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SIMULATED POWER FUNCTIONS[1]

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Tests for a null hypothesis whose specification involves an unknown nuisance parameter may be obtained by inverting a bootstrap confidence region for the parameter being tested or by constructing a simulated null distribution for the test statistic. The power of either test against certain alternatives involving the same unknown nuisance parameter can itself be estimated by simulation.

1. Introduction. Bootstrap distribution estimates generate confidence regions of approximate level $1 - \alpha$ in a variety of statistical models, including some models for which alternative constructions of confidence regions encounter substantial technical difficulties [cf. Efron (1979), Bickel and Freedman (1981, 1982), Beran (1984), and Beran and Srivastava (1985)]. Underlying the bootstrap is the concept of simulation: the fitting of a mathematical model to observations on a system and the subsequent use of the fitted model to mimic, or simulate, the system. Simulation is a well-established technique in disciplines as diverse as numerical weather forecasting, the calculation of tide tables, and economic forecasting.

Simulation ideas also have application in statistical hypothesis testing. Tests for a null hypothesis whose specification involves an unknown nuisance parameter may be obtained by inverting a bootstrap confidence region for the parameter being tested or by constructing a simulated null distribution for the test statistic. The power of either test against certain alternatives involving the same unknown nuisance parameter can itself be estimated by simulation. The uniform consistency of such simulated power functions is the main result of this paper.

Consider the following general situation. Suppose Ξ and Θ are metric spaces with metrics m_1, m_2, respectively. The observations $X_1, X_2, \ldots, X_n$ are independent identically distributed random vectors with joint distribution $P^n_{\xi_A, \theta_A}$, which belongs to a parametric family $\{P^n_{\xi, \theta}: (\xi, \theta) \in \Omega\}$. The parameter space Ω is a subset of $\Xi \times \Theta$. The subscript "A" in (ξ_A, θ_A) designates the "actual" parameter values which are supposed to underlie the experiment. Both ξ_A and θ_A are unknown. Let ξ_0 be a specified element of Ξ. We wish to test the null hypothesis that $\xi_A = \xi_0$ against alternative hypotheses in which ξ_A differs from ξ_0, with θ_A being viewed as a fixed nuisance parameter.

More formally, the testing problem under consideration is

$$(1.1) \qquad H_{n,0}\text{: the } \{X_i; 1 \le i \le n\} \text{ have distribution } P^n_{\xi_0, \theta_A}, (\xi_0, \theta_A) \in \Omega$$

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versus alternatives of the form

(1.2) $H_{n,\xi}$: the $\{X_i; 1 \le i \le n\}$ have distribution P^n_{ξ,θ_A}, $(\xi, \theta_A) \in \Omega$, $\xi \ne \xi_0$,

with θ_A being the unknown actual value of θ in the experiment. The alternatives (1.2) are of particular interest because they include the actual distribution $P^n_{\xi_A,\theta_A}$ of the data when the null hypothesis $H_{n,0}$ is false; and because the power function of a test over the alternatives (1.2) is typically estimable while power against $P^n_{\xi_A,\theta_A}$ is not. Tacit is the requirement that the model $\{P^n_{\xi,\theta}: (\xi, \theta) \in \Omega\}$ be general enough to contain, or reasonably approximate, the actual distribution of the data.

The test statistic approach. Suppose $T_n = T_n(X_1, X_2, \dots, X_n)$ is a test statistic for the null hypothesis $H_{n,0}$. Let $K_{n,T}(\xi, \theta)$ denote $\mathscr{L}[T_n | P^n_{\xi,\theta}]$, the distribution of T_n under $P^n_{\xi,\theta}$, and let

$$K_{n,T}(x; \xi, \theta) = P^n_{\xi,\theta}[T_n > x] \tag{1.3}$$

be the corresponding survival function. For $\alpha \in (0, 1)$, let

$$\begin{aligned} d_{n,L}(\alpha; \xi_0, \theta) &= \inf\{x: K_{n,T}(x; \xi_0, \theta) \le \alpha\}, \\ d_{n,u}(\alpha; \xi_0, \theta) &= \sup\{x: K_{n,T}(x; \xi_0, \theta) \ge \alpha\}. \end{aligned} \tag{1.4}$$

Suppose $\hat{\theta}_n$ is a consistent estimate of θ_A and $d_n(\alpha; \xi_0, \hat{\theta}_n)$ is any random variable lying between $d_{n,L}(\alpha; \xi_0, \hat{\theta}_n)$ and $d_{n,u}(\alpha; \xi_0, \hat{\theta}_n)$. Define the test φ_n by

$$\varphi_n(\mathbf{X}) = \begin{cases} 1 & \text{if } T_n > d_n(\alpha; \xi_0, \hat{\theta}_n), \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

The critical value $d_n(\alpha; \xi_0, \hat{\theta}_n)$ is an upper α-point of $K_{n,T}(\xi_0, \hat{\theta}_n)$, the simulated null distribution of T_n. In practice, $d_n(\alpha; \xi_0, \hat{\theta}_n)$ can often be approximated by performing a Monte Carlo simulation of the distribution $K_{n,T}(\xi_0, \hat{\theta}_n)$. This calculation is an extension of the more familiar Monte Carlo technique for finding a critical value when testing a simple hypothesis. Under conditions to be described in Theorem 2.1, the test φ_n has asymptotic level α under $H_{n,0}$.

The power of φ_n against the alternative $H_{n,\xi}$ is

$$\beta_{n,\varphi}(\alpha; \xi, \theta_A) = P^n_{\xi,\theta_A}[T_n > d_n(\alpha; \xi_0, \hat{\theta}_n)]. \tag{1.6}$$

To estimate $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$, we might seek an analytical asymptotic approximation to $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$; and then replace θ_A by $\hat{\theta}_n$ wherever θ_A appears in this approximation. Unfortunately, the available asymptotic approximations to $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$ are not always trustworthy for every value of ξ. Consider, for instance, local asymptotic power approximations when ξ is an infinite dimensional parameter.

The simulation estimate of the power $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$ is $\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_n)$. Evaluation of $\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_n)$ typically requires Monte Carlo simulation of the distribution $T_n - d_n(\alpha; \xi_0, \hat{\theta}_n)$, the simulation samples being drawn from $P^n_{\xi,\hat{\theta}_n}$. For each sample $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_n^*)$ drawn from $P^n_{\xi,\hat{\theta}_n}$, the critical value $d_n(\alpha; \xi_0, \hat{\theta}_n(\mathbf{X}^*))$ must itself be recalculated, usually by nested Monte Carlo simulation of the distribution $K_n(\xi_0, \hat{\theta}_n(\mathbf{X}^*))$. If ξ is a euclidean parameter and other requirements to be specified in Theorem 2.2 are met, then

$$\lim_{n\to\infty} \sup_{\xi} \left|\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_n) - \beta_{n,\varphi}(\alpha; \xi, \theta_A)\right| = 0 \tag{1.7}$$

with $P^{\infty}_{\xi_A, \theta_A}$-probability one. If ξ is an infinite dimensional parameter, a uniform convergence result slightly weaker than (1.7) can still be proved (Theorem 2.3).

Another estimate of $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$, which relies on asymptotic constancy of the critical value $d_n(\alpha; \xi_0, \hat{\theta}_n)$, is $K_{n,T}[d_n(\alpha; \xi_0, \hat{\theta}_n); \xi, \hat{\theta}_n]$. This computationally simpler estimate also converges to $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$, uniformly in ξ, under conditions to be described in Theorems 2.2 and 2.3. Evaluation of $K_{n,T}[d_n(\alpha; \xi_0, \hat{\theta}_n); \xi, \hat{\theta}_n]$ typically requires the initial calculation of the critical value $d_n(\alpha; \xi_0, \hat{\theta}_n)$ and Monte Carlo simulation of the distribution $K_{n,T}(\xi, \hat{\theta}_n)$.

Section 2 of this paper describes a numerical study wherein the actual power function of the bootstrap t-test was compared with the two power function estimates described above and with the power function of the classical t-test.

While the function $\beta_{n,\varphi}(\alpha; \cdot, \theta_A)$ is estimable, as indicated above, it does not seem possible to estimate $\beta_{n,\varphi}(\alpha; \xi_A, \theta_A)$ itself. Suppose $\hat{\xi}_n, \hat{\theta}_n)$ are consistent estimates of (ξ_A, θ_A). In general, $\beta_{n,\varphi}(\alpha; \hat{\xi}_n, \hat{\theta}_n)$ does *not* converge to $\beta_{n,\varphi}(\alpha; \xi_A, \theta_A)$. For example, consider the following case: $P_{\xi,\theta}$ is the $N(\xi, \theta)$ distribution; $(\hat{\xi}_n, \hat{\theta}_n)$ are the usual estimates of mean and variance; T_n is the t-statistic; $H_{n,0}$ is the hypothesis that $\xi_A = 0$. Small perturbations in ξ_A affect the value of $\beta_{n,\varphi}(\alpha; \xi_A, \theta_A)$ far more than do small perturbations in the nuisance parameter θ_A.

Equation (1.7) immediately implies that $\beta_{n,\varphi}(\alpha; \xi_0, \hat{\theta}_n)$ is a consistent estimate of the actual level of the test φ_n. Moreover, suppose we are interested in the performance of φ_n against alternatives $H_{n,\xi}$ indexed by $\xi \in \Xi_n$, where $\Xi_n = \{\xi \in \Xi\colon m_1(\xi, \xi_0) \geq \varepsilon_n\}$ and $\{\varepsilon_n\}$ is a sequence of positive constants tending to zero in such a way that $\inf\{\beta_{n,\varphi}(\alpha; \xi, \theta_A); \xi \in \Xi_n\}$ has a limit in $(0, 1)$. (Under the assumptions for Theorem 2.2, $\varepsilon_n = n^{-1/2}\varepsilon$ will do.) Because of (1.7), $\inf\{\beta_{n,\varphi}(\alpha; \xi, \theta_A); \xi \in \Xi_n\}$ is estimated consistently by $\inf\{\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_n)\colon \xi \in \Xi_n\}$. Thus, asymptotically correct comparisons between tests based on test statistics $T_{n,1}$ and $T_{n,2}$ can be made by referring to the estimated level and the estimated minimum power over Ξ_n of each test.

Implied by this result is a technique for constructing adaptive tests based upon a finite collection of tests $\varphi_{n,1}, \varphi_{n,2}, \dots, \varphi_{n,k}$ of asymptotic level α: use the test $\varphi_{n,j}$ for which $\inf\{\beta_{n,\varphi_j}(\alpha; \xi, \hat{\theta}_n); \xi \in \Xi_n\}$ is greatest. In view of the preceding paragraph, this procedure defines a test φ_n^* which has asymptotic level α and the property that $\inf\{\beta_{n,\varphi^*}(\alpha; \xi, \theta_A); \xi \in \Xi_n\}$ converges to $\max_j \inf\{\beta_{n,\varphi_j}(\alpha; \xi, \theta_A); \xi \in \Xi_n\}$.

A related problem is the estimation of test power in future experiments on the basis of current information. Suppose the $\{X_i; 1 \leq i \leq n\}$ are observations to be taken in a future experiment. Let $\hat{\theta}_m$ be a consistent estimate of θ_A based on an independent training sample of size m. Under conditions similar to those for Theorems 2.2 and 2.3, $\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_m)$ is a uniformly consistent estimate of $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$ as $\min(m, n)$ tends to infinity. The performance of the test φ_n in the proposed experiment can therefore be assessed by examining the simulated power function $\beta_n(\alpha; \xi, \hat{\theta}_m)$, as in the previous paragraphs.

