, p_{[2]})$ with $p_{[1]} + p_{[2]} \geq 1$ we take a nonrandomized Bayes optimal procedure $\mathscr{P} \in \mathscr{C}$ and modify it by steps to obtain a procedure in $\mathscr{C}^*$. We show that at each step, $E\{N|\bar{\mathbf{p}}\}$ decreases or remains equal (in fact, it must remain equal), and thus the resulting procedure is optimal for $\bar{\mathbf{p}}$. It is easily seen that $E\{N|\bar{\mathbf{p}}\}$ is the same for all procedures in $\mathscr{C}^*$ and hence, all procedures in $\mathscr{C}^*$ are optimal for all $\bar{\mathbf{p}}$ with $p_{[1]} + p_{[2]} \geq 1$.

Remark 3.1. It is easily seen that $E_{\mathscr{P}^*}\{N|\bar{\mathbf{p}}\} = E_{\mathscr{P}}\{N|\bar{\mathbf{p}}\}$ for all $\mathscr{P} \in \mathscr{C}^*$; thus $\mathscr{P}^* = (\mathscr{R}^*, \mathscr{S}^*, \mathscr{T}^*)$ is also optimal among procedures in $\mathscr{C}$, whenever $p_{[1]} + p_{[2]} \geq 1$.

In order to prove Theorem 3.1 and for the proofs of later results, we require the following lemmas:

Lemma 3.1. *For a given nonrandomized procedure $\mathscr{P} \in C$ and a given state $\bar{y}_{\bar{m}}$, suppose that after being in state $\bar{y}_{\bar{m}}$ it is impossible to stop, using $\mathscr{P}$, without eventually taking an observation from Π_i. Then there is a nonrandomized procedure $\mathscr{P}' \in \mathscr{C}$ such that*

$$d(y_m; \mathscr{P}') = d(y_m; \mathscr{P}) \quad \text{for } m < \bar{m},$$

$$d(y_{\bar{m}}; \mathscr{P}') = d(y_{\bar{m}}; \mathscr{P}) \quad \text{for } y_{\bar{m}} \neq \bar{y}_{\bar{m}},$$

$$d(\bar{y}_{\bar{m}}; \mathscr{P}') = i$$

and $E_{\mathscr{P}'}\{N_{(1)}|\bar{\mathbf{p}}\} \leq E_{\mathscr{P}}\{N_{(1)}|\bar{\mathbf{p}}\}$ and $E_{\mathscr{P}'}\{N_{(2)}|\bar{\mathbf{p}}\} \leq E_{\mathscr{P}}\{N_{(2)}|\bar{\mathbf{p}}\}$ for all $\bar{\mathbf{p}}$.

Proof. (i) If $d(\bar{y}_{\bar{m}}; \mathscr{P}) = i$, take $\mathscr{P}' = \mathscr{P}$.

(ii) If $d(\bar{y}_{\bar{m}}; \mathscr{P}) \neq i$, define $\mathscr{P}'$ as follows: "Take the first $\bar{m}$ observations using the sampling rule of $\mathscr{P}$, and if $y_{\bar{m}} \neq \bar{y}_{\bar{m}}$, continue to use $\mathscr{P}$. If $y_{\bar{m}} = \bar{y}_{\bar{m}}$, take an observation from Π_i and then continue to use the sampling rule $\mathscr{P}$, ignoring the observation on Π_i, until $\mathscr{P}$ calls for an observation from Π_i; now use the sampling rule $\mathscr{P}$, behaving as if the earlier observation from Π_i had been taken at this point. At all times use the stopping rule $\mathscr{S}^*$, based on *all* of the observations that have been taken up to that stage."

For a fixed pair of sequences of observations on Π_1 and Π_2, $\mathscr{P}$ must take at least as many observations as $\mathscr{P}'$ on each of Π_1 and Π_2 and the result follows. □

LEMMA 3.2. *For a given nonrandomized procedure $\mathscr{P} \in \mathscr{C}$ and a given state $\bar{y}_{\bar{m}} \in \Omega_B$ (i.e., $\bar{s}_1 = \bar{s}_2$ and $\bar{f}_1 = \bar{f}_2$) there are nonrandomized procedures $\mathscr{P}_1$ and $\mathscr{P}_2 \in \mathscr{C}$ such that*

$$d(y_m; \mathscr{P}_1) = d(y_m; \mathscr{P}_2) = d(y_m; \mathscr{P}) \quad \text{for } m < \bar{m},$$

$$d(y_{\bar{m}}; \mathscr{P}_1) = d(y_{\bar{m}}; \mathscr{P}_2) = d(y_{\bar{m}}; \mathscr{P}) \quad \text{for } y_{\bar{m}} \neq \bar{y}_{\bar{m}},$$

$$d(\bar{y}_{\bar{m}}; \mathscr{P}_1) = 1, \qquad d(\bar{y}_{\bar{m}}; \mathscr{P}_2) = 2$$

and $E_{\mathscr{P}_1}\{N_{(1)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}_2}\{N_{(1)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}}\{N_{(1)}|\bar{\mathbf{p}}\}$ *and* $E_{\mathscr{P}_1}\{N_{(2)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}_2}\{N_{(2)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}}\{N_{(2)}|\bar{\mathbf{p}}\}$ *for all* $\bar{\mathbf{p}}$.

PROOF. We construct a procedure $\mathscr{P}_1$ with the required properties. A procedure $\mathscr{P}_2$ can be constructed similarly. Let i^c denote the complement of i in $\{1,2\}$.

(i) If $d(\bar{y}_{\bar{m}}; \mathscr{P}) = 1$, take $\mathscr{P}_1 = \mathscr{P}$.

