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A NOTE ON THE CHARACTERIZATION OF OPTIMAL RETURN FUNCTIONS AND OPTIMAL STRATEGIES FOR GAMBLING PROBLEMS

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We consider finite state gambling problems with the Dubins and Savage payoff and with the lim inf payoff. For these models we show that the optimal return function with respect to all stationary strategies can be characterized similarly to the optimal return function. This enables us then to characterize those stationary strategies which are optimal within the set of all stationary strategies in the same way as it was done for optimal strategies by Dubins and Savage.

1. Introduction and summary. Let (F, Γ, u) be a gambling problem as defined by Dubins and Savage [3]. That is, F is a nonempty set and, for each $f \in F$, $\Gamma(f)$ is a nonempty set of probability measures (called gambles) defined on subsets of the state space F. The utility function u is a bounded function from F into the real numbers.

In this paper we will always tacitly assume that F is a *finite* set and, to keep the paper as short as possible, we will use, with minor changes, notation and terminology as in [3] and [6]. Thus

$$u(\sigma) = \limsup_{\tau} \int u(f_\tau)\, d\sigma,$$

where the lim sup is taken over the directed set of stop rules, denotes the payoff of the strategy σ and V denotes the optimal return function, which is given by

$$V(f) = \sup\{u(\sigma) \mid \sigma \text{ at } f\}.$$

Further a strategy σ is called *optimal* at f if $u(\sigma) = V(f)$, *thrifty* at f if $V(\sigma) = V(f)$ and *equalizing* if $V(\sigma) = u(\sigma)$.

Two main results in [3], Theorem 3.3.1 and Theorem 3.5.1, are:

1) V is the smallest of those functions Q that are excessive for Γ and for which $Q(\sigma) \geq u(\sigma)$ for every strategy σ.
2) A strategy σ is optimal if and only if σ is thrifty and equalizing.

A Γ-selector is a function γ with domain F such that $\gamma(f) \in \Gamma(f)$ for all f. Such a selector determines a *stationary family of strategies* γ^∞. Since a stationary strategy $\gamma^\infty(f)$ is thrifty if and only if $\gamma(f)$ conserves V, i.e. $\gamma(f)V \geq V(f)$, for all f, we can reformulate (2) for stationary strategies as follows:

2′) A stationary family of strategies γ^∞ is optimal if and only if γ is conserving and $\gamma^\infty(f)$ is equalizing for all f.

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Now our main result is that $\tilde{V}$, the optimal return function with respect to all stationary strategies which is given by

$$\tilde{V}(f) = \sup\{u(\sigma) \mid \sigma \text{ is a stationary strategy at } f\},$$

allows for a similar characterization as V and that stationary strategies which are optimal within the set of all stationary strategies can also be characterized by similar expressions as conserving and equalizing.

DEFINITION. A stationary family of strategies γ^∞ is called *$\tilde{V}$-conserving* if $\gamma(f)\tilde{V} \geq \tilde{V}(f)$ for all f and *$\tilde{V}$-equalizing* if $\tilde{V}(\gamma^\infty) = u(\gamma^\infty)$.

THEOREM 1.

1) *$\tilde{V}$ is the smallest of those functions Q that are excessive for Γ and for which $Q(\gamma^\infty) \geq u(\gamma^\infty)$ for every stationary family γ^∞.*
2) *A stationary family γ^∞ satisfies $u(\gamma^\infty) = \tilde{V}$ if and only if γ^∞ is $\tilde{V}$-conserving and $\gamma^\infty(f)$ is $\tilde{V}$-equalizing for all f.*

This Theorem is not true for general state space F, since the function $\tilde{V}$ is not excessive for the examples considered in [4] and [5].

2. Proofs. Examining the proofs in [3] reveals that we only have to prove that $\tilde{V}$ is excessive for Γ, i.e. $\gamma\tilde{V} \leq \tilde{V}(f)$ for all $\gamma \in \Gamma(f), f \in F$, since using only stationary strategies in these proofs all the other arguments can be used unaltered.

LEMMA 1. *For each positive ε there exists some stationary family α^∞ such that $u(\alpha^\infty) \geq \tilde{V} - \varepsilon$.*

PROOF. By definition of $\tilde{V}$ there exists for each positive ε and each $f \in F$ some stationary strategy γ_f^∞ such that $u(\gamma_f^\infty) \geq \tilde{V}(f) - \varepsilon$. Define a new gambling problem $(\hat{F}, \hat{\Gamma}, \hat{u})$ with $\hat{F} = F$, $\hat{u} = u$ and $\hat{\Gamma}(f) = \cup_{f' \in F}\{\gamma_{f'}(f)\}$ for all f. Denote the optimal return function for the new problem by $\hat{V}$, which obviously satisfies $\hat{V} \geq \tilde{V} - \varepsilon$. By Theorem 3.9.1 in [3] there exists some stationary family α^∞ in $\hat{\Gamma}$ such that $u(\alpha^\infty) = \hat{V}$. Since $\hat{\Gamma}(f) \subseteq \Gamma(f)$ for all f, α^∞ is also a stationary family in the original model and $u(\alpha^\infty) = \hat{V} \geq \tilde{V} - \varepsilon$.