The confidence region approach. Suppose $R_n(\mathbf{X}, \xi)$ is a pivot for ξ, a random function depending on $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and on the parameter ξ. Let $J_{n,R}(\xi, \theta)$ denote $\mathscr{L}[R_n(\mathbf{X}, \xi)|P^n_{\xi,\theta}]$ and let $J_{n,R}(x; \xi, \theta)$ be the corresponding survival function, defined as in (1.3). For $\alpha \in (0, 1)$, let

$$c_{n,L}(\alpha; \xi, \theta) = \inf\{x: J_{n,R}(x; \xi, \theta) \le \alpha\},$$
$$c_{n,u}(\alpha; \xi, \theta) = \sup\{x: J_{n,R}(x; \xi, \theta) \ge \alpha\}. \tag{1.8}$$

Suppose $(\hat{\xi}_n, \hat{\theta}_n)$ is a consistent estimate of (ξ_A, θ_A) and $c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n)$ is any random variable lying between $c_{n,L}(\alpha; \hat{\xi}_n, \hat{\theta}_n)$ and $c_{n,u}(\alpha; \hat{\xi}_n, \hat{\theta}_n)$. The set $\{\xi: R_n(\mathbf{X}, \xi) \le c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n)\}$ is a bootstrap confidence region for ξ_A of ostensible level $1 - \alpha$. Under circumstances to be described in Theorem 3.1, the corresponding test

$$\psi_n(\mathbf{X}) = \begin{cases} 1 & \text{if } R_n(\mathbf{X}, \xi_0) > c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n), \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

has asymptotic level α under $H_{n,0}$. The tests φ_n and ψ_n are related when $T_n(\mathbf{X}) = R_n(\mathbf{X}, \xi_0)$, but even then will usually have different critical values and therefore different power functions.

The power of the test ψ_n against the alternative $H_{n,\xi}$,

$$\beta_{n,\psi}(\alpha; \xi, \theta_A) = P^n_{\xi,\theta_A}[R_n(\mathbf{X}, \xi_0) > c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n)], \tag{1.10}$$

can be estimated by $\beta_{n,\psi}(\alpha; \xi, \hat{\theta}_n)$. Let $K_{n,R}(\xi, \theta)$ denote $\mathscr{L}[R_n(\mathbf{X}, \xi_0)|P^n_{\xi,\theta}]$ and let $K_{n,R}(x; \xi, \theta)$ be the associated survival function. An alternative estimate for $\beta_{n,\psi}(\alpha; \xi, \theta_A)$ is $K_{n,R}[c_n(\alpha; \xi, \hat{\theta}_n); \xi, \hat{\theta}_n]$. Note that the critical value $c_n(\alpha; \xi, \hat{\theta}_n)$ appearing in this power estimate varies with ξ. Evaluation of $K_{n,R}[c_n(\alpha; \xi, \hat{\theta}_n); \xi, \hat{\theta}_n]$ generally requires Monte Carlo simulation of $J_{n,R}(\xi, \hat{\theta}_n)$, to obtain $c_n(\alpha; \xi, \hat{\theta}_n)$, and of $K_{n,R}(\xi, \hat{\theta}_n)$. Evaluation of $\beta_{n,\psi}(\alpha; \xi, \hat{\theta}_n)$ typically requires a nested two-stage Monte Carlo simulation. The convergence of $\beta_{n,\psi}(\alpha; \xi, \hat{\theta}_n)$ and of $K_{n,R}[c_n(\alpha; \xi, \hat{\theta}_n); \xi, \hat{\theta}_n]$ to $\beta_{n,\psi}(\alpha; \xi, \theta_A)$, uniformly in ξ, is the subject of Theorems 3.2 and 3.3.

REMARK. The results in this paper do not contradict those of Bahadur and Savage (1956) because level and power are defined over a smaller model here. The level of φ_n is $E_{\xi_0,\theta_A}(\varphi_n)$ for us, but is $\sup_\theta E_{\xi_0,\theta}(\varphi_n)$ for Bahadur and Savage. Similarly, the power of φ_n is $E_{\xi,\theta_A}(\varphi_n)$ for us rather than $\sup_\theta E_{\xi,\theta}(\varphi_n)$. Considering test performance only over the distributions in the family $\{P^n_{\xi,\theta_A}: \xi \in \Xi\}$ is reasonable because the actual distribution $P^n_{\xi_A,\theta_A}$ of the sample falls within this family.

2. The test statistic approach: asymptotics and examples.

Do the tests φ_n and ψ_n defined in Section 1 have approximate level α? Are the associated power function estimates consistent, uniformly in ξ? This section addresses these questions for the test φ_n; Section 3 does the same for ψ_n. The notation of Section 1 is retained throughout. All theorem proofs are deferred to Section 4.

The first theorem gives sufficient conditions under which the asymptotic level of φ_n is α.

THEOREM 2.1. *Suppose the following requirements are met*:

A.1. $\lim_{n\to\infty} P^n_{\xi_0, \theta_A}[m_2(\hat{\theta}_n, \theta_A) > \varepsilon] = 0$ *for every positive* ε.

A.2. *If* $\{\theta_n\}$ *is any sequence such that* $\{(\xi_0, \theta_n) \in \Omega\}$ *and* $\lim_{n\to\infty}\theta_n = \theta_A$, *then* $K_{n,T}(\xi_0, \theta_n)$ *converges weakly to a unique limit distribution* $K_T(\xi_0, \theta_A)$.

Let $K_T(x; \xi_0, \theta_A)$ *be the survival function associated with* $K(\xi_0, \theta_A)$ *and let*

$$d_L(\alpha; \xi_0, \theta_A) = \inf\{x\colon K_T(x; \xi_0, \theta_A) \le \alpha\},$$
$$d_u(\alpha; \xi_0, \theta_A) = \sup\{x\colon K_T(x; \xi_0, \theta_A) \ge \alpha\}. \tag{2.1}$$

Then

$$\begin{aligned} K_T[d_u(\alpha; \xi_0, \theta_A); \xi_0, \theta_A] &\le \liminf_{n\to\infty} \beta_{n,\varphi}(\alpha; \xi_0, \theta_A) \\ &\le \limsup_{n\to\infty} \beta_{n,\varphi}(\alpha; \xi_0, \theta_A) \\ &\le K_T[d_L(\alpha; \xi_0, \theta_A) - ; \xi_0, \theta_A]. \end{aligned} \tag{2.2}$$

If $K_T(x; \xi_0, \theta_A)$ *is continuous in* x, *then*

$$\lim_{n\to\infty} \beta_{n,\varphi}(\alpha; \xi_0, \theta_A) = \alpha. \tag{2.3}$$

If the convergence in A.1 is uniform over m_2-compacts of Θ, so is the convergence (2.3) to asymptotic level α [cf. the derivation of (4.6) in the proof of Theorem 2.2]. Each example in this section exhibits this type of uniform convergence.

EXAMPLE 1: *Minimum distance tests.* Suppose $\{F_\theta\colon \theta \in \Theta\}$ is a parametric family of c.d.f.'s on the real line, Θ being an open subset of R^k. The null hypothesis $H_{n,0}$ asserts that the observed random vectors $\{X_i; 1 \le i \le n\}$ are i.i.d. with c.d.f. F_{θ_A}, the value of θ_A being unknown. Let $\hat{F}_n$ be the empirical c.d.f. and let $\|\cdot\|$ denote supremum norm. Consider the test which rejects $H_{n,0}$ if the statistic $T_n = n^{1/2}\inf_{\theta\in\Theta}\|\hat{F}_n - F_\theta\|$ is sufficiently large. Bootstrap critical values for the test can be found as in (1.6), by identifying P_{ξ_0, θ_A} with the distribution determined by F_{θ_A}; the definition of ξ_0 is arbitrary here.

Conditions A.1, A.2 for Theorem 3.1 will be verified under the following assumptions on the parametric model, which are made for every $\theta_0 \in \Theta$:

Identifiability. For every neighborhood N of θ_0, $\inf\{\|F_\theta - F_{\theta_0}\|; \theta \notin N\} > 0$.

Continuous norm differentiability. There exists a $k \times 1$ vector function η_{θ_0} such that the components of η_{θ_0} are bounded, $\|F_\theta - F_{\theta_0} - (\theta - \theta_0)'\eta_{\theta_0}\| = o(|\theta - \theta_0|)$, and $\lim_{\theta\to\theta_0}\|\eta_\theta - \eta_{\theta_0}\| = 0$.

Nonsingularity. The components of η_{θ_0} are linearly independent.

Let G be a c.d.f. on the real line. By an argument similar to that in Pollard (1980),

$$(2.4)\qquad \inf_{\theta \in \Theta} \|G - F_\theta\| = \inf_{t \in R^k} \|G - F_{\theta_0} - t'\eta_{\theta_0}\| + o(\|G - F_{\theta_0}\|)$$

as $\|G - F_{\theta_0}\|$ tends to zero.

A.1. Let m_2 be euclidean metric on R^k and let $\hat{\theta}_n$ be a minimum distance estimate of θ_A, satisfying the requirement $\|\hat{F}_n - F_{\hat{\theta}_n}\| \le \inf_{\theta \in \Theta}\|\hat{F}_n - F_\theta\| + n^{-1}$. It is well known that this choice of $\hat{\theta}_n$ is consistent under F_{θ_A}.

A.2. Let $\{\theta_n \in \Theta\}$ be any sequence which converges to θ_A. Let B_n denote the empirical Brownian bridge based on n i.i.d. random variables which are uniformly distributed on (0,1). Let B denote the Brownian bridge process. Since $\mathscr{L}[n^{1/2}(\hat{F}_n - F_{\theta_n})|P^n_{\xi_0, \theta_n}] = \mathscr{L}[B_n \cdot F_{\theta_n}]$, it follows from (2.4) and the assumptions on the parametric model that $K_{n,T}(\xi_0, \theta_n) = \mathscr{L}[n^{1/2}\inf_{\theta \in \Theta}\|\hat{F}_n - F_\theta\|\,|P^n_{\xi_0, \theta_n}]$ converges weakly to $K_T(\xi_0, \theta_A) = \mathscr{L}[\inf_t\|B \cdot F_{\theta_A} + t'\eta_{\theta_A}\|]$.

Thus, the bootstrap test based on the minimum distance statistic T_n has approximate size α, in the sense of (2.2).

The next theorem establishes uniform consistency of the two power function estimates for φ_n. A key assumption is finite-dimensionality of the parameter ξ. Conditions B.2 and B.3 in the statement of Theorem 2.2 imply the weaker conditions A.1 and A.2 used in Theorem 2.1.

THEOREM 2.2. *Suppose Ξ is R^k and the following requirements are met:*

B.1. $P^\infty_{\xi_A, \theta_A}[\lim_{n \to \infty}\hat{\theta}_n = \theta_A] = 1$.

B.2. $\lim_{n \to \infty}\sup_\xi P^n_{\xi, \theta_A}[m_2(\hat{\theta}_n, \theta_A) > \varepsilon] = 0$ *for every positive* ε.

B.3. *If* $\{(\xi_n, \theta_n) \in \Omega;\ n \ge 1\}$ *is any sequence such that* $\lim_{n \to \infty} n^{1/2}(\xi_n - \xi_0) = h$ *for some* $h \in R^k$ *and* $\lim_{n \to \infty}\theta_n = \theta_A$, *then* $K_{n,T}(\xi_n, \theta_n) \Rightarrow K_T^{(h)}(\xi_0, \theta_A)$, *a limit distribution which is continuous and does not depend upon the particular sequence* $\{(\xi_n, \theta_n)\}$ *chosen. Moreover,* $K_T^{(0)}(\xi_0, \theta_A)$ *has a strictly monotone survival function.*

B.4. *If* $\{(\xi_n, \theta_n) \in \Omega;\ n \ge 1\}$ *is any sequence such that* $\lim_{n \to \infty} n^{1/2}|\xi_n - \xi_0| = \infty$ *and* $\lim_{n \to \infty}\theta_n = \theta_A$, *then* $\lim_{n \to \infty} K_{n,T}(x; \xi_n, \theta_n) = 1$ *for every finite real* x.