(ii) If $d(\bar{y}_{\bar{m}}, \mathscr{P}) = 2$, define $\mathscr{P}_1$ as follows: "Take the first $\bar{m}$ observations using the sampling rule of $\mathscr{P}$, and if $y_{\bar{m}} \neq \bar{y}_{\bar{m}}$ continue to use $\mathscr{P}$. If $y_{\bar{m}} = \bar{y}_{\bar{m}}$, take $d(y_m; \mathscr{P}_1) = \{d(y'_m; \mathscr{P})\}^c$ for $m \geq \bar{m}$, where y'_m is formed from y_m by changing i_r to i_r^c for all $r \geq \bar{m} + 1$." Thus, when $y_{\bar{m}} = \bar{y}_{\bar{m}}$, $\mathscr{P}_1$ behaves like $\mathscr{P}$ would behave if the labels of the two populations had been interchanged after stage $\bar{m}$. By symmetry, $E_{\mathscr{P}_1}\{N_{(1)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}}\{N_{(1)}|\bar{\mathbf{p}}\}$ and $E_{\mathscr{P}_1}\{N_{(2)}|\bar{\mathbf{p}}\} = E_{\mathscr{P}}\{N_{(2)}|\bar{\mathbf{p}}\}$ for all $\mathbf{p}$. □

Let $\Omega_C = \{y \in \Omega_A\colon s_1 + f_2 = n - 1 \text{ or } s_2 + f_1 = n - 1\}$. Thus Ω_C is the set of states for which either $s_1 \neq s_2$ or $f_1 \neq f_2$, and the next observation will be the last one if it takes the appropriate value.

LEMMA 3.3. *For a given nonrandomized procedure $\mathscr{P} \in \mathscr{C}$, define $\mathscr{P}'$ by*

$$d(y; \mathscr{P}') = d(y; \mathscr{P}) \quad \text{for } y \notin \Omega_C,$$

$$d(y; \mathscr{P}') = d(y; \mathscr{C}^*) \quad \text{for } y \in \Omega_C.$$

Then $E_{\mathscr{P}'}\{N|\bar{\mathbf{p}}\} \leqq E_{\mathscr{P}}\{N|\bar{\mathbf{p}}\}$ *whenever* $p_{[1]} + p_{[2]} \geq 1$.

PROOF. Without loss of generality we consider a state $\bar{y} \in \Omega_C$ with $x(\bar{y}) = (\bar{s}_1, \bar{f}_1, \bar{n}_1; \bar{s}_2, \bar{f}_2, \bar{n}_2)$ and for which $\bar{s}_1 + \bar{f}_2 = n - 1$. If on any future observation a success is obtained from Π_1 or a failure from Π_2, then the experiment terminates. Thus from this point, the application of a *particular* sampling rule is determined by the fixed sequence, containing $(n - \bar{n}_1)$ 1's and $(n - \bar{n}_2)$ 2's which give the order of the population indices for those of the remaining $(2n - \bar{n}_1 - \bar{n}_2)$ possible observations which need to be taken. Note that under the stopping rule $\mathscr{S}^*$ the $2n$th observation is never taken, i.e., $N \leq 2n - 1$.

Suppose there are m_1 1's and m_2 2's in the first r terms of such a sequence ($m_1 + m_2 = r$, $\bar{n}_1 + \bar{n}_2 + r \leq 2n - 2$). Then for $\omega = 1$, $(p_1, p_2) = (p_{[2]}, p_{[1]})$ and, conditional on the occurrence of state $\bar{y}$,

$$P\{N > \bar{n}_1 + \bar{n}_2 + r\} = \left(1 - p_{[2]}\right)^{m_1}\left(p_{[1]}\right)^{r - m_1}.$$

This expression decreases as m_1 increases if $p_{[1]} + p_{[2]} \geq 1$. Similarly, for $\omega = 2$,

$$P\{N > \bar{n}_1 + \bar{n}_2 + r\} = (1 - p_{[1]})^{m_1}(p_{[2]})^{r-m_1},$$

which also decreases as m_1 increases if $p_{[1]} + p_{[2]} \geq 1$. If the experiment has not stopped by stage $2n - 2$, it does not matter which population is sampled next since the next observation will be the last. Thus, if $p_{[1]} + p_{[2]} \geq 1$, an optimal sampling rule from state $\bar{y}$ on is to take observations from Π_1 when possible; $\mathscr{P}'$ therefore samples optimally from state $\bar{y}$ on and the result follows. □

LEMMA 3.4. *For a given nonrandomized procedure* $\mathscr{P} \in \mathscr{C}$ *such that* $d(y; \mathscr{P}) = d(y; \mathscr{C}^*)$ *for all* $y \in \Omega_C$, *suppose there is a state* $\bar{y}_{\bar{m}} \in \Omega_A$ *with* $x(\bar{y}_{\bar{m}}) = (\bar{s}_1, \bar{f}_1, \bar{n}_1; \bar{s}_2, \bar{f}_2, \bar{n}_2)$ *and* $d(\bar{y}_{\bar{m}}; \mathscr{P}) \neq d(\bar{y}_{\bar{m}}; \mathscr{C}^*)$. *Then there is a nonrandomized procedure* $\mathscr{P}' \in \mathscr{C}$ *such that*

$$d(y; \mathscr{P}') = d(y; \mathscr{C}^*) \quad \text{for all } y \in \Omega_C,$$

$$d(y_m; \mathscr{P}') = d(y_m; \mathscr{P}) \quad \text{for } m < \bar{m},$$

$$d(y_{\bar{m}}; \mathscr{P}') = d(y_{\bar{m}}; \mathscr{P}) \quad \text{for } y_{\bar{m}} \neq \bar{y}_{\bar{m}},$$

$$d(\bar{y}_{\bar{m}}; \mathscr{P}') = d(\bar{y}_{\bar{m}}; \mathscr{C}^*),$$

and $E_{\mathscr{P}'}\{N|\bar{\mathbf{p}}\} \leq E_{\mathscr{P}}\{N|\bar{\mathbf{p}}\}$ *when* $p_{[1]} + p_{[2]} \geq 1$.

PROOF. First, note that the procedures $\mathscr{P}'$ constructed in Lemma 3.1 can be modified by application of Lemma 3.3 to give procedures $\mathscr{P}''$ for which $d(y; \mathscr{P}'') = d(y; \mathscr{C}^*)$ for all $y \in \Omega_C$, with no increase in $E\{N|\bar{\mathbf{p}}\}$. Also, the procedures $\mathscr{P}'$ constructed in Lemma 3.2 preserve the property $d(y; \mathscr{P}) = d(y; \mathscr{C}^*)$ for all $y \in \Omega_C$. To prove the lemma we consider the following cases:

CASE 1a: $\bar{s}_1 > \bar{s}_2$ and $\bar{f}_1 > \bar{f}_2$.
In this case $d(\bar{y}_{\bar{m}}; \mathscr{C}^*) = 2$. Neither (3.1) nor (3.2) can be satisfied without a further observation from Π_2 and the required procedure $\mathscr{P}'$ can be constructed by application of Lemmas 3.1 and 3.3.