Lemma 1 paves the way to prove the excessivity of $\tilde{V}$.

THEOREM 2. *$\tilde{V}$ is excessive for Γ.*

PROOF. Fix $f_0 \in F$ and $\gamma_0 \in \Gamma(f_0)$. For each positive ε, let α^∞ be the stationary family of Lemma 1. Define a new gambling problem $(\hat{F}, \hat{\Gamma}, \hat{u})$ by setting $\hat{F} = F$, $\hat{u} = u$ and $\hat{\Gamma}(f) = \{\alpha(f)\}$ for $f \neq f_0$ and $\hat{\Gamma}(f) = \{\gamma_0\} \cup \{\alpha(f)\}$ for $f = f_0$. Since the optimal return function $\hat{V}$ for this new model is excessive for $\hat{\Gamma}$, $\gamma_0 \in \hat{\Gamma}(f_0)$ and $\hat{V} \geq \tilde{V} - \varepsilon$, we have $\hat{V}(f_0) \geq \gamma_0\hat{V} \geq \gamma_0\tilde{V} - \varepsilon$. Furthermore, again by Theorem 3.9.1 in [3], there exists an optimal stationary family for $(\hat{F}, \hat{\Gamma}, \hat{u})$, and since this stationary family is also available in the original model, we have $\tilde{V} \geq \hat{V}$.

Combining these results gives $\tilde{V}(f_0) \geq \gamma_0 \tilde{V} - \varepsilon$ for all positive ε and all $\gamma_0 \in \Gamma(f_0)$ and $f_0 \in F$. Thus $\tilde{V}$ is excessive for Γ.

3. Extensions and concluding remarks. In [6] Sudderth introduced the lim inf payoff

$$\underline{u}(\sigma) = \liminf_\tau \int u(f_\tau)\, d\sigma$$

with the optimal return function

$$W(f) = \sup\{\underline{u}(\sigma) \mid \sigma \text{ at } f\}.$$

For this model a strategy is called *optimal* at f if $\underline{u}(\sigma) = W(f)$, *thrifty* at f if $W(\sigma) = W(f)$ and *equalizing* if $W(\sigma) = \underline{u}(\sigma)$. Furthermore, a stationary family γ^∞ is called *conserving* if γ conserves W. With these definitions and replacing V by W and $u(\sigma)$ by $\underline{u}(\sigma)$ (1), (2) and (2′) of Section 1 remain true for lim inf payoff problems (see [6], Lemma 1).

If one defines now $\tilde{W}$, $\tilde{W}$-conserving and $\tilde{W}$-equalizing according to the related expressions for gambling problems with the Dubins and Savage payoff, then our Theorem is also true for lim inf payoff gambling problems. For the proof one has only to replace in Section 2 Theorem 3.9.1 in [3] by Theorem 3.1 in [6].

One may be tempted to think that all results which are true for classical gambling problems remain true for lim inf payoff problems (at least for a finite state space), but this is not the case. Dubins and Savage proved that if there is an optimal strategy then there exists a stationary family of optimal strategies ([3], Theorem 3.9.3). This is not valid for lim inf payoff problems.

Example. $F = \{0, 1\}$; $u(0) = 0$, $u(1) = 1$; $\Gamma(0) = \{\delta(1)\}$, $\Gamma(1) = \{(n-1)/n)\delta(1) + (1/n)\delta(0) \mid n \in \mathbb{N}\}$. Thus, $W(0) = W(1) = 1$ but $\tilde{W}(0) = \tilde{W}(1) = 0$ and, as can easily be seen, there exists some strategy σ such that $\underline{u}(\sigma) = W$.

Remark 1. If $\Gamma(f)$ is finite for all f then $V = \tilde{V}$ and $W = \tilde{W}$, and if Γ is leavable, i.e. the point mass $\delta(f) \in \Gamma(f)$ for all f, then even $V = \tilde{V} = W = \tilde{W}$ (see [6], §4). Other conditions which guarantee $V = \tilde{V}$ or $W = \tilde{W}$ can be found in [1].

2. Related results for dynamic programming problems are contained in [2].

REFERENCES

[1] Dawen, van, R. (1985). Gambling problems with finite state space. To appear in *Z. Angew. Math. Mech.* Heft 5.

[2] Dawen, van, R. (1984). Finite state dynamic programming with the total reward criterion. Submitted for publication.

[3] Dubins, L. E. and Savage, L. J. (1976). *Inequalities for Stochastic Processes: How to Gamble if You Must.* Dover, New York.

[4] Dubins, L. E. and Sudderth, W. D. (1975). An example where stationary strategies are not adequate. *Ann. Probab.* **3** 722–725.

[5] ORNSTEIN, D. (1969). On the existence of stationary optimal strategies. *Proc. Amer. Math. Soc.* **20** 563–569.
[6] SUDDERTH, W. D. (1983). Gambling problems with a limit inferior payoff. *Math. Oper. Res.* **8** 287–297.

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