Then

$$(2.5)\quad P^\infty_{\xi_A, \theta_A}\left[\lim_{n \to \infty}\sup_\xi\left|K_{n,T}[d_n(\alpha; \xi_0, \hat{\theta}_n); \xi, \hat{\theta}_n] - \beta_{n,\varphi}(\alpha; \xi, \theta_A)\right| = 0\right] = 1.$$

Suppose B.2 *is strengthened to*

B.2′. *If* $\{(\xi_n, \theta_n) \in \Omega;\ n \ge 1\}$ *is any sequence such that* $\lim_{n \to \infty}\theta_n = \theta_A$, *then* $\lim_{n \to \infty} P^n_{\xi_n, \theta_n}[m_2(\hat{\theta}_n, \theta_A) > \varepsilon] = 0$ *for every positive* ε.

Then also

$$(2.6)\qquad P^\infty_{\xi_A, \theta_A}\left[\lim_{n \to \infty}\sup_\xi\left|\beta_{n,\varphi}(\alpha; \xi, \hat{\theta}_n) - \beta_{n,\varphi}(\alpha; \xi, \theta_A)\right| = 0\right] = 1.$$

The suprema in (2.5) and (2.6) are taken over $\{\xi \in \Xi: (\xi, \theta_A) \in \Omega \text{ and } (\xi, \hat{\theta}_n) \in \Omega\}$. Similarly, the supremum in B.2 is taken over $\{\xi \in \Xi: (\xi, \theta_A) \in \Omega\}$.

EXAMPLE 2: *Testing a mean.* Suppose the $\{X_i; 1 \le i \le n\}$ are i.i.d. random $k \times 1$ vectors with c.d.f. $G_A(x - \xi_A)$, where G_A has mean zero and nonsingular covariance matrix $\Sigma(G_A)$. Both ξ_A and G_A are unknown. Consider testing the null hypothesis $\xi_A = \xi_0$ versus the alternatives $\xi_A \ne \xi_0$, the c.d.f. G_A being regarded as a fixed unknown nuisance parameter. The test statistic to be used is $T_n = |S_n^{-1/2} n^{1/2}(\overline{X}_n - \xi_0)|$, where $|\cdot|$ is any norm on R^k and $\overline{X}_n$, S_n are the sample mean and sample covariance matrix respectively. When the norm $|\cdot|$ is euclidean, T_n^2 is a multiple of Hotelling's T^2-statistic.

Let δ_L denote Lévy metric and estimate G_A by $\hat{G}_n$, the empirical c.d.f. of the residuals $\{X_i - \overline{X}_n; 1 \le i \le n\}$. For any $k \times k$ matrix A, let $|A| = \sup\{|Ax|: |x| = 1\}$. Define the metric m_2 by

$$m_2(\hat{G}_n, G_A) = \delta_L(\hat{G}_n, G_A) + |\Sigma(\hat{G}_n) - \Sigma(G_A)|. \tag{2.7}$$

We will verify that conditions B.1, B.2′, B.3, and B.4 are satisfied in this example. Consequently, the bootstrap test φ_n defined in (1.6) has asymptotic size α (Theorem 2.1) and both bootstrap power function estimates are uniformly consistent (Theorem 2.2).

B.1. Without loss of generality, because of location invariance, take $\xi_A = 0$. Let $\hat{F}_n$ be the empirical c.d.f. of the $\{X_i; 1 \le i \le n\}$. With P_{0, G_A}^{∞}-probability one, $\delta_L[\hat{G}_n(x), G_A(x + \overline{X}_n)] \le \|\hat{F}_n - G_A\| \to 0$ by the vector version of the Glivenko–Cantelli theorem; and $\delta_L[G_A(x + \overline{X}_n), G_A(x)] \to 0$ and $\Sigma(\hat{G}_n) \to \Sigma(G_A)$ by the strong law of large numbers. Condition B.1 follows, in view of the definition (2.7) for m_2.

B.2′. Let G_n converge to G_A in the metric m_2 and, without loss of generality, take $\xi_n = 0$. In P_{0, G_n}^{n}-probability, $\delta_L[\hat{G}_n(x), G_n(x + \overline{X}_n)] \le \|\hat{F}_n - G_n\| \to 0$ by the Dvoretzky–Kiefer–Wolfowitz inequality; and $\delta_L[G_n(x + \overline{X}_n), G_A(x)] \to 0$ and $\Sigma(\hat{G}_n) \to \Sigma(G_A)$ by a triangular array version of Khintchine's weak law of large numbers.

B.3. Let $\{(\xi_n, G_n)\}$ be any sequence such that $\lim_{n \to \infty} n^{1/2}(\xi_n - \xi_0) = h$ and $\lim_{n \to \infty} m_2(G_n, G_A) = 0$. The latter convergence is equivalent to saying $G_n \Rightarrow G_A$ and $\Sigma(G_n) \to \Sigma(G_A)$. Then $K_{n,T}(\xi_n, G_n) \Rightarrow \mathscr{L}[|Z + \Sigma^{-1/2}(G_A)h|] \equiv K_T^{(h)}(\xi_0, G_A)$, where Z is a $k \times 1$ normal random vector with mean zero and identity covariance matrix. Indeed, $\mathscr{L}[n^{1/2}(\overline{X}_n - \xi_0)|P_{\xi_n, G_n}^n] \Rightarrow N(h, \Sigma(G_A))$ by the Lindeberg central limit theorem; and S_n converges in P_{ξ_n, G_n}^n-probability to $\Sigma(G_A)$, by Khintchine's weak law of large numbers. The limit distribution $K_T^{(h)}(\xi_0, G_A)$ does not depend on ξ_0 in this example because of location invariance. The limit distribution is continuous because the set $\{z \in R^k: |z| = c\}$ has Lebesgue measure zero for every norm $|\cdot|$ on R^k. Thus B.3 holds.

B.4. Let $\{(\xi_n, G_n)\}$ be such that $\lim_{n \to \infty} n^{1/2}|\xi_n - \xi_0| = \infty$ and $\lim_{n \to \infty} m_2(G_n, G_A) = 0$. The inequality $|x| \le |A| \cdot |A^{-1}x|$ implies

$$\begin{aligned} T_n &= |S_n^{-1/2}(\xi_n - \xi_0) + S_n^{-1/2} n^{1/2}(\overline{X}_n - \xi_n)| \\ &\ge |S_n^{1/2}|^{-1} n^{1/2} |\xi_n - \xi_0| - |S_n^{-1/2} n^{1/2}(\overline{X}_n - \xi_n)|. \end{aligned} \tag{2.8}$$

As in the previous paragraph, $\mathscr{L}[S_n^{-1/2}n^{1/2}(\overline{X}_n - \xi_n)|P^n_{\xi_n, G_n}] \Rightarrow \mathscr{L}(Z)$ and $S_n \to \Sigma(G_A)$ in $P^n_{\xi_n, G_n}$-probability. In view of (2.8), condition B.4 holds.

A numerical study of the univariate bootstrap t-test and of the two associated power function estimates yielded some additional information. In this study, the data was taken to be normally distributed with unit variance. The null hypothesis value of ξ was $\xi_0 = 0$; the nominal test level $\alpha = 0.05$; and the sample size $n = 20$. Table 1 compares the power function of the bootstrap t-test with the power function of the classical t-test under standard normal shift alternatives. The critical value of the bootstrap test was obtained from 200 bootstrap samples. The power of the bootstrap t-test was approximated by Monte Carlo simulation, using 1000 standard normal samples. Even at sample size 20, the power function of the bootstrap test is almost indistinguishable from that of the classical t-test.

For a single standard normal sample of size 20, Table 2 records the two power function estimates described earlier in this example and the normal approximation to the power function based on the sample standard deviation. The calculation of $\beta_{n,\varphi}(\alpha; \xi, \hat{G}_n)$ used 200 bootstrap samples for the critical value loop and 1000 bootstrap samples for the outer loop. The calculation of $K_{n,T}[d_n(\alpha; \xi_0, \hat{G}_n); \xi, \hat{G}_n]$ used 1000 bootstrap samples for both the critical value and for $K_{n,T}(\xi, \hat{G}_n)$. Two points stand out:

(a) The three power function estimates in Table 2 are roughly similar, especially when $|\xi|$ is near zero or is large, even though the second estimate is more asymmetric.
(b) Each of the estimated power functions usually underestimates the actual power function reported in Table 1.

Point (a) is not surprising, since each of the power function estimates converges uniformly in ξ to the actual power function as n increases. Point (b) is attributable to the particular $N(0,1)$ sample of size 20 from which the power function estimates were computed; the estimated standard deviation of this sample happened to be 1.140, which is larger than the population standard deviation.

EXAMPLE 3: *Testing correlation.* Suppose the $\{X_i; 1 \le i \le n\}$ are i.i.d. 2×1 random vectors with c.d.f. $G_A[D^{-1}(\xi_A, \sigma_{1A}, \sigma_{2A})x]$, where

$$D(\xi, \sigma_1, \sigma_2) = \begin{pmatrix} \sigma_1 & 0 \\ \xi\sigma_2 & (1-\xi^2)^{1/2}\sigma_2 \end{pmatrix} \tag{2.9}$$

and G_A has mean zero, identity covariance matrix, and finite fourth moments. The covariance matrix of each X_i is therefore

$$\Sigma(\xi_A, \sigma_{1A}, \sigma_{2A}) = \begin{pmatrix} \sigma_{1A}^2 & \xi_A\sigma_{1A}\sigma_{2A} \\ \xi_A\sigma_{1A}\sigma_{2A} & \sigma_{2A}^2 \end{pmatrix}. \tag{2.10}$$

The values of the correlation $\xi_A \in (-1, 1)$, of the standard deviations $\sigma_{1A} > 0$, $\sigma_{2A} > 0$, and of the c.d.f. G_A are unknown. Consider testing the null hypothesis $\xi_A = \xi_0$ versus alternatives $\xi_A \neq \xi_0$, with $\theta_A = (\sigma_{1A}, \sigma_{2A}, G_A)$ regarded as a fixed

TABLE 1

Power comparisons at nominal level $\alpha = 0.05$ of the bootstrap t-test with the classical t-test

ξ		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
Power of the bootstrap t-test	at ξ	0.047	0.067	0.123	0.244	0.408	0.591	0.729	0.849	0.926	0.965	0.990	0.994
	at $-\xi$	0.047	0.070	0.135	0.242	0.382	0.544	0.728	0.849	0.925	0.971	0.986	0.997
Power of the classical t-test at $\pm\xi$		0.050	0.071	0.136	0.247	0.397	0.565	0.721	0.844	0.924	0.968	0.989	0.997

Data is normal with unit variance. Sample size n is 20. Null hypothesis is $\xi = 0$.

TABLE 2

Three estimates for the power of the bootstrap t-test with nominal level $\alpha = 0.05$

ξ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$\beta_{n,\varphi}(\alpha; \xi, \hat{G}_n)$	0.046	0.062	0.115	0.191	0.299	0.480	0.629	0.786	0.873	0.960	0.984	0.998
$\beta_{n,\varphi}(\alpha; -\xi, \hat{G}_n)$	0.046	0.069	0.116	0.190	0.326	0.463	0.620	0.753	0.854	0.918	0.958	0.986
$K_{n,T}[d_n(\alpha; \xi_0, \hat{G}_n); \xi, \hat{G}_n]$	0.050	0.064	0.110	0.203	0.351	0.511	0.686	0.819	0.916	0.967	0.992	0.994
$K_{n,T}[d_n(\alpha; \xi_0, \hat{G}_n); -\xi, \hat{G}_n]$	0.050	0.082	0.153	0.267	0.382	0.542	0.686	0.794	0.889	0.937	0.969	0.986
Normal power approximation at $\pm\xi$ based on sample standard deviation	0.050	0.068	0.123	0.218	0.348	0.501	0.653	0.784	0.881	0.942	0.975	0.991

Each power estimate was computed from a single $N(0, 1)$ sample size 20 whose estimated standard deviation was 1.140. Null hypothesis is $\xi = 0$.

unknown nuisance parameter. The test statistic to be used is $T_n = n^{1/2}|\hat{\xi}_n - \xi_0|$, where $\hat{\xi}_n$ is the sample correlation.