CASE 2a: $\bar{s}_1 \leq \bar{s}_2$ and $\bar{f}_1 > \bar{f}_2$.
In this case $d(\bar{y}_{\bar{m}}; \mathscr{C}^*) = 2$. Since $d(y; \mathscr{P}) = d(y; \mathscr{C}^*)$ for $y \in \Omega_C$, it is impossible to terminate after being in state $\bar{y}_{\bar{m}}$, without an observation from Π_2. The required procedure $\mathscr{P}'$ can therefore be constructed by application of Lemmas 3.1 and 3.3.

CASE 3a: $\bar{s}_1 > \bar{s}_2$ and $\bar{f}_1 = \bar{f}_2$.
In this case $d(\bar{y}_{\bar{m}}; \mathscr{C}^*) = 1$. First define $\mathscr{P}_1$ as in Lemma 3.2, so that if the next $(\bar{s}_1 - \bar{s}_2)$ observations are taken from Π_2 and they are all successes, then the next observation is taken from Π_1. Since $d(y; \mathscr{P}_1) = d(y; \mathscr{C}^*)$ for $y \in \Omega_C$, it is impossible to terminate under $\mathscr{P}_1$, after being in state $\bar{y}_{\bar{m}}$, without an observa-

tion from Π_1 and the required procedure $\mathscr{P}'$ can be constructed from $\mathscr{P}_1$ by application of Lemmas 3.1 and 3.3.

The proofs for Case 1b: $\bar{s}_2 > \bar{s}_1$ and $\bar{f}_2 > \bar{f}_1$, Case 2b: $\bar{s}_2 \le \bar{s}_1$ and $\bar{f}_2 > \bar{f}_1$, and Case 3b: $\bar{s}_2 > \bar{s}_1$ and $\bar{f}_1 = \bar{f}_2$ are as above with the indices 1 and 2 interchanged. These six cases cover all $\bar{y}_{\bar{m}} \in \Omega_A$ and the lemma is proved. □

PROOF OF THEOREM 3.1. The proof uses Lemmas 3.3 and 3.4. Suppose $\mathscr{P} \in \mathscr{C}$ is a nonrandomized Bayes optimal procedure for a given $\bar{\mathbf{p}} = (p_{[1]}, p_{[2]})$ with $p_{[1]} + p_{[2]} \ge 1$. Then by Lemma 3.3, $\mathscr{P}$ can be modified to give a procedure $\mathscr{P}'$ for which $d(y; \mathscr{P}') = d(y; \mathscr{C}^*)$ for all $y \in \Omega_C$ and, in turn, using Lemma 3.4, $\mathscr{P}'$ can be modified gradually, starting with $\bar{m} = 1$ etc., to obtain a procedure $\mathscr{P}'' \in \mathscr{C}^*$ for which $E_{\mathscr{P}''}\{N|\bar{\mathbf{p}}\} \le E_{\mathscr{P}}\{N|\bar{\mathbf{p}}\}$. It follows that all procedures in $\mathscr{C}^*$ are Bayes optimal for this problem and since the procedures do not depend on $\bar{\mathbf{p}}$ they are optimal whenever $p_{[1]} + p_{[2]} \ge 1$. □

The conjugate procedure $\bar{\mathscr{P}}^* = (\bar{\mathscr{R}}^*, \mathscr{S}^*, \mathscr{T}^*)$ uses the following sampling rule.

DEFINITION OF SAMPLING RULE $\bar{\mathscr{R}}^*$. "Sample Π_1 next when $s_1 < s_2$ or $(s_1 = s_2$ and $f_1 > f_2)$. Sample Π_2 next when $s_1 > s_2$ or $(s_1 = s_2$ and $f_1 < f_2)$. Sample Π_1 or Π_2 next with probability $\frac{1}{2}$ each if $s_1 = s_2$ and $f_1 = f_2$."

It is a consequence of Theorem 3.1 that $\bar{\mathscr{P}}^*$ and all procedures $\mathscr{P}$, which agree with $\bar{\mathscr{P}}^*$ when $s_1 \ne s_2$ or $f_1 \ne f_2$, minimize $E\{N|\bar{\mathbf{p}}\}$ whenever $p_{[1]} + p_{[2]} \le 1$. For $k = 2$, we denote by $\bar{\mathscr{C}}^*$ the class of nonrandomized procedures $\mathscr{P} \in \mathscr{C}$, which agree with $\bar{\mathscr{P}}^*$ when $s_1 \ne s_2$ or $f_1 \ne f_2$.

REMARK 3.2. The sampling rules for procedures in $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ do not depend on n.

4. Minimizing the expected number of observations from the inferior population and the expected total number of failures: $k = 2$.

THEOREM 4.1. *Amongst all procedures in $\mathscr{C}$, those in $\mathscr{C}^*$ minimize $E\{N_{(1)}|(p_{[1]}, p_{[2]})\}$, the expected number of observations from the inferior population, if and only if*

$$(4.1)\qquad p_{[2]} \ge \max\left\{\left(1 + p_{[2]}/p_{[1]}\right)^{-1}, \left(1 + (1 - p_{[1]})/(1 - p_{[2]})\right)^{-1}\right\}.$$

This condition reduces to $p_{[2]} \ge [3 - p_{[1]} - \{(3 - p_{[1]})^2 - 4\}^{1/2}]/2$; *a sufficient condition is* $p_{[2]} \ge \frac{1}{2}$.

METHOD OF PROOF. The proof is similar to that of Theorem 3.1. The following lemma corresponds to Lemma 3.3.

LEMMA 4.1. For a given nonrandomized procedure $\mathscr{P} \in \mathscr{C}$, define $\mathscr{P}'$ by

$$d(y; \mathscr{P}') = d(y; \mathscr{P}) \quad \text{for } y \notin \Omega_C,$$
$$d(y; \mathscr{P}') = d(y; \mathscr{C}^*) \quad \text{for } y \in \Omega_C.$$

Then $E_{\mathscr{P}'}\{N_{(1)}|\bar{\mathbf{p}}\} \le E_{\mathscr{P}}\{N_{(1)}|\bar{\mathbf{p}}\}$ if (4.1) is true.