Let

$$S_n = \begin{pmatrix} s_{n,1}^2 & s_{n,12} \\ s_{n,12} & s_{n,2}^2 \end{pmatrix} \tag{2.11}$$

be the sample covariance matrix. Take $\hat{G}_n(x) = \hat{F}_n[D(\hat{\xi}_n, s_{n,1}, s_{n,2})x]$ as the estimate of G_A and let $\hat{\theta}_n = (s_{n,1}, s_{n,2}, \hat{G}_n)$. Suppose that $\mu_{r_1,r_2}(G_A) = \int x_1^{r_1}x_2^{r_2}\,dG_A(x_1, x_2)$ is finite whenever r_1, r_2 are nonnegative integers such that $r_1 + r_2 \le 4$. Define the metric m_2 by

$$m_2(\hat{\theta}_n, \theta_A) = \delta_L(\hat{G}_n, G_A) + \sum_{r_1+r_2\le 4} \left|\mu_{r_1,r_2}(\hat{G}_n) - \mu_{r_1,r_2}(G_A)\right|. \tag{2.12}$$

We will verify that conditions B.1, B.2′, B.3, and B.4 are satisfied in this example, under the additional assumption that $\Xi = [-b, b]$ for some b in $(0,1)$.

B.2′. Let $\{\xi_n\}$ be any sequence in $[-b, b]$ and let $\{\theta_n = (\sigma_{n,1}, \sigma_{n,2}, G_n)\}$ be any sequence which converges to θ_A. Suppose B.2′ does not hold. By going to a subsequence, we can assume that, for some positive ε, the sequence $\{P^n_{\xi_n,\theta_n}[m_2(\hat{\theta}_n, \theta_A) > \varepsilon]\}$ remains bounded away from zero and $\lim_{n\to\infty}\xi_n = \xi^* \in [-b, b]$. From Khintchine's weak law of large numbers, $\hat{\xi}_n \to \xi^*$ and $s_{n,i} \to \sigma_{iA}$ in $P^n_{\xi_n,\theta_n}$-probability. By the definition of $\hat{G}_n$,

$$\begin{aligned} &\delta_L(\hat{G}_n, G_A) \\ &\le \delta_L\{\hat{F}_n[D(\hat{\xi}_n, s_{n,1}, s_{n,2})x], G_n[D^{-1}(\xi_n, \sigma_{n,1}, \sigma_{n,2})D(\hat{\xi}_n, s_{n,1}, s_{n,2})x]\} \\ &\quad + \delta_L\{G_n[D^{-1}(\xi_n, \sigma_{n,1}, \sigma_{n,2})D(\hat{\xi}_n, s_{n,1}, s_{n,2})x], G_A(x)\}. \end{aligned} \tag{2.13}$$

The first term on the right side of (2.13) is bounded above by $\|\hat{F}_n(x) - G_n[D^{-1}(\xi_n, \sigma_{n,1}, \sigma_{n,2})x]\|$, which converges to zero in $P^n_{\xi_n,\theta_n}$-probability. The second term on the right side of (2.13) also tends to zero in $P^n_{\xi_n,\theta_n}$-probability because both D and D^{-1} are continuous at $(\xi^*, \sigma_{1A}, \sigma_{2A})$. Moreover, $\mu_{r_1,r_2}(\hat{G}_n) \to \mu_{r_1,r_2}(G_A)$ in $P^n_{\xi_n,\theta_n}$-probability because $\mu_{r_1,r_2}(\hat{F}_n)$ converges to the corresponding moment of P_{ξ^*,θ_A} by the weak law of large numbers and because $D^{-1}(\hat{\xi}_n, s_{n,1}, s_{n,2}) \to_p D^{-1}(\xi^*, \sigma_{1A}, \sigma_{2A})$. It follows that $m_2(\hat{\theta}_n, \theta_A) \to 0$ in $P^n_{\xi_n,\theta_n}$-probability, contradicting the supposition to the contrary made at the start of this argument.

B.1. The verification of B.1 is similar to that of B.2′. Note that, with $P^\infty_{\xi_A,\theta_A}$-probability one, $\lim_{n\to\infty}\|\hat{F}_n(x) - G_A[D^{-1}(\xi_A, \sigma_{1A}, \sigma_{2A})]\| = 0$, by the Glivenko–Cantelli theorem for two-dimensional c.d.f.'s, and

$$\lim_{n\to\infty}(\hat{\xi}_n, s_{n,1}, s_{n,2}) = (\xi_A, \sigma_{1A}, \sigma_{2A}),$$

by the strong law of large numbers.

B.3. Let $\{(\xi_n, \theta_n)\}$ be any sequence such that $\lim_{n\to\infty}n^{1/2}(\xi_n - \xi_0) = h$ and $\lim_{n\to\infty}\theta_n = \theta_A$. The latter convergence is equivalent to saying $G_n \Rightarrow G_A$ and $\mu_{r_1,r_2}(G_n) \to \mu_{r_1,r_2}(G_A)$ whenever $r_1 + r_2 \le 4$. In particular, $\lim_{n\to\infty}\sigma_{n,i} = \sigma_{iA}$ for $i = 1, 2$. By the Lindeberg central limit theorem, $\mathscr{L}[n^{1/2}\{S_n - \Sigma(\xi_n, \sigma_{n,1}, \sigma_{n,2})\}|P^n_{\xi_n,\theta_n}]$ converges weakly to a singular normal distribution with mean zero and covariance structure depending only on (ξ_0, θ_A). Since $\hat{\xi}_n = s_{n,12}/(s_{n,1}s_{n,2})$ is a continuously differentiable function of S_n, $\mathscr{L}[n^{1/2}(\hat{\xi}_n - \xi_n)|P^n_{\xi_n, \theta_n}]$ converges to a normal distribution with mean zero and variance $\sigma^2(\xi_0, \theta_A)$, say. The weak limit of $\{K_{n,T}(\xi_n, \theta_n)\}$ is therefore $\mathscr{L}[|\sigma(\xi_0, \theta_A)Z + h|]$, Z being a $N(0,1)$ random variable. This limit distribution fulfils the requirements of B.3.

B.4. Let $\{(\xi_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty} n^{1/2}|\xi_n - \xi_0| = \infty$ and $\lim_{n\to\infty}\theta_n = \theta_A$. Then $\lim_{n\to\infty} K_{n,T}(x; \xi_n, \theta_n) = 1$ for every finite positive x. If not, by going to a subsequence, we can assume that, for some positive x, $K_{n,T}(x; \xi_n, \theta_n)$ does not converge to one while $\lim_{n\to\infty}\xi_n = \xi^* \in [-b, b]$. From the argument in the previous paragraph, $\mathscr{L}[n^{1/2}(\hat{\xi}_n - \xi_n)|P^n_{\xi_n, \theta_n}] \Rightarrow N(0, \sigma^2(\xi^*, \theta_A))$. Since

$$T_n \geq n^{1/2}|\xi_n - \xi_0| - n^{1/2}|\hat{\xi}_n - \xi_n|, \tag{2.14}$$

it follows that $\lim_{n\to\infty} K_{n,T}(x; \xi_n, \theta_n) = 1$ for every positive x, contradicting the initial assumption to the contrary and thereby establishing B.4.

The conditions for Theorem 2.2 are adapted to euclidean Ξ and to local asymptotic power calculations. The next theorem identifies more general circumstances under which the two bootstrap power function estimates approximate $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$. Let δ_p denote the Prohorov metric on the extended real line $\overline{R}$.

THEOREM 2.3. *Suppose the following requirement is met in addition to conditions* B.1 *and* B.2 *of Theorem* 2.2:

C.1. *If* $\{(\xi_n, \theta_n) \in \Omega;\ n \geq 1\}$ *is any sequence such that* $\lim \theta_n = \theta_A$, *then* $\lim_{n\to\infty}\delta_p[K_{n,T}(\xi_n, \theta_n), K_{n,T}(\xi_n, \theta_A)] = 0$.

Then the following assertion is true with $P^{\infty}_{\xi_A, \theta_A}$*-probability one: for every sufficiently small positive* ε, *there exists* $n_0(\varepsilon)$ *such that for every* $n \geq n_0(\varepsilon)$,

$$\begin{aligned} &K_{n,T}\big[d_{n,u}(\alpha - \varepsilon; \xi_0, \hat{\theta}_n) + \varepsilon; \xi, \hat{\theta}_n\big] - \varepsilon \\ &\quad \leq \beta_{n,\varphi}(\alpha; \xi, \theta_A) \leq K_{n,T}\big[d_{n,L}(\alpha + \varepsilon; \xi_0, \hat{\theta}_n) - \varepsilon; \xi, \hat{\theta}_n\big] + \varepsilon \end{aligned} \tag{2.15}$$

simultaneously for every possible ξ. *Suppose* B.2 *is strengthened to* B.2′. *Let*

$$\begin{aligned} \beta_{n,\varphi,L}(\alpha; \xi, \theta, \varepsilon) &= P^n_{\xi,\theta}\big[T_n > d_{n,u}(\alpha - \varepsilon; \xi_0, \hat{\theta}_n) + \varepsilon\big] - \varepsilon, \\ \beta_{n,\varphi,u}(\alpha; \xi, \theta, \varepsilon) &= P^n_{\xi,\theta}\big[T_n > d_{n,L}(\alpha + \varepsilon; \xi_0, \hat{\theta}_n) - \varepsilon\big] + \varepsilon. \end{aligned} \tag{2.16}$$

Then (2.15) *may be replaced by*

$$\beta_{n,\varphi,L}(\alpha; \xi, \hat{\theta}_n, \varepsilon) \leq \beta_{n,\varphi}(\alpha; \xi, \theta_A) \leq \beta_{n,\varphi,u}(\alpha; \xi, \hat{\theta}_n, \varepsilon). \tag{2.17}$$

This theorem asserts that certain small perturbations of either bootstrap power function estimate will bracket $\beta_{n,\varphi}(\alpha; \xi, \theta_A)$, uniformly in ξ, provided the sample size n is large enough. While the conclusions are weaker than those of Theorem 2.2, Theorem 2.3 can be applied to examples where the parameter ξ is infinite dimensional.

EXAMPLE 4: *Testing for symmetry.* Suppose the $\{X_i; 1 \le i \le n\}$ are i.i.d. random variables with unknown c.d.f. F_A. Set $\xi_A(x) = \frac{1}{2}[F_A(x) + F_A(-x) - 1]$ and $\theta_A(x) = \frac{1}{2}[F_A(x) - F_A(-x) + 1]$. Then $\xi_A(x) = \xi_A(-x)$, $\theta_A(x) + \theta_A(-x) = 1$, θ_A is the c.d.f. of a random variable distributed symmetrically about the origin, and $F_A(x) = \xi_A(x) + \theta_A(x)$. Consider the problem of testing the null hypothesis $\xi_A = 0$ (that is, the null hypothesis that X_i is symmetrically distributed about the origin) versus the alternatives $\xi_A \ne 0$. The nuisance parameter θ_A is unknown but fixed by the experiment.

The space Θ consists of all c.d.f.s associated with random variables symmetrically distributed about the origin. The space Ξ consists of all real-valued functions ξ on R such that $\xi(x) = \xi(-x)$ and $\|\xi\| \le 2^{-1}$. Let

$$\begin{aligned} \hat{\xi}_n(x) &= \tfrac{1}{2}[\hat{F}_n(x) + \hat{F}_n(-x) - 1], \\ \hat{\theta}_n(x) &= \tfrac{1}{2}[\hat{F}_n(x) - \hat{F}_n(-x) + 1]. \end{aligned} \tag{2.18}$$

The test statistic to be used is $T_n = n^{1/2}\|\hat{\xi}_n\|$. We will show that the conditions for Theorems 2.1 and 2.3 hold when the metric m_2 on Θ is defined by

$$m_2(\hat{\theta}_n, \theta_A) = \|\hat{\theta}_n - \theta_A\|. \tag{2.19}$$

B.1. By Glivenko–Cantelli, $\lim_{n\to\infty}\|\hat{F}_n - F_A\| = 0$ with $P^{\infty}_{\xi_A, \theta_A}$-probability one. This convergence and the definitions of $\hat{\theta}_n$, θ_A imply B.1.

B.2′ and A.1. Let $\{(\xi_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$. Under $P^n_{\xi_n, \theta_n}$, the empirical c.d.f. $\hat{F}_n$ may be represented as $F_{\xi_n, \theta_n} + n^{1/2}B_n \cdot F_{\xi_n, \theta_n}$, where $F_{\xi, \theta}(x) = \xi(x) + \theta(x)$ and B_n is the empirical Brownian bridge process. Thus, from (2.18),

$$\hat{\theta}_n(x) = \theta_n(x) + \tfrac{1}{2}n^{-1/2}\left[B_n \cdot F_{\xi_n, \theta_n}(x) - B_n \cdot F_{\xi_n, \theta_n}(-x)\right], \tag{2.20}$$

which implies B.2′ and therefore A.1.

C.1. Let $\{(\xi_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$. Let

$$S_n(\xi, \theta) = \left\|\tfrac{1}{2}\left[B_n \cdot F_{\xi, \theta}(x) + B_n \cdot F_{\xi, \theta}(-x)\right] + n^{1/2}\xi(x)\right\|. \tag{2.21}$$

Since $T_n = \|n^{1/2}(\hat{\xi}_n - \xi_n) + \xi_n\|$, it follows from (2.18) that $K_{n,T}(\xi_n, \theta_n) = \mathscr{L}[S_n(\xi_n, \theta_n)]$ and $K_{n,T}(\xi_n, \theta_A) = \mathscr{L}[S_n(\xi_n, \theta_A)]$. By Skorokhod's theorem, there exist versions of $\{B_n\}$ and of the Brownian bridge B such that $\lim_{n\to\infty}\|B_n - B\| = 0$ w.p. 1 and B has uniformly continuous sample paths. Since $\|F_{\xi_n, \theta_n} - F_{\xi_n, \theta_A}\| = \|\theta_n - \theta_A\| \to 0$, the corresponding versions of $\{S_n(\xi_n, \theta_n)\}$ and $\{S_n(\xi_n, \theta_A)\}$ have the property that $\lim_{n\to\infty}[S_n(\xi_n, \theta_n) - S_n(\xi_n, \theta_A)] = 0$ w.p. 1. This convergence implies C.1.

A.2. Let $\{\theta_n \in \Theta\}$ be any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$. From the previous paragraph, $K_{n,T}(0, \theta_n) \Rightarrow \mathscr{L}[\frac{1}{2}\|B \cdot \theta_A(x) + B \cdot \theta_A(-x)\|]$, a limit law satisfying the requirements of A.2.

EXAMPLE 5: *Testing for multivariate normality.* Suppose the $\{X_i; 1 \le i \le n\}$ are i.i.d. $p \times 1$ random vectors with c.d.f. $G_A[\Sigma_A^{-1/2}(x - \mu_A)]$, where μ_A is a $p \times 1$ vector, Σ_A is a $p \times p$ positive definite symmetric matrix, and G_A is a continuous c.d.f. on R^p with mean zero, identity covariance matrix, and finite fourth moments. Identify ξ_A with G_A and θ_A with (μ_A, Σ_A). Consider the problem of testing the null hypothesis $G_A = \Phi$, where Φ is the standard normal c.d.f. on R^p versus alternatives $G_A \neq \Phi$. The nuisance parameters (μ_A, Σ_A) are unknown but fixed by the experiment.

Let $\bar{X}_n$, S_n be the sample mean vector and sample covariance matrix, respectively. Set $\hat{\theta}_n = (\bar{X}_n, S_n)$ and $\hat{G}_n(x) = \hat{F}_n(S_n^{1/2}x + \bar{X}_n)$, where $\hat{F}_n$ is the empirical c.d.f. The test statistic to be considered is

$$T_n = n^{1/2}\|\hat{F}_n(x) - \Phi[S_n^{-1/2}(x - \bar{X}_n)]\| = n^{1/2}\|\hat{G}_n - \Phi\|. \tag{2.22}$$

The space Θ consists of all pairs (μ, Σ) such that μ is a $p \times 1$ vector and Σ is a $p \times p$ positive definite symmetric matrix. Let m_2 be euclidean metric on Θ. For any c.d.f. G on R^p, let $\mu(r_1, r_2, \dots, r_p; G) = \int x_1^{r_1} \dots x_p^{r_p}\, dG(x_1, \dots, x_p)$. Define the metric m_1 by

$$m_1(\hat{G}_n, G_A) = \|\hat{G}_n - G_A\| + \Sigma_r\left|\mu(r_1, \dots, r_p; \hat{G}_n) - \mu(r_1, \dots, r_p; G_A)\right|, \tag{2.23}$$

the sum being taken over all sets of $\{r_j; 1 \le j \le p\}$ such that every r_j is a nonnegative integer and $\Sigma_{j=1}^p r_j = 4$. Let Ξ be any set of continuous c.d.f.s G on R^p such that the mean of G is zero, the covariance matrix of G is the identity matrix, Ξ contains Φ and G_A, and Ξ is compact in the metric m_1. Then, the conditions for Theorems 2.1 and 2.3 hold.

B.1. By the strong law of large numbers, $\hat{\theta}_n = (\bar{X}_n, S_n)$ converges with $P^{\infty}_{G_A, \theta_A}$-probability one to $\theta_A = (\mu_A, \Sigma_A)$ in the metric m_2.

B.2′ and A.1. Let $\{(G_n, \theta_n) \in \Omega\}$, where $\theta_n = (\mu_n, \Sigma_n)$, be any sequence such that $\lim_{n\to\infty}\mu_n = \mu_A$ and $\lim_{n\to\infty}\Sigma_n = \Sigma_A$. By Khintchine's weak law of large numbers, $\theta_n \to \theta_A$ in $P^n_{G_n, \theta_n}$-probability. This implies both B.2′ and A.1.

C.1. Let $\{(G_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$. Since Ξ is compact, we may assume without loss of generality that G_n converges to some c.d.f. $G^* \in \Xi$ in the metric m_1. Evidently

$$\begin{aligned} K_{n,T}(G_n, \theta_n) = \mathscr{L}\Big[\Big\| n^{1/2}\big[\hat{F}_n(\Sigma_n^{1/2}x + \mu_n) - G_n(x)\big] \\ + n^{1/2}\big[\Phi(x) - \Phi\{S_n^{-1/2}(\Sigma_n^{1/2}x + \mu_n - \bar{X}_n)\}\big] \\ + n^{1/2}\big[G_n(x) - \Phi(x)\big]\Big\| \Big| P^n_{G_n, \theta_n}\Big]. \end{aligned} \tag{2.24}$$

Standard weak convergence arguments based on the Lindeberg central limit theorem establish the following fact: there exists a gaussian process $W(G^*, \theta_A)$ such that

$$\begin{aligned} \mathscr{L}\Big[n^{1/2}\big[\hat{F}_n(\Sigma_n^{1/2}x + \mu_n) - G_n(x)\big] \\ + n^{1/2}\big[\Phi(x) - \Phi\{S_n^{-1/2}(\Sigma_n^{1/2}x + \mu_n - \bar{X}_n)\}\big] \Big| P^n_{G_n, \theta_n}\Big] \end{aligned}$$

converges weakly to $\mathscr{L}[W(G^*, \theta_A)]$ on $C(R^p)$. Consequently,

$$\lim_{n\to\infty} \delta_p\left\{K_{n,T}(G_n, \theta_n), \mathscr{L}\big[\|W(G^*, \theta_A) + n^{1/2}(G_n - \Phi)\|\big]\right\} = 0. \tag{2.25}$$

Since (2.25) remains true if θ_n is replaced by θ_A, the fulfilment of condition C.1 follows.

3. The confidence region approach: asymptotics and examples. The main difference between the tests φ_n and ψ_n, defined in (1.6) and (1.12), respectively, lies in their critical values. The critical value of φ_n depends on $\hat{\theta}_n$ and the null hypothesis value ξ_0, while the critical value of ψ_n depends on estimates $\hat{\xi}_n$ and $\hat{\theta}_n$ of both parameters. As a result, the asymptotic theory for ψ_n described in this section requires conditions on $\hat{\xi}_n$ as well as $\hat{\theta}_n$.

THEOREM 3.1. *Suppose the following requirements are met:*

D.1. $\lim_{n\to\infty} P^n_{\xi_0,\theta_A}[m_1(\hat{\xi}_n, \xi_0) > \varepsilon] = \lim_{n\to\infty} P^n_{\xi_0,\theta_A}[m_2(\hat{\theta}_n, \theta_A) > \varepsilon] = 0$ *for every positive* ε.

D.2. *If* $\{(\xi_n, \theta_n) \in \Omega\}$ *is any sequence such that* $\lim_{n\to\infty}\xi_n = \xi_0$ *and* $\lim_{n\to\infty}\theta_n = \theta_A$, *then* $J_{n,R}(\xi_n, \theta_n)$ *converges weakly to a unique limit distribution* $J_R(\xi_0, \theta_A)$.

Let $J_R(x; \xi_0, \theta_A)$ *be the survival function associated with* $J_R(\xi_0, \theta_A)$ *and let*

$$c_L(\alpha; \xi_0, \theta_A) = \inf\{x: J_R(x; \xi_0, \theta_A) \le \alpha\},$$
$$c_u(\alpha; \xi_0, \theta_A) = \sup\{x: J_R(x; \xi_0, \theta_A) \ge \alpha\}. \tag{3.1}$$

Then

$$\begin{aligned} J_R[c_u(\alpha; \xi_0, \theta_A); \xi_0, \theta_A] &\le \liminf_{n\to\infty} \beta_{n,\psi}(\alpha; \xi_0, \theta_A) \\ &\le \limsup_{n\to\infty} \beta_{n,\psi}(\alpha; \xi_0, \theta_A) \\ &\le J_R[c_L(\alpha; \xi_0, \theta_A) - ; \xi_0, \theta_A]. \end{aligned} \tag{3.2}$$

If $J_R(x; \xi_0, \theta_A)$ *is continuous in* x, *then*

$$\lim_{n\to\infty} \beta_{n,\psi}(\alpha; \xi_0, \theta_A) = \alpha. \tag{3.3}$$

EXAMPLE 6: *Testing mean orientation.* Let $\xi = (\xi_1, \xi_2, \xi_3)'$ be a unit vector in R^3 other than the vector $e = (1,0,0)'$. Let $|\cdot|_2$ denote euclidean metric. Define the orthogonal matrix

$$O(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ 0 & (1-\xi_1^2)^{-1/2}\xi_3 & -(1-\xi_1^2)^{-1/2}\xi_2 \\ (1-\xi_1^2)^{1/2} & -(1-\xi_1^2)^{-1/2}\xi_1\xi_2 & -(1-\xi_1^2)^{-1/2}\xi_1\xi_3 \end{pmatrix}, \tag{3.4}$$

noting that $O(\xi)\xi = e$. Suppose the $\{X_i; 1 \le i \le n\}$ are i.i.d. random unit vectors with c.d.f. $G_A[O(\xi_A)x]$, where G_A is the c.d.f. of a singular distribution in R^3 whose support is the unit sphere and whose normalized mean vector $E_{G_A}(X)/|E_{G_A}(X)|_2 = e$. Then the mean orientation of X_i is $E(X_i)/|E(X_i)|_2 = \xi_A$. Consider testing the null hypothesis $\xi_A = \xi_0 \ne e$, with G_A being regarded as a fixed unknown nuisance parameter. (A different choice of the orthogonal matrix

$O(\xi)$ handles the case $\xi_0 = e$.) Let $\hat{\xi}_n = \bar{X}_n/|\bar{X}_n|_2$, the unit vector in the direction of the sample resultant. The pivot to be used is $R_n(X, \xi) = 2^{-1}n|\hat{\xi}_n - \xi|_2^2 = n(1 - \hat{\xi}_n'\xi)$.