PROOF. Without loss of generality we consider a state $\bar{y} \in \Omega_C$ with $x(\bar{y}) = (\bar{s}_1, \bar{f}_1, \bar{n}_1; \bar{s}_2, \bar{f}_2, \bar{n}_2)$ and for which $\bar{s}_1 + \bar{f}_2 = n - 1$ and $\bar{n}_i < n$ $(i = 1,2)$; it follows that $\bar{s}_1 \ge \bar{s}_2$ and $\bar{f}_1 \le \bar{f}_2$ with at least one strict inequality. As before, a success from Π_1 or a failure from Π_2 leads to termination of the experiment, and the application of a particular sampling rule from this point, is determined by the fixed sequence, containing $(n - \bar{n}_1)$ 1's and $(n - \bar{n}_2)$ 2's which gives the order of the population indices for the remaining observations. Denote this sequence by $z = (i_1, i_2, i_3, \dots)$. Under $\mathscr{P}'$, the sequence consists of $(n - \bar{n}_1)$ 1's followed by $(n - \bar{n}_2)$ 2's. The sequence z corresponding to $\mathscr{P}$ can be transformed into that corresponding to $\mathscr{P}'$ by successively interchanging pairs of elements in z. This may be done in such a way that each transition is of the form z_2 to z_1 where the sequences z_1 and z_2 consist of

(i) a sequence containing a_1 1's and a_2 2's, for some a_1 and a_2, which is the same for both z_1 and z_2, followed by
(ii) (1, 2) in z_1 and (2, 1) in z_2, and then
(iii) the same final terms in both z_1 and z_2.

Let $E_z\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\}$ denote the expected number of further observations on the inferior population, starting from state $\bar{y}$ with sampling according to the sequence z. We shall show that for z_1 and z_2 as described above, $E_{z_1}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\} \le E_{z_2}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\}$ if (4.1) holds; hence $E_{\mathscr{P}'}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\} \le E_{\mathscr{P}}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\}$ and the lemma follows.

If the difference between the sequences z_1 and z_2 is in the last two elements, then the only difference in sampling occurs at a state $y \in \Omega_B$ and hence $E_{z_1}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\} \le E_{z_2}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\}$. Suppose now that the difference in the two sequences occurs earlier. Note that this implies $a_2 \le \bar{s}_1 - \bar{s}_2$ and $a_1 \le \bar{f}_2 - \bar{f}_1$ with at least one strict inequality. Under $\omega = 1$, Π_2 is the inferior population and by considering the pairs of sequences of observations from Π_1 and Π_2, which lead to different values for $N_{(1)}^{\bar{y}}$ under z_1 and z_2, we have

$$E_{z_2}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}, \omega = 1\} - E_{z_1}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}, \omega = 1\} = (1 - p_{[2]})^{a_1}(p_{[1]})^{a_2} p_{[2]}.$$

Similarly,

$$E_{z_2}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}, \omega = 2\} - E_{z_1}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}, \omega = 2\} = -(1 - p_{[1]})^{a_1}(p_{[2]})^{a_2}(1 - p_{[2]}).$$

The posterior distribution of ω in state $\bar{y}$ is given by $P\{\omega = 1\} = \alpha_1$, $P\{\omega = 2\} = \alpha_2$ where $\alpha_1 + \alpha_2 = 1$ and

$$\alpha_1/\alpha_2 = (p_{[2]}/p_{[1]})^{\bar{s}_1 - \bar{s}_2}((1 - p_{[1]})/(1 - p_{[2]}))^{\bar{f}_2 - \bar{f}_1}.$$

Hence

$$E_{z_2}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\} - E_{z_1}\{N_{(1)}^{\bar{y}}|\bar{\mathbf{p}}\} = \alpha_1(1-p_{[2]})^{a_1}(p_{[1]})^{a_2}p_{[2]} \\ -\alpha_2(1-p_{[1]})^{a_1}(p_{[2]})^{a_2}(1-p_{[2]})$$

and this is positive if

$$(4.2)\qquad p_{[2]} \geq \left\{1 + \frac{\alpha_1(1-p_{[2]})^{a_1}(p_{[1]})^{a_2}}{\alpha_2(1-p_{[1]})^{a_1}(p_{[2]})^{a_2}}\right\}^{-1} = \left\{1 + \left(\frac{p_{[2]}}{p_{[1]}}\right)^{\bar{s}_1-\bar{s}_2-a_2}\left(\frac{1-p_{[1]}}{1-p_{[2]}}\right)^{\bar{f}_2-\bar{f}_1-a_1}\right\}^{-1}.$$

But $\bar{s}_1 - \bar{s}_2 - a_2 \geq 0$ and $\bar{f}_2 - \bar{f}_1 - a_1 \geq 0$ with at least one strict inequality; thus (4.2) holds for all the required $\bar{y}$ if it holds both when $\bar{s}_1 - \bar{s}_2 - a_2 = 1$ and $\bar{f}_2 - \bar{f}_1 - a_1 = 0$ and when $\bar{s}_1 - \bar{s}_2 - a_2 = 0$ and $\bar{f}_2 - \bar{f}_1 - a_1 = 1$. Substituting these two cases into (4.2) gives condition (4.1). □

PROOF OF THEOREM 4.1. The proof that procedures in $\mathscr{C}^*$ minimize $E\{N_{(1)}|\bar{\mathbf{p}}\}$ if (4.1) holds uses Lemmas 3.1, 3.2, and 4.1 in the same way that Lemmas 3.1 to 3.3 were used to prove Theorem 3.1.

The necessity of condition (4.1) is proved by showing that if it does not hold, procedures in $\mathscr{C}^*$ do not minimize $E\{N_{(1)}^{y}|\bar{\mathbf{p}}\}$ for certain states y. The superior sampling rules are found by interchanging a pair of elements in the vector z discussed in Lemma 4.1. □

Let $N^F = E\{\sum_{i=1}^{2}(1-p_{[i]})N_{(i)}|\bar{\mathbf{p}}\}$, the total number of failures from both populations at termination.