The space $\Xi = \{x \in R^3: |x|_2 = 1\} - e$ while Θ consists of all c.d.f.'s supported on the unit sphere whose mean orientation vector is e. Take $\hat{G}_n(x) = \hat{F}_n[O'(\hat{\xi}_n)x]$, which is the empirical c.d.f. of the rotated sample $\{O(\hat{\xi}_n)X_i; 1 \le i \le n\}$. Let m_1 and m_2 be, respectively, euclidean metric and Lévy metric. In this example, $\hat{G}_n$ and G_A play the roles of $\hat{\theta}_n$ and θ_A, respectively.

D.1. The weak law of large numbers and continuity of $O(\xi)$ at $\xi = \xi_0$ imply that condition D.1 holds.

D.2. Let $\{(\xi_n, G_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty}\xi_n = \xi_0$ and $G_n \Rightarrow G_A$. Let $F_{\xi,G}(x) = G[O(\xi)x]$ and designate the mean vector and covariance matrix of $F_{\xi,G}$ by $\mu(\xi, G)$ and $\Sigma(\xi, G)$, respectively. Since $F_{\xi_n, G_n} \Rightarrow F_{\xi_0, G_A}$ and $O(\xi)$ is continuous at $\xi = \xi_0$,

$$\mathscr{L}\left[n^{1/2}(\bar{X}_n - \mu(\xi_n, G_n))|P^n_{\xi_n, G_n}\right] \Rightarrow N(0, \Sigma(\xi_0, G_A)).$$

Thus, $J_{n,R}(\xi_n, \theta_n) = \mathscr{L}[n^{1/2}(\hat{\xi}_n - \xi_n)|P^n_{\xi_n, G_n}]$ converges weakly to a limit distribution which is normal with mean zero and covariance matrix depending on ξ_0 and G_A. Consequently, D.2 holds.

Because $P_{\hat{\xi}_n, \hat{G}_n}$ is the empirical distribution of the sample in this example, the reparametrization by ξ and G is not necessary in constructing the bootstrap confidence region which generates the test. (For instance, Theorem 1 in Beran (1984) could be applied.) However, the (ξ, G)-parametrization is useful in estimating the power of the test, a question to which we return after the next theorem.

THEOREM 3.2. *Suppose Ξ is R^k and the following requirements are met in addition to conditions* B.1 *and* B.2 *of Theorem* 2.2:

E.1. $\lim_{n\to\infty}\sup_{n^{1/2}|\xi-\xi_0|\le c}P^n_{\xi,\theta_A}|\hat{\xi}_n - \xi_0| > \varepsilon = 0$ *for every positive ε and c.*

E.2. *If $\{(\xi_n, \theta_n) \in \Omega\}$ is any sequence such that $\lim_{n\to\infty}n^{1/2}(\xi_n - \xi_0) = h$ for some $h \in R^k$ and $\lim_{n\to\infty}\theta_n = \theta_A$, then $K_{n,R}(\xi_n, \theta_n) \Rightarrow K_R^{(h)}(\xi_0, \theta_A)$ and $J_{n,R}(\xi_n, \theta_n) \Rightarrow J_R(\xi_0, \theta_A)$. Both limit distributions are continuous and do not depend upon the particular sequence $\{(\xi_n, \theta_n)\}$ chosen. Moreover, $J(\xi_0, \theta_A)$ has a strictly monotone survival function.*

E.3. *If $\{(\xi_n, \theta_n) \in \Omega\}$ is any sequence such that $\lim_{n\to\infty}n^{1/2}|\xi_n - \xi_0| = \infty$ and $\lim_{n\to\infty}\theta_n = \theta_A$, then $\lim_{n\to\infty}K_{n,R}(x; \xi_n, \theta_n) = 1$ for every finite real x and the distributions $\{J_{n,R}(\xi_n, \theta_n); n \ge 1\}$ are tight.*

Then

$$(3.5)\quad P^\infty_{\xi_A, \theta_A}\left[\lim_{n\to\infty}\sup_{\xi}\left|K_{n,R}\left[c_n(\alpha; \xi, \hat{\theta}_n); \xi, \hat{\theta}_n\right] - \beta_{n,\psi}(\alpha; \xi, \theta_A)\right| = 0\right] = 1.$$

Suppose B.2 *and* E.1 *are strengthened respectively to condition* B.2′ *of Theorem* 2.2 *and to*

E.1′. *If $\{(\xi_n, \theta_n) \in \Omega\}$ is any sequence such that $\sup_n n^{1/2}|\xi_n - \xi_0| \le c$ and $\lim_{n\to\infty}\theta_n = \theta_A$, then $\lim_{n\to\infty}P^n_{\xi_n,\theta_n}[|\hat{\xi}_n - \xi_0| > \varepsilon] = 0$ for every positive ε and c.*

Then also

$$P^{\infty}_{\xi_A,\theta_A}\left[\lim_{n\to\infty}\sup_{\xi}\left|\beta_{n,\psi}(\alpha;\xi,\hat{\theta}_n)-\beta_{n,\psi}(\alpha;\xi,\theta_A)\right|=0\right]=1. \tag{3.6}$$

EXAMPLE 2 *continued.* Consider the pivot $R_n(X,\xi)=|S_n^{-1/2}n^{1/2}(\overline{X}_n-\xi)|$. The test statistic T_n discussed earlier in this example coincides with $R_n(X,\xi_0)$. Because of location invariance,

$$J_{n,R}(\xi,\theta)=\mathscr{L}\left[|S_n^{-1/2}n^{1/2}\overline{X}_n|\,|P^n_{0,\theta}\right]=K_{n,T}(\xi_0,\theta).$$

Thus, $c_n(\alpha;\xi,\theta)$ does not depend on ξ and coincides with $d_n(\alpha;\xi_0,\theta)$. The tests φ_n and ψ_n are identical. Since $K_{n,T}(\xi,\theta)=K_{n,R}(\xi,\theta)$, the bootstrap power estimates $K_{n,T}[d_n(\alpha;\xi_0,\hat{\theta}_n);\xi,\hat{\theta}_n]$ and $K_{n,R}[c_n(\alpha;\xi,\hat{\theta}_n);\xi,\hat{\theta}_n]$ coincide, as do the estimates $\beta_{n,\varphi}(\alpha;\xi,\hat{\theta}_n)$ and $\beta_{n,\psi}(\alpha;\xi,\hat{\theta}_n)$. Theorem 3.2 is not needed for this example.

EXAMPLE 3 *continued.* Consider the pivot $R_n(X,\xi)=n^{1/2}|\hat{\xi}_n-\xi|$, where $\hat{\xi}_n$ is the sample correlation. Retaining definitions made in the earlier discussion of this example, we will show that conditions D.2, E.1′, E.2 and E.3 of Theorems 3.1 and 3.2 hold. Condition D.1 for Theorem 3.1 follows from B.2 and E.1.

E.1′. If $\{(\xi_n,\theta_n)\in\Omega\}$ is any sequence which converges to (ξ_0,θ_A), $\hat{\xi}_n$ converges in $P^n_{\xi_n,\theta_n}$-probability to ξ_0, by Khintchine's weak law of large numbers applied separately to the sample covariance and the two sample variances. In particular, E.1′ holds.

E.2. Let $\{(\xi_n,\theta_n)\in\Omega\}$ be any sequence such that $\lim_{n\to\infty}n^{1/2}(\xi_n-\xi_0)=h$ and $\lim_{n\to\infty}\theta_n=\theta_A$. Since $R_n(X,\xi_0)$ equals the test statistic T_n discussed earlier in the example, B.3 implies the desired weak convergence for $\{K_{n,R}(\xi_n,\theta_n)\}$. The argument for B.3 also shows that $J_{n,R}(\xi_n,\theta_n)\Rightarrow\mathscr{L}[|\sigma(\xi_0,\theta_0)Z|]$, where Z is a $N(0,1)$ random variable.

D.2 is checked like the second part of E.2.

E.3. Let $\{(\xi_n,\theta_n)\in\Omega\}$ be such that $\lim_{n\to\infty}n^{1/2}|\xi_n-\xi_0|=\infty$ and $\lim\theta_n=\theta_A$. Condition B.4 implies the first part of E.3 because $R_n(X,\xi_0)=T_n$. Suppose the distributions $\{J_{n,R}(\xi_n,\theta_n);\,n\geq 1\}$ were not tight, hence not relatively compact. By going to a subsequence, we can assume that the $\{J_{n,R}(\xi_n,\theta_n)\}$ do not converge to a limit distribution while $\lim_{n\to\infty}\xi_n=\xi^*\in[-b,b]$ exists. On the other hand, $\{J_{n,R}(\xi_n,\theta_n)\}$ converges weakly to $\mathscr{L}[|\sigma(\xi^*,\theta_A)Z|]$ by the Lindeberg central limit theorem, as in B.3. The contradiction establishes E.3.

EXAMPLE 6 *continued.* Let Ξ be any compact subset of $\{x\in R^3:\,|x|_2=1\}-e$. Retain other definitions made in the earlier discussion of this example. An argument similar to the one used in example 3 shows that the conditions for Theorem 3.2 are satisfied in this case as well.

THEOREM 3.3. *Suppose the following requirements are met in addition to conditions B.1 and B.2 of Theorem 2.2:*

F.1. $\lim_{n\to\infty}\sup_\xi P^n_{\xi,\theta_A}[m_1(\hat{\xi}_n, \xi) > \varepsilon] = 0$ *for every positive* ε.

F.2. *If* $\{(\xi_n, \theta_n) \in \Omega\}$ *is any sequence such that* $\lim_{n\to\infty}\theta_n = \theta_A$, *then* $\lim_{n\to\infty}\delta_p[K_{n,R}(\xi_n, \theta_n), K_{n,R}(\xi_n, \theta_A)] = 0$.

F.3. *If* $\{(\xi_n, \theta_n) \in \Omega\}$ *and* $\{(\bar{\xi}_n, \bar{\theta}_n) \in \Omega\}$ *are any two sequences such that* $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\bar{\theta}_n = \theta_A$ *and* $\lim_{n\to\infty}m_1(\xi_n, \bar{\xi}_n) = 0$, *then* $\lim_{n\to\infty}\delta_p[J_{n,R}(\xi_n, \theta_n), J_{n,R}(\bar{\xi}_n, \bar{\theta}_n)] = 0$.