THEOREM 4.2. *Amongst all procedures in* $\mathscr{C}$, *those in* $\mathscr{C}^*$ *minimize* $E\{N^F|(p_{[1]}, p_{[2]})\}$ *if and only if* either $p_{[1]} + p_{[2]} \geq 1$ or $p_{[1]} + p_{[2]} < 1$ *and*

$$(4.3)\qquad p_{[2]} \geq \frac{2 - 4p_{[1]} + p_{[1]}^2 - (1-p_{[1]})\sqrt{(2 - 4p_{[1]} + p_{[1]}^2)}}{1 - 2p_{[1]}}.$$

PROOF. The proof is similar to that of Theorem 4.1. With state $\bar{y}$ and sequences z_1 and z_2 as described in Lemma 4.1, let $E_z\{N_F^{\bar{y}}|\bar{\mathbf{p}}\}$ denote the expected number of further failures, starting from $\bar{y}$, sampling according to the sequence z. Then

$$(4.4)\qquad E_{z_2}\{N_F^{\bar{y}}|\bar{\mathbf{p}}\} - E_{z_1}\{N_F^{\bar{y}}|\bar{\mathbf{p}}\} = \alpha_1(1-p_{[2]})^{a_1}(p_{[1]})^{a_2}(1-p_{[1]})(2p_{[2]}-1) \\ + \alpha_2(1-p_{[1]})^{a_1}(p_{[2]})^{a_2}(1-p_{[2]})(2p_{[1]}-1).$$

This expression is positive if $p_{[1]} + p_{[2]} \geq 1$. Consideration of the cases $\{\bar{s}_1 - \bar{s}_2 - a_2 = 1$ and $\bar{f}_2 - \bar{f}_1 - a_1 = 0\}$ and $\{\bar{s}_1 - \bar{s}_2 - a_2 = 0$ and $\bar{f}_2 - \bar{f}_1 - a_1 = 1\}$

gives the necessary and sufficient condition for (4.4) always to be positive when $p_{[1]} + p_{[2]} < 1$, which reduces to (4.3). □

THEOREM 4.3. *Amongst all procedures in* $\mathscr{C}$, *those in* $\bar{\mathscr{C}}^*$ *minimize* $E\{N^F|(p_{[1]}, p_{[2]})\}$ *if* $p_{[2]} \le \frac{1}{2}$.

PROOF. The proof is similar to that of Theorem 4.2. Since procedures in $\bar{\mathscr{C}}^*$ are associated with the sequence z_2, they minimize $E\{N^F|\bar{\mathbf{p}}\}$ for values of $p_{[1]}$ and $p_{[2]}$ for which (4.4) is always negative. A sufficient condition for this is $p_{[2]} \le \frac{1}{2}$ but the necessary condition, obtained by substituting extreme values of α_1 and α_2 in (4.4) depends on n. □

5. Extensions to three or more populations. In this section we extend the results of Sections 3 and 4 to the case $k \ge 3$. We continue to restrict attention to the class $\mathscr{C}$ of procedures using the stopping rule $\mathscr{S}^*$ defined by (2.1), the terminal decision rule $\mathscr{T}^*$, and a sampling rule which takes at most n observations from any one of the k populations. We first generalize the classes $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ for $k \ge 3$.

Let O_1 be the ordering in which Π_i precedes Π_j if $f_i < f_j$ or if $f_i = f_j$ and $s_i > s_j$, and Π_i is tied with Π_j if both $f_i = f_j$ and $s_i = s_j$. Let A_1 denote the set of populations tied in first place under O_1. $\mathscr{C}^*$ consists of those nonrandomized procedures in $\mathscr{C}$ which always take the next observation from a member of A_1: the procedure $\mathscr{P}^* = (\mathscr{R}^*, \mathscr{S}^*, \mathscr{T}^*)$ of Bechhofer and Kulkarni (1982a) chooses one of these populations at random.

The ordering O_2 is defined as follows: first place is given to the population with most successes, with ties broken according to the smallest number of failures; if there is still a tie, then all of these tied populations are tied for first place. Let r be the number of populations tied for first place; then the remaining $(k - r)$ populations occupy places $(r + 1)$ to k and are ordered as in O_1. If $r = 1$, let A_2 denote the population in second place under O_2, or the set of such populations if there is a tie; if $r > 1$ and the $(r + 1)$st population under O_2 has at least as many failures as the populations tied for first place, let A_2 denote the set of populations tied for first place under O_2; if $r > 1$ and the $(r + 1)$st population under O_2 has less failures (and therefore less successes) than the populations tied for first place, let A_2 denote the population in $(r + 1)$st place under O_2, or the set of such populations if there is a tie. $\bar{\mathscr{C}}^*$ consists of those nonrandomized procedures in $\mathscr{C}$ which always take the next observation from a member of A_2. In this context the natural generalization of $\bar{\mathscr{R}}^*$ is the sampling rule which chooses one of these populations at random. This should be distinguished from the generalization $\bar{\mathscr{R}}^*$ given by Bechhofer and Kulkarni (1982a). Although, at first sight, $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ appear to have quite different forms they are in fact members of a single family of procedures; see Jennison and Kulkarni (1984).

We note that both $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ as generalized above, agree with the previous definitions for $k = 2$. We shall prove that procedures in $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ minimize $E\{N|\bar{\mathbf{p}}\}$ in two different regions of the parameter space $\{\bar{\mathbf{p}} = (p_{[1]}, \dots, p_{[k]})\}$.

When $k \geq 3$ the concept of elimination is useful. Populations are eliminated successively: if $f_j + s_i \geq n$ and Π_i is not yet eliminated, then we say Π_i eliminates Π_j; once a population has been eliminated it is unnecessary to take any further observations from it since at best it can only tie with its eliminator for the most successes.

THEOREM 5.1. *Amongst all procedures in* $\mathscr{C}$, *those in* $\mathscr{C}^*$ *minimize* $E\{N|(p_{[1]}, \ldots, p_{[k]})\}$ *if* $p_{[1]} + (\sum_{i=2}^{k} p_{[i]})/(k-1) \geq 1$.

REMARK 5.1. We note that this is a stronger result than the conjecture of Bechhofer and Kulkarni (1982a, Conjecture 7.1), namely that $\mathscr{P}^*$ minimizes $E\{N|\bar{\mathbf{p}}\}$ for $p_{[1]} + p_{[2]} \geq 1$.

METHOD OF PROOF. The proof is similar to that of Theorem 3.1. The major difference occurs in the following generalization of Lemma 3.3.

LEMMA 5.1. *Suppose that all but two populations have been eliminated, with populations* Π_i *and* Π_j *remaining; suppose also that* $s_i + f_j = n - 1$, $s_i + f_i < n$, $s_j + f_j < n$, *and either* $s_i > s_j$ *or* $f_i < f_j$. *Then if* $p_{[1]} + (\sum_{i=2}^{k} p_{[i]})/(k-1) \geq 1$, *an optimal sampling rule from this point is to sample from population* Π_i *until the first success, switching to* Π_j *only if* $n - (s_i + f_i)$ *failures are obtained from* Π_i.

PROOF. Using the notation of Lemma 4.1, suppose that in the above situation, a procedure $\mathscr{P}$ which does not sample from eliminated populations gives rise to a sequence z of is and js which contains the pair (j, i) as two consecutive elements and that these are not the last two elements in the sequence. Let $\bar{y}$ denote the state of the experiment when the procedure $\mathscr{P}$ is about to take the observation on Π_j corresponding to the "j" of the pair (j, i). Define the procedure $\mathscr{P}'$ which agrees with $\mathscr{P}$ except that the pair (j, i) is replaced by (i, j). For fixed p_i and p_j, if $N^{\bar{y}}$ denotes the total number of further observations starting from $\bar{y}$, then $E_{\mathscr{P}'}\{N^{\bar{y}}|\mathbf{p}\} - E_{\mathscr{P}}\{N^{\bar{y}}|\mathbf{p}\} = 1 - p_i - p_j$. Let α_{i_1, i_2} be the posterior probability, when in state $\bar{y}$, that the two populations, Π_i and Π_j which have not been eliminated, have success probabilities $p_{[i_1]}$ and $p_{[i_2]}$, respectively. Then

$$E_{\mathscr{P}'}\{N^{\bar{y}}|\bar{\mathbf{p}}\} - E_{\mathscr{P}}\{N^{\bar{y}}|\bar{\mathbf{p}}\} = \sum_{i_1} \sum_{i_2 \neq i_1} \alpha_{i_1, i_2}\left(1 - p_{[i_1]} - p_{[i_2]}\right). \tag{5.1}$$

For given $\bar{\mathbf{p}}$, the prior distribution for $\mathbf{p}$ assigns probability $1/k!$ to each permutation of $\bar{\mathbf{p}}$. In state $\bar{y}$, Π_i has at least as many successes and at most as many failures as each other population, with at least one strict inequality in each case: thus, α_{i_1, i_2} is an increasing function of i_1 for fixed i_2, and the right-hand side of (5.1) is negative if $p_{[1]} + (\sum_{i=2}^{k} p_{[i]})/(k-1) \geq 1$. It is easily seen that there is no advantage in sampling from eliminated populations and the result follows. □

PROOF OF THEOREM 5.1. Let $\mathscr{P}$ be a Bayes optimal nonrandomized procedure for a particular $\bar{\mathbf{p}}$ with $p_{[1]} + (\sum_{i=2}^{k} p_{[i]})/(k-1) \geq 1$. Then $\mathscr{P}$ can be modified to agree with $\mathscr{C}^*$ in the situations described in Lemma 5.1, with no increase in $E\{N|\bar{\mathbf{p}}\}$; it can then be seen that in any state a future observation from a population in the set A_1 is essentially inevitable (the possibility of a new population tying with those in A_1 is handled using a generalization of Lemma 3.2) and hence $\mathscr{P}$ can be modified to give a procedure in $\mathscr{C}^*$, again with no increase in $E\{N|\bar{\mathbf{p}}\}$. It follows that all procedures in $\mathscr{C}^*$ minimize $E\{N|\bar{\mathbf{p}}\}$ for the specified $\bar{\mathbf{p}}$. □

THEOREM 5.2. *Amongst all procedures in* $\mathscr{C}$, *those in* $\bar{\mathscr{C}}^*$ *minimize* $E\{N|(p_{[1]}, \dots, p_{[k]})\}$ *if*

$$p_{[k]}\left\{1 + \sum_{i=1}^{k-2} \prod_{j=1}^{i} (1 - p_{[j]})\right\} \leqq \prod_{i=1}^{k-1} (1 - p_{[i]}); \tag{5.2}$$

values of $\bar{\mathbf{p}}$ *satisfying this condition include those for which either* $\sum_{i=1}^{k-1} p_{[i]} + (k-1)p_{[k]} \leqq 1$ *or* $p_{[k]} \leq 1 - (\frac{1}{2})^{1/(k-1)}$.

METHOD OF PROOF. Again, the proof is similar to that of Theorem 3.1. The following lemma replaces Lemma 3.3.

LEMMA 5.2. *Suppose that in state* $\bar{y}$ *there is a unique leader in the ordering* O_2; *without loss of generality let this be population* Π_1. *Suppose also that* $\bar{s}_1 + \bar{f}_i \geq n - 1$ *for* $i \neq 1$ *so that a single success from* Π_1 *terminates the experiment with the selection of* Π_1. *Then if* (5.2) *holds, there is an optimal procedure for minimizing* $E\{N|\bar{\mathbf{p}}\}$ *which does not take the next observation from* Π_1.

PROOF. First consider a state $\bar{y}$ for which no populations have been eliminated and for which $\bar{f}_j = \bar{f}_1$ for all $j \neq 1$. In this case $\bar{f}_j = n - \bar{s}_1 - 1$, $(n-1)$ observations have been taken from Π_1 and $n - (r_j + 1)$ observations, say, have been taken from Π_j $(j \neq 1)$ where $r_j \geq 1$. Without loss of generality suppose $1 \leq r_2 \leq r_3 \leq \cdots \leq r_k$. Suppose further that $\mathscr{P}$ is a Bayes optimal procedure for a given $\bar{\mathbf{p}}$ and that in state $\bar{y}$, $\mathscr{P}$ takes the next observation from Π_1. If this observation is a success the experiment stops; if not it can be seen by the "inevitable observation" argument that $\mathscr{P}$ must sample from $\Pi_2, \dots, \Pi_k$ in order, in each case sampling from Π_j until it is eliminated by the occurrence of a failure or until another $(r_j + 1)$ successes are obtained from Π_j and the experiment stops with Π_j selected as best. We define the procedure $\mathscr{P}'$ as follows: sample as under $\mathscr{P}$ but omit the initial observation from Π_1; if r_j successes are obtained from Π_j take an observation from Π_1 and then proceed again as under $\mathscr{P}$ behaving as if the observation from Π_1 had been taken when in state $\bar{y}$. Let $N^{\bar{y}}$ denote the total number of further observations starting from $\bar{y}$, and let $X = E_{\mathscr{P}}\{N^{\bar{y}}|\mathbf{p}\} -$