Then the following assertion is true with $P^\infty_{\xi_A,\theta_A}$*-probability one: For every sufficiently small positive* ε, *there exists* $n_0(\varepsilon)$ *such that for every* $n \ge n_0(\varepsilon)$,

$$\begin{aligned} &K_{n,R}[c_{n,u}(\alpha - \varepsilon; \xi, \hat{\theta}_n) + \varepsilon; \xi, \hat{\theta}_n] - \varepsilon \\ &\le \beta_{n,\psi}(\alpha; \xi, \theta_A) \\ &\le K_{n,R}[c_{n,L}(\alpha + \varepsilon; \xi, \hat{\theta}_n) - \varepsilon; \xi, \hat{\theta}_n] + \varepsilon \end{aligned} \tag{3.7}$$

simultaneously for every possible ξ. *Suppose* B.2 *and* F.1 *are strengthened respectively to condition* B.2′ *of Theorem* 2.2 *and to*

F.1′. *If* $\{(\xi_n, \theta_n) \in \Omega\}$ *is any sequence such that* $\lim_{n\to\infty}\theta_n = \theta_A$, *then* $\lim_{n\to\infty}P^n_{\xi_n,\theta_n}[m_1(\hat{\xi}_n, \xi_n) > \varepsilon] = 0$ *for every positive* ε *and c.*

Let

$$\begin{aligned} \beta_{n,\psi,L}(\alpha; \xi, \theta, \varepsilon) &= P^n_{\xi,\theta}[R_n(\mathbf{X}, \xi_0) > c_{n,u}(\alpha - \varepsilon; \hat{\xi}_n, \hat{\theta}_n) + \varepsilon] - \varepsilon, \\ \beta_{n,\psi,u}(\alpha; \xi, \theta, \varepsilon) &= P^n_{\xi,\theta}[R_n(\mathbf{X}, \xi_0) > c_{n,L}(\alpha + \varepsilon; \hat{\xi}_n, \hat{\theta}_n) - \varepsilon] + \varepsilon. \end{aligned} \tag{3.8}$$

Then (3.7) *may be replaced by*

$$\beta_{n,\psi,L}(\alpha; \xi, \hat{\theta}_n, \varepsilon) \le \beta_{n,\psi}(\alpha; \xi, \theta_A) \le \beta_{n,\psi,u}(\alpha; \xi, \hat{\theta}_n, \varepsilon). \tag{3.9}$$

EXAMPLE 4 *continued.* Consider the pivot $R_n(X, \xi) = n^{1/2}\|\hat{\xi}_n - \xi\|$, where $\hat{\xi}_n$, ξ are defined by (2.18) and the surrounding discussion. Let both m_1 and m_2 be supremum norm metric. The conditions for Theorem 3.1 and 3.3 hold for the following reasons:

F.1′ and D.1. Let $\{(\xi_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$. Under $P^n_{\xi_n,\theta_n}$, $\hat{\xi}_n$ has the representation

$$\hat{\xi}_n(x) = \xi_n(x) + \tfrac{1}{2}n^{-1/2}[B_n \cdot F_{\xi_n,\theta_n}(x) + B_n \cdot F_{\xi_n,\theta_n}(-x)], \tag{3.10}$$

which implies F.1′. Moreover, F.1′ and B.2′ (verified earlier) imply D.1.

F.2. Since $K_{n,R}(\xi, \theta) = K_{n,T}(\xi, \theta)$ in this example, condition F.2 coincides with the previously verified condition C.1.

F.3. Let $\{(\xi_n, \theta_n) \in \Omega\}$ and $\{(\bar{\xi}_n, \bar{\theta}_n) \in \Omega\}$ be any two sequences such that $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\bar{\theta}_n = \theta_A$ and $\lim_{n\to\infty}\|\xi_n - \bar{\xi}_n\| = 0$. Let

$$V_n(\xi, \theta) = \tfrac{1}{2}\|B_n \cdot F_{\xi,\theta}(x) + B_n \cdot F_{\xi,\theta}(-x)\|. \tag{3.11}$$

In view of (3.10), $J_{n,R}(\xi,\theta) = \mathscr{L}[V_n(\xi,\theta)]$. Let $\{B_n\}$ be versions of the empirical Brownian bridge processes which converge almost surely to a Brownian bridge process B. Since $\lim_{n\to\infty}\|F_{\xi_n,\theta_n} - F_{\bar{\xi}_n,\bar{\theta}_n}\| = 0$, the corresponding versions of $\{V_n(\xi_n,\theta_n)\}$ and $\{V_n(\bar{\xi}_n,\bar{\theta}_n)\}$ have the property that $\lim_{n\to\infty}[V_n(\xi_n,\theta_n) - V_n(\bar{\xi}_n,\bar{\theta}_n)] = 0$ with probability one. This implies F.3.

D.2. Let $\{(\xi_n,\theta_n)\in\Omega\}$ be any sequence such that $\lim_{n\to\infty}\xi_n = \xi_0$ and $\lim_{n\to\infty}\theta_n = \theta_A$. By the argument for F.3, $J_{n,R}(\xi_n,\theta_n)$ converges weakly to a limit distribution

$$J_R(\xi_0,\theta_A) = \mathscr{L}\left[\tfrac{1}{2}\left\|B\cdot F_{\xi_0,\theta_A}(x) + B\cdot F_{\xi_0,\theta_A}(-x)\right\|\right],$$

as required.

4. Theorem proofs.

PROOF OF THEOREM 2.1. Similar to Theorem 1 in Beran (1984).

PROOF OF THEOREM 2.2. Let $\{\theta_n\in\Theta;\ n\ge 1\}$ be any sequence which converges to θ_A. It follows from B.3, specialized to the sequence $\{(\xi_0,\theta_n)\}$, that

$$\lim_{n\to\infty} d_n(\alpha;\xi_0,\theta_n) = d(\alpha;\xi_0,\theta_A), \tag{4.1}$$

the upper α-point of $K_T^{(0)}(\xi_0,\theta_A)$.

Let $\{(\xi_n,\theta_n)\in\Omega;\ n\ge 1\}$ be any sequence such that $\lim_{n\to\infty} n^{1/2}(\xi_n-\xi_0) = h$ for some $h\in R^k$ and $\{\theta_n\}$ converges to θ_A. From B.3 and (4.1),

$$\lim_{n\to\infty} K_{n,T}[d_n(\alpha;\xi_0,\theta_n);\xi_n,\theta_n] = K_T^{(h)}[d(\alpha;\xi_0,\theta_A);\xi_0,\theta_A]. \tag{4.2}$$

On the other hand, it follows from B.2 and (4.1) that $\{d_n(\alpha;\xi_0,\hat{\theta}_n)\}$ converges in $P^n_{\xi_n,\theta_A}$-probability to $d(\alpha;\xi_0,\theta_A)$. From this and B.3,

$$\lim_{n\to\infty}\beta_{n,\varphi}(\alpha;\xi_n,\theta_A) = K_T^{(h)}[d(\alpha;\xi_0,\theta_A);\xi_0,\theta_A]. \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$\lim_{n\to\infty}\sup_{n^{1/2}|\xi-\xi_0|\le c}\left|K_{n,T}[d_n(\alpha;\xi_0,\theta_n);\xi,\theta_n] - \beta_{n,\varphi}(\alpha;\xi,\theta_A)\right| = 0 \tag{4.4}$$

for every positive c and for every sequence $\{\theta_n\}$ converging to θ_A.

Let $\{(\xi_n,\theta_n)\in\Omega;\ n\ge 1\}$ be any sequence such that $\lim_{n\to\infty} n^{1/2}|\xi_n-\xi_0| = \infty$ and $\{\theta_n\}$ converges to θ_A. The critical values $\{d_n(\alpha;\xi_0,\hat{\theta}_n)\}$ still converge in $P^n_{\xi_n,\theta_A}$-probability to $d(\alpha;\xi_0,\theta_A)$ because of B.2 and (4.1). In view of B.4, both $\lim_{n\to\infty}\beta_{n,\varphi}(\alpha;\xi_n,\theta_A)$ and $\lim_{n\to\infty}K_{n,T}[d_n(\alpha;\xi_0,\theta_n);\xi_n,\theta_n]$ are equal to 1. This fact, together with (4.4), implies that

$$\lim_{n\to\infty}\sup_{\xi}\left|K_{n,T}[d_n(\alpha;\xi_0,\theta_n);\xi,\theta_n] - \beta_{n,\theta}(\alpha;\xi,\theta_A)\right| = 0 \tag{4.5}$$

for every sequence $\{\theta_n\}$ converging to θ_A. The theorem assertion (2.5) follows from (4.5) and B.1.

Let $\{(\xi_n, \theta_n) \in \Omega;\, n \geq 1\}$ be any sequence such that $\{\theta_n\}$ converges to θ_A. From B.2′ and (4.1), it follows that the critical values $\{d_n(\alpha; \xi_0, \hat{\theta}_n)\}$ converge in $P^n_{\xi_n, \theta_n}$-probability to $d(\alpha; \xi_0, \theta_A)$. Thus, by essentially the same arguments as above,

$$\lim_{n\to\infty} \beta_{n,\varphi}(\alpha; \xi_n, \theta_n) = K_T^{(h)}[d(\alpha; \xi_0, \theta_A); \xi_0, \theta_A] \tag{4.6}$$

if

$$\lim_{n\to\infty} n^{1/2}(\xi_n - \xi_0) = h; \text{ and } \lim_{n\to\infty} \beta_{n,\varphi}(\alpha; \xi_n, \theta_n) = 1 \text{ if } \lim_{n\to\infty} n^{1/2}|\xi_n - \xi_0| = \infty.$$

Consequently,

$$\lim_{n\to\infty} \sup_{\xi} |\beta_{n,\varphi}(\alpha; \xi, \theta_n) - \beta_{n,\varphi}(\alpha; \xi, \theta_A)| = 0, \tag{4.7}$$

which implies (2.6), in view of B.1.

PROOF OF THEOREM 2.3. Fix $\alpha \in (0,1)$. Without loss of generality, assume that the test statistic T_n takes its values in $[0,1]$; if necessary, replace T_n by a strictly monotone function of T_n to achieve this end. Let $\{\theta_n\}, \{\bar{\theta}_n\}$ be any two sequences in Θ such that $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\bar{\theta}_n = \theta_A$. First we show that, for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that

$$\begin{aligned} d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon &< d_{n,L}(\alpha; \xi_0, \bar{\theta}_n) \\ &\leq d_{n,u}(\alpha; \xi_0, \bar{\theta}_n) \\ &\leq d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon \end{aligned} \tag{4.8}$$

for every $n \geq n_0(\varepsilon)$. Indeed, from the definition (1.4) of the upper and lower α-points,

$$\begin{aligned} K_{n,T}[d_{n,u}(\alpha; \xi_0, \theta_n) + \varepsilon/2; \xi_0, \theta_n] &< \alpha \\ &< K_{n,T}[d_{n,L}(\alpha; \xi_0, \theta_n) - \varepsilon/2; \xi_0, \theta_n] \end{aligned} \tag{4.9}$$

for all sufficiently large n. In view of C.1 and the compact support of T_n, $\lim \delta_L[K_{n,T}(\xi_0, \bar{\theta}_n), K_{n,T}(\xi_0, \theta_n)] = 0$, where δ_L denotes Lévy metric. Hence,

$$\begin{aligned} K_{n,T}[d_{n,u}(\alpha; \xi_0, \theta_n) &+ \varepsilon/2; \xi_0, \theta_n] \\ &\geq K_{n,T}[d_{n,u}(\alpha; \xi_0, \theta_n) + \varepsilon; \xi_0, \bar{\theta}_n] - \varepsilon \end{aligned} \tag{4.10}$$

for all sufficiently large n. Combining (4.9) with (4.10) yields

$$K_{n,T}[d_{n,u}(\alpha; \xi_0, \theta_n) + \varepsilon; \xi_0, \bar{\theta}_n] < \alpha + \varepsilon, \tag{4.11}$$

which implies $d_{n,u}(\alpha; \xi_0, \theta_n) + \varepsilon \geq d_{n,u}(\alpha + \varepsilon; \xi_0, \bar{\theta}_n)$ and therefore the right half of (4.8). The left half of (4.8) is proved similarly.