$E_{\mathscr{P}'}\{N^{\bar{y}}|\mathbf{p}\}$. For fixed $\mathbf{p} = (p_1, \ldots, p_k)$ it can be shown that

$$X = (1 - p_2^{r_2})(1 - p_3^{r_3})\ldots(1 - p_k^{r_k})$$
$$-p_1\Bigg\{\frac{1 - p_2^{r_2}}{1 - p_2} + (1 - p_2^{r_2})\frac{1 - p_3^{r_3}}{1 - p_3}$$
$$+(1 - p_2^{r_2})(1 - p_3^{r_3})\frac{1 - p_4^{r_4}}{1 - p_4} + \cdots \tag{5.3}$$
$$+(1 - p_2^{r_2})(1 - p_3^{r_3})\ldots(1 - p_{k-1}^{r_{k-1}})\frac{1 - p_k^{r_k}}{1 - p_k}\Bigg\}.$$

As a function of r_j with r_i fixed $(i \neq j)$, X is of the form

$$X = A(1 - p_j^{r_j}) - B - C(1 - p_j^{r_j})/(1 - p_j),$$

where $B \geq 0$. If $X \geq 0$ for $r_j = 1$, then $A(1 - p_j) - C \geq B$, hence

$$\{A(1 - p_j) - C\}(1 - p_j^{r_j})/(1 - p_j) \geq B$$

and $X \geq 0$ for all $r_j \geq 1$. Thus $X \geq 0$ for all $1 \leq r_2 \leq r_3 \leq \cdots \leq r_k$ if

$$\prod_{i=2}^{k}(1 - p_i) - p_1\left\{1 + \sum_{i=2}^{k-1}\prod_{j=2}^{i}(1 - p_j)\right\} \geq 0. \tag{5.4}$$

For given $\bar{\mathbf{p}} = (p_{[1]}, \ldots, p_{[k]})$ the left-hand side of (5.4) is smallest when $p_1 = p_{[k]}$, $p_2 = p_{[1]}$, $p_3 = p_{[2]}, \ldots, p_k = p_{[k-1]}$, and hence (5.4) is certainly satisfied if (5.2) holds; this establishes a contradiction and we conclude that an optimal policy does not sample from Π_1 when in state $\bar{y}$.

We now consider states $\bar{y}$ in which no population has been eliminated and for which $\bar{f}_1 < \bar{f}_j$ $(j \neq 1)$ and the values $\bar{f}_j$ are equal for all $j \neq 1$. We argue inductively on the value of $\bar{f}_j - \bar{f}_1$. Again suppose there is a Bayes optimal procedure $\mathscr{P}$ which samples from Π_1 when in state $\bar{y}$ and use the inductive hypothesis to find the optimal sampling procedure when a single failure on population Π_1 has been observed after being in state $\bar{y}$. A procedure $\mathscr{P}'$ can then be constructed as before for which $X = E_{\mathscr{P}}\{N^{\bar{y}}|\mathbf{p}\} - E_{\mathscr{P}'}\{N^{\bar{y}}|\mathbf{p}\}$ is given by (5.3), where now $r_j = n - n_j \geq 1$, and the result follows.

Finally, we consider states $\bar{y}$ in which one or more populations have been eliminated. The proof follows the same lines using the fact that for given $\mathbf{p}$ the expression for $E_{\mathscr{P}}\{N^{\bar{y}}|\mathbf{p}\} - E_{\mathscr{P}'}\{N^{\bar{y}}|\mathbf{p}\}$ is positive as long as X defined by (5.3) with $r_2 = r_3 = \cdots = r_k = 1$ is positive. □

PROOF OF THEOREM 5.2. Let $\mathscr{P}$ be a Bayes optimal nonrandomized procedure for a known $\bar{\mathbf{p}}$ which satisfies (5.2). In a state where there is a unique leader under O_2, it is a consequence of Lemma 5.2 that $\mathscr{P}$ can be modified so that an observation on a population in A_2 is essentially inevitable and hence $\mathscr{P}$ can be modified to give a procedure which agrees with $\mathscr{C}^*$ in such situations, with no increase in $E\{N|\bar{\mathbf{p}}\}$. Similarly, if populations Π_i and Π_j, say, are tied for first place under O_2 and $f_i = f_j \leq f_l$ $(1 \leq l \leq k)$ then an observation on one of the leading populations is essentially inevitable and $\mathscr{P}$ can be further modified to agree with $\bar{\mathscr{C}}^*$ in these situations also, with no increase in $E\{N|\bar{\mathbf{p}}\}$. Finally, if $r > 1$ populations, including Π_i and Π_j, are tied for first place under O_2 and $f_i = f_j > f_u$ for some u, then an observation on a population in $(r+1)$th place under O_2 is essentially inevitable and $\mathscr{P}$ can be modified once more, again with no increase in $E\{N|\bar{\mathbf{p}}\}$, to agree with $\bar{\mathscr{C}}^*$ in these situations also. The resulting procedure is in $\bar{\mathscr{C}}^*$ and therefore the procedures in $\bar{\mathscr{C}}^*$ minimize $E\{N|\bar{\mathbf{p}}\}$ for the specified $\bar{\mathbf{p}}$.

If $\sum_{i=1}^{k-1} p_{[i]} + (k-1)p_{[k]} \leqq 1$, then $\prod_{j=i}^{k-1}(1 - p_{[j]}) \geqq (k-1)p_{[k]}$ for $1 \leqq i \leqq k-1$, and hence

$$\left\{\prod_{i=1}^{k-1}(1 - p_{[i]})\right\}^{-1} p_{[k]}\left\{1 + \sum_{i=1}^{k-2}\prod_{j=1}^{i}(1 - p_{[j]})\right\}$$

$$= p_{[k]}\sum_{i=1}^{k-1}\left\{\prod_{j=i}^{k-1}(1 - p_{[j]})\right\}^{-1} \leq 1,$$

and thus (5.2) holds. By considering the case $p_{[1]} = p_{[2]} = \cdots = p_{[k]}$, it is seen that (5.2) holds whenever $p_{[k]} \leqq 1 - (\frac{1}{2})^{1/(k-1)}$. □

We note that the methods of proof in Theorems 5.1 and 5.2 show that procedures in $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ sample optimally for the specified values of $\bar{\mathbf{p}}$, starting from any state y regardless of whether it can be reached using a procedure in $\mathscr{C}^*$ or $\bar{\mathscr{C}}^*$, respectively.

The procedures in $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ behave in two quite different ways. Those in $\mathscr{C}^*$ are appropriate when the $p_{[i]}$s are large; they aim to reach a conclusion by obtaining a large number of successes from the leading population. Procedures in $\bar{\mathscr{C}}^*$ are appropriate when the $p_{[i]}$s are small; they aim to reach a conclusion by obtaining a large number of failures from *every* losing population. The sets of values of $\bar{\mathbf{p}}$ for which procedures in $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ minimize $E\{N|\bar{\mathbf{p}}\}$ are larger than those given in the statements of Theorems 5.1 and 5.2; further specification of these two sets can be found from the details of the proofs of the theorems. We showed in Section 3 that for $k = 2$ the regions of optimality of $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ together span the entire parameter space; this is not the case for $k \geq 3$, and it does not seem possible to characterize optimal procedures in this remaining region using the techniques of Sections 3 and 4. The disadvantage of procedures in $\bar{\mathscr{C}}^*$ is that a failure on each of $(k-1)$ losing populations is needed to do the work of a single success on the leading population, and this is a serious problem for large k. The problem is less serious if several populations have been eliminated, and this suggests that a combination of $\mathscr{C}^*$ and $\bar{\mathscr{C}}^*$ may be optimal when neither is optimal by itself. In practice $\bar{\mathbf{p}}$ usually is not known and we suggest that either $\mathscr{R}^*$ or $\bar{\mathscr{R}}^*$ be used as a sampling rule, according to the current estimated success probabilities of the uneliminated populations.

An extension of Theorem 4.1 requires a generalization of the number of observations from the inferior population to the case $k \geq 3$. In a medical study, for instance, one wishes to allocate patients to treatments with high success probabilities but at the same time an early result is desirable and the total number of observations taken should also be small. A general objective might be to minimize $E\{\sum_{i=1}^{k}\lambda_i N_{(i)}|\bar{\mathbf{p}}\}$ where $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ is possibly a function of $\bar{\mathbf{p}}$. Special cases of this goal are the minimization of $E\{N_{(1)}|\bar{\mathbf{p}}\}$, $E\{\sum_{i=1}^{k-1}N_{(i)}|\bar{\mathbf{p}}\}$, and $E\{N|\bar{\mathbf{p}}\}$; if $\lambda_i = 1 - p_{[i]}$, $E\{\sum_{i=1}^{k}\lambda_i N_{(i)}|\bar{\mathbf{p}}\}$ is the expected total number of failures, discussed in Section 4 for $k = 2$. We say that a procedure is *fully optimal* for a particular $\bar{\mathbf{p}}$ if it minimizes $E\{\sum_{i=1}^{k}\lambda_i N_{(i)}|\bar{\mathbf{p}}\}$ for all λ satisfying $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$. An equivalent requirement is that the procedure minimize $E\{\sum_{i=1}^{r}N_{(i)}|\bar{\mathbf{p}}\}$ for all r $(1 \leq r \leq k)$. The following theorem shows that $\mathscr{P}^* = (\mathscr{R}^*, \mathscr{S}^*, \mathscr{T}^*)$ is fully optimal if $p_{[1]} + p_{[2]} \geq 1$.

THEOREM 5.3. *Amongst all procedures in $\mathscr{C}$, those in $\mathscr{C}^*$ minimize $E\{\sum_{i=1}^{r}N_{(i)}|(p_{[1]}, \ldots, p_{[k]})\}$ for all r $(1 \leq r \leq k)$ whenever $p_{[1]} + p_{[2]} \geq 1$.*

METHOD OF PROOF. Again, the proof is similar to that of Theorem 3.1. This time the following lemma replaces Lemma 3.3.

LEMMA 5.3. *Suppose $\bar{y}$ is a state in which all except two populations have been eliminated and these have success probabilities $p_{[i_1]}$ and $p_{[i_2]}$; we condition on knowing $p_{[i_1]}$ and $p_{[i_2]}$ but not their pairing with the two populations. Let $N_{(i)}^{\bar{y}}$ denote the number of further observations taken on the population associated with $p_{[i]}$, after being in state $\bar{y}$.*

(i) *If $r < \min\{i_1, i_2\}$ then $E\{\sum_{i=1}^{r}N_{(i)}^{\bar{y}}|p_{[i_1]}, p_{[i_2]}\} = 0$ for all procedures in $\mathscr{C}^*$.*

(ii) *If $i_1 \leq r \leq i_2$ then a sufficient condition for procedures in $\mathscr{C}^*$ to minimize $E\{\sum_{i=1}^{r}N_{(i)}^{\bar{y}}|p_{[i_1]}, p_{[i_2]}\}$ is $p_{[i_2]} \geq \frac{1}{2}$.*

(iii) *If $\max\{i_1, i_2\} \leq r$ then a sufficient condition for procedures in $\mathscr{C}^*$ to minimize $E\{\sum_{i=1}^{r}N_{(i)}^{\bar{y}}|p_{[i_1]}, p_{[i_2]}\}$ is $p_{[i_1]} + p_{[i_2]} \geq 1$.*

PROOF. The proof follows directly from the 2-population results of Theorems 3.1 and 4.1. □

PROOF OF THEOREM 5.3. It follows from Lemma 5.3 that $p_{[1]} + p_{[2]} \geq 1$ is a sufficient condition for procedures in $\mathscr{C}^*$ to minimize $E\{\sum_{i=1}^{r}N_{(i)}^{\bar{y}}|\bar{\mathbf{p}}\}$ for all states $\bar{y}$ in which all but two populations have been eliminated and for all $1 \leq r \leq k$. The results of Lemma 5.3 can be applied as in the proofs of previous theorems to show that procedures in $\mathscr{C}^*$ are fully optimal whenever $p_{[1]} + p_{[2]} \geq 1$. □

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