Let $\{\xi_n\}$ be any sequence in Ξ such that both $\{(\xi_n, \theta_n)\} \in \Omega$ and $\{(\xi_n, \theta_A)\} \in \Omega$. We prove next that, for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that for every $n \geq n_0(\varepsilon)$

$$\begin{aligned} &K_{n,T}\left[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon; \xi_n, \theta_A\right] - \varepsilon \\ &\qquad \leq \beta_{n,\varphi}(\alpha; \xi_n, \theta_A) \leq K_{n,T}\left[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon; \xi_n, \theta_A\right] + \varepsilon. \end{aligned} \tag{4.12}$$

Let

$$\begin{aligned} A_n(\varepsilon) = \{d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon \\ < d_{n,L}(\alpha; \xi_0, \hat{\theta}_n) \leq d_{n,u}(\alpha; \xi_0, \hat{\theta}_n) \\ \leq d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon\}. \end{aligned} \tag{4.13}$$

Since $\{\hat{\theta}_n\}$ converges to θ_A in $P^n_{\xi_n, \theta_A}$-probability by B.2, there exist versions of the $\{\hat{\theta}_n\}$ such that $\lim_{n\to\infty}\hat{\theta}_n = \theta_A$ w.p. 1 [Wichura (1970)]. For these versions, $P[\liminf_{n\to\infty} A_n(\varepsilon)] = 1$ because of (4.8). Hence $\lim_{n\to\infty} P[A_n(\varepsilon)] = 1$, which implies

$$\lim_{n\to\infty} P^n_{\xi_n, \theta_A}[A_n(\varepsilon)] = 1 \tag{4.14}$$

for the original estimates $\{\hat{\theta}_n\}$. Consequently,

$$\begin{aligned} &P^n_{\xi_n, \theta_A}\left[T_n > d_{n,u}(\alpha; \xi_0, \hat{\theta}_n)\right] \\ &\qquad \geq K_{n,T}\left[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon; \xi_n, \theta_A\right] + o(1), \\ &P^n_{\xi_n, \theta_A}\left[T_n > d_{n,L}(\alpha; \xi_0, \hat{\theta}_n)\right] \\ &\qquad \leq K_{n,T}\left[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon; \xi_n, \theta_A\right] + o(1) \end{aligned} \tag{4.15}$$

as n tends to infinity. The definition (1.6) of φ_n implies that $\beta_{n,\varphi}(\alpha; \xi_n, \theta_A)$ lies between the two probabilities on the left side of (4.15). Hence (4.12) follows.

The third step is to show that θ_A can be replaced by θ_n in the two bounds in (4.12). Since $\lim_{n\to\infty}\delta_L[K_{n,T}(\xi_n, \theta_n), K_{n,T}(\xi_n, \theta_A)] = 0$, by C.1 and the compact support of T_n,

$$\begin{aligned} &K_{n,T}\left[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon; \xi_n, \theta_A\right] \\ &\qquad \geq K_{n,T}\left[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + 2\varepsilon; \xi_n, \theta_n\right] - \varepsilon, \\ &K_{n,T}\left[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon; \xi_n, \theta_A\right] \\ &\qquad \leq K_{n,T}\left[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - 2\varepsilon; \xi_n, \theta_n\right] + \varepsilon. \end{aligned} \tag{4.16}$$

Combining (4.16) with (4.12) and using the monotonicity of $d_{n,L}(\alpha; \xi, \theta)$, $d_{n,u}(\alpha; \xi, \theta)$ in α yields the following conclusion: For every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that for every $n \geq n_0(\varepsilon)$

$$\begin{aligned} &K_{n,T}\left[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon; \xi, \theta_n'\right] - \varepsilon \\ &\qquad \leq \beta_{n,\varphi}(\alpha; \xi, \theta_A) \leq K_{n,T}\left[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon; \xi, \theta_n\right] + \varepsilon \end{aligned} \tag{4.17}$$

simultaneously for every possible ξ. The theorem assertion (2.15) follows from (4.17) and condition B.1.

Finally, we show that for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that for every $n \geq n_0(\varepsilon)$,

$$\beta_{n,\varphi,L}(\alpha; \xi_n, \theta_n, \varepsilon) \leq \beta_{n,\varphi}(\alpha; \xi_n, \theta_A) \leq \beta_{n,\varphi,u}(\alpha; \xi_n, \theta_n, \varepsilon). \tag{4.18}$$

Let

$$\begin{aligned} B_n(\varepsilon) = \{d_{n,L}(\alpha + 2\varepsilon; \xi_0, \hat{\theta}_n) - \varepsilon < d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) \\ \leq d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) \leq d_{n,u}(\alpha - 2\varepsilon; \xi_0, \hat{\theta}_n) + \varepsilon\}. \end{aligned} \tag{4.19}$$

Under condition B.2′, $\lim_{n\to\infty} P^n_{\xi_n,\theta_n}[B_n(\varepsilon)] = 1$, by an argument based on (4.8) and similar to that for (4.14). Thus,

$$\begin{aligned} &K_{n,T}[d_{n,L}(\alpha + \varepsilon; \xi_0, \theta_n) - \varepsilon; \xi_n, \theta_n] \\ &\quad \leq P^n_{\xi_n,\theta_n}[T_n > d_{n,L}(\alpha + 2\varepsilon; \xi_0, \hat{\theta}_n) - 2\varepsilon; \xi_n, \theta_n] + o(1), \\ &K_{n,T}[d_{n,u}(\alpha - \varepsilon; \xi_0, \theta_n) + \varepsilon; \xi_n, \theta_n] \\ &\quad \geq P^n_{\xi_n,\theta_n}[T_n > d_{n,u}(\alpha - 2\varepsilon; \xi_0, \hat{\theta}_n) + 2\varepsilon; \xi_n, \theta_n] + o(1) \end{aligned} \tag{4.20}$$

as n tends to infinity. Combining (4.20) with (4.17) yields (4.18). The theorem assertion (2.17) follows from (4.18) and condition B.1.

PROOF OF THEOREM 3.1. Essentially the same as Theorem 1 in Beran (1984).

PROOF OF THEOREM 3.2. Let $\{(\xi_n, \theta_n) \in \Omega\}$ be any sequence such that $\lim_{n\to\infty} n^{1/2}(\xi_n - \xi_0) = h \in R^k$ and $\lim_{n\to\infty}\theta_n = \theta_A$. From E.2,

$$\lim_{n\to\infty} c_n(\alpha; \xi_n, \theta_n) = c(\alpha; \xi_0, \theta_A) \tag{4.21}$$

the upper α-point of $J(\xi_0, \theta_A)$. Moreover, in view of E.1 and B.2, $\{c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n)\}$ converges in $P^n_{\xi_n,\theta_A}$-probability to $c(\alpha; \xi_0, \theta_A)$. It follows from these two convergences and E.2 that

$$\lim_{n\to\infty} \sup_{n^{1/2}|\xi - \xi_0| \leq c} |K_{n,R}[c_n(\alpha; \xi, \theta_n); \xi, \theta_n] - \beta_{n,\psi}(\alpha; \xi, \theta_A)| = 0 \tag{4.22}$$

for every positive c; the argument parallels that for (4.4).

If $\{(\xi_n, \theta_n) \in \Omega\}$ is any sequence such that $\lim_{n\to\infty}\theta_n = \theta_A$, conditions E.2 and E.3 imply that the distributions $\{J_{n,R}(\xi_n, \theta_n)\}$ are tight; consequently $\sup_n|c_n(\alpha; \xi_n, \theta_n)|$ is finite for every $\alpha \in (0, 1)$. Suppose $\lim_{n\to\infty} n^{1/2}|\xi_n - \xi_0| = \infty$. It follows from E.3 and the above that $\lim_{n\to\infty} K_{n,R}[c_n(\alpha; \xi_n, \theta_n); \xi_n, \theta_n] = 1$. Since $R_n \to \infty$ and $\hat{\theta}_n \to \theta_A$ in $P^n_{\xi_n,\theta_A}$-probability, there exists versions of $\{(\hat{\xi}_n, \hat{\theta}_n, R_n)\}$ such that these convergences occur with probability one. For these versions, $\sup_n|c_n(\alpha; \hat{\xi}_n, \hat{\theta}_n)|$ is finite with probability one. Hence $\lim_{n\to\infty}\beta_{n,\psi}(\alpha; \xi_n, \theta_A) = 1$.

From (4.4) and the previous paragraph,

$$\lim_{n\to\infty} \sup_{\xi} |K_{n,R}[c_n(\alpha; \xi, \theta_n); \xi, \theta_n)] - \beta_{n,\psi}(\alpha; \xi, \theta_A)| = 0 \tag{4.23}$$

for every sequence $\{\theta_n\}$ converging to θ_A. Theorem assertion (3.5) follows from

(4.23) and B.1. If conditions B.2 and E.1 are strengthened to B.2′ and E.1′, then θ_A may be replaced by θ_n in (4.23). Consequently,

$$\lim_{n\to\infty} \sup_{\xi} |\beta_{n,\psi}(\alpha; \xi_n, \theta_n) - \beta_{n,\varphi}(\alpha; \xi, \theta_A)| = 0, \tag{4.24}$$

which implies (3.6), in view of B.1.

PROOF OF THEOREM 3.3. The argument is a modification of the proof for Theorem 2.3. Without loss of generality, assume that the pivot $R_n(\mathbf{X}, \xi)$ takes its values in $[0, 1]$ for every ξ. Let $\{(\xi_n, \theta_n)\}$ and $\{(\bar{\xi}_n, \bar{\theta}_n)\}$ be any two sequences in Ω such that $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\bar{\theta}_n = \theta_A$ and $\lim_{n\to\infty} m_1(\xi_n, \bar{\xi}_n) = 0$. Then, for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that

$$\begin{aligned} c_{n,L}(\alpha + \varepsilon; \xi_n, \theta_n) - \varepsilon < c_{n,L}(\alpha; \bar{\xi}_n, \bar{\theta}_n) & \\ \le c_{n,u}(\alpha; \bar{\xi}_n, \bar{\theta}_n) & \\ \le c_{n,u}(\alpha - \varepsilon; \xi_n, \theta_n) + \varepsilon & \end{aligned} \tag{4.25}$$

for every $n \ge n_0(\varepsilon)$. This follows from F.3 by an argument similar to that for (4.8).

Conditions B.2, F.1, and the reasoning for (4.12) now imply: for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that

$$\begin{aligned} &K_{n,R}[c_{n,u}(\alpha - \varepsilon; \xi_n, \theta_n) + \varepsilon; \xi_n, \theta_A] - \varepsilon \\ &\quad \le \beta_{n,\psi}(\alpha; \xi_n, \theta_A) \le K_{n,R}[c_{n,L}(\alpha + \varepsilon; \xi_n, \theta_n) - \varepsilon; \xi_n, \theta_A] + \varepsilon \end{aligned} \tag{4.26}$$

for every $n \ge n_0(\varepsilon)$.

A further argument, drawing on F.2 and similar to that for (4.17), shows that θ_A can be replaced by θ_n in the two bounds in (4.26). Hence, for every sufficiently small positive ε, there exists $n_0(\varepsilon)$ such that for every $n \ge n_0(\varepsilon)$

$$\begin{aligned} &K_{n,R}[c_{n,u}(\alpha - \varepsilon; \xi, \theta_n) + \varepsilon; \xi, \theta_n] - \varepsilon \\ &\quad \le \beta_{n,\psi}(\alpha; \xi, \theta_A) \le K_{n,R}[c_{n,L}(\alpha + \varepsilon; \xi, \theta_n) - \varepsilon; \xi, \theta_n] + \varepsilon. \end{aligned} \tag{4.27}$$

The theorem assertion (3.7) follows from (4.27) and B.1.

Let

$$\begin{aligned} C_n(\varepsilon) = \{c_{n,L}(\alpha + 2\varepsilon; \hat{\xi}_n, \hat{\theta}_n) - \varepsilon < c_{n,L}(\alpha + \varepsilon; \xi_n, \theta_n) & \\ \le c_{n,u}(\alpha - \varepsilon; \xi_n, \theta_n) & \\ \le c_{n,u}(\alpha - 2\varepsilon; \hat{\xi}_n, \hat{\theta}_n) + \varepsilon\}. & \end{aligned} \tag{4.28}$$

Under conditions B.2′ and F.1′, $\lim_{n\to\infty} P^n_{\xi_n,\theta_n}[C_n(\varepsilon)] = 1$ for every positive sufficiently small ε; the argument is based upon (4.25) and is similar to that for (4.14). Continuing along the lines of the last paragraph in the proof of Theorem 2.3 yields (3.9).

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