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FAMILIES OF A-OPTIMAL BLOCK DESIGNS FOR COMPARING TEST TREATMENTS WITH A CONTROL[1]

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A-optimal designs for comparing each of v test treatments simultaneously with a control, in b blocks of size k each are considered. It is shown that several families of BIB designs in the test treatments augmented by t replications of a control in each block are A-optimal. In particular these designs with $t = 1$ are optimal whenever $(k-2)^2 + 1 \le v \le (k-1)^2$ irrespective of the number of blocks. This includes BIB designs associated with finite projective and Euclidean geometries.

1. Introduction. We shall consider the problem of obtaining optimal designs for comparing several test treatments with a control in incomplete blocks. Here the term control is used in the sense of a special or standard treatment, rather than a check on the experiment as advocated by Fisher and other experts in the theory of experimental design. To fix notation, we shall henceforth denote by v the number of test treatments, by b the number of blocks and by k the size of each block.

A serious study of the problem of comparing test treatments with a control was done by Pearce (1960), who proposed a class of suitable designs and gave their analysis. Bechhofer and Tamhane (1981) systematically investigated the problem of finding optimal designs for simultaneous interval estimation. Majumdar and Notz (1983) looked at the optimal design theory as postulated by Kiefer and other researchers, and gave a method of obtaining ϕ-optimal designs for a large class of criteria ϕ. In particular, the A-optimality criterion possesses considerable statistical content. Ture (1982) has also considered the problem of determination and construction of A-optimal designs. Notz and Tamhane (1983) and Hedayat and Majumdar (1984) have considered the problem of constructing efficient designs.

Even before these investigations, Cox (1958, page 238) had advocated augmenting a BIB design with one or more replications of the control in each block as a means of getting good designs. Pearce (1960) gave an example of a real life experiment in which a balanced block design was augmented by 2 replications of the control in each block. Pesěk (1974) investigated the efficiency of BIB designs augmented by one control in each block and Constantine (1983) established their A-optimality in a restricted class of competing designs. The catalog of A-optimal designs given by Hedayat and Majumdar (1984) for reasonably small

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values of the parameters v, b and k, shows that designs of this type are in fact A-optimal in many cases. One benefit of considering this type of optimal designs is that one may appeal to the vast literature on BIB designs for their existence and construction. This is an appreciable advantage over the more general class of BTIB designs of Bechhofer and Tamhane (1981), which are not very widely studied as yet. It also seems reasonable to assume that these designs will be quite efficient for comparisons among the test treatments.

The purpose of this paper is to obtain A-optimal designs explicitly. In particular, we investigate the optimality of augmented BIB designs. At the very outset we warn the reader that there are situations when an augmented BIB design is not optimal. Examples may be found in Constantine (1983) ($v = b = 11$, $k = 6$) and Hedayat and Majumdar (1984) ($v = 9$, $b = 48$, $k = 7$). The A-optimality of several families of augmented BIB designs is established here. In deriving these results, we have utilized the technique of Majumdar and Notz (1983). Indeed, no more A-optimal augmented BIB designs can be obtained by that technique. It may be noted that Majumdar and Notz's method involved minimizing a function over finite set of positive integers. In this paper, we have been successful in solving the problem analytically for those cases where the minimum corresponds to an augmented BIB design.

In the process, many interesting families of A-optimal designs have been obtained. Some examples are the families of BIB designs corresponding to finite projective and Euclidean geometries augmented by one replication of a control in each block, as well as the union of any number of copies of such designs. We would like to point out that these are the first known families of A-optimal designs for comparing test treatments with a control.

In Section 2 we prove general results on the optimality of BIB designs augmented by t replications of a control in each block. The special case $t = 1$ has many interesting features. These are studied in depth in Section 3. Examples of optimal designs for some other values of t are also provided.

2. Optimality of BIB designs augmented by t replications of a control in each block.

v test treatments are to be compared with a special treatment called the control in b blocks of size k each. We label the test treatments by $1, \cdots, v$ and the control by 0. Let Y_{ijl} denote the observation on treatment i $(0 \le i \le v)$ in block j $(1 \le j \le b)$ in plot l $(1 \le l \le k)$. We assume the usual additive linear model without interaction, namely

$$Y_{ijl} = \mu + \tau_i + \beta_j + e_{ijl},$$

where e_{ijl} are assumed to be uncorrelated random variables with mean 0 and common variance σ^2. The unknown constants μ, τ_i and β_j represent the general mean, the effect of treatment i and the effect of block j respectively. An experimental design is an allocation of treatments to blocks. Let $\mathscr{D}(v, b, k)$ be the set of all possible experimental designs. We want to choose an experimental design from $\mathscr{D}(v, b, k)$ which attains the minimum of

$$\sum_{i=1}^{v} \operatorname{var}(\hat{\tau}_{d0} - \hat{\tau}_{di})$$

as d varies over all of $\mathscr{D}(v, b, k)$. Here $\hat{\tau}_{d0} - \hat{\tau}_{di} (1 \le i \le v)$ denote the BLUE's of

$\tau_0 - \tau_i$ under the design d. A design which achieves the minimum is called an A-optimal design. We shall only deal with the incomplete block set up, $v \geq k$. A-optimal designs for $k = 2$ have been extensively studied, theoretically and numerically, in Hedayat and Majumdar (1984). In this paper we shall concentrate only on parameters v, k which satisfy

$$k \geq 3 \tag{2.1}$$

$$v \geq k. \tag{2.2}$$

In Theorem 2.1, due to Majumdar and Notz (1983), we state a method for finding A-optimal designs. Before stating it, we need some definitions and notation.

A design d in $\mathscr{D}(v, b, k)$ for which all $\text{var}(\hat{\tau}_{d0} - \hat{\tau}_{di})$, $(1 \leq i \leq v)$ are equal, and all $\text{Cov}(\hat{\tau}_{d0} - \hat{\tau}_{di}, \hat{\tau}_{d0} - \hat{\tau}_{di'})$ $(1 \leq i, i' \leq v, i \neq i')$ are equal is called a Balanced Test treatment Incomplete Block (BTIB) design. These were defined by Bechhofer and Tamhane (1981), who also gave a combinatorial definition of such designs. An important subclass of BTIB designs are the augmented BIB designs. We shall denote by BIB (v, b, k) a BIB design based on v treatments in b blocks of size k each. A BIB $(v, b, k - t)$ based on the v test treatments and augmented by t replications of a control in each block will be denoted by ABIB $(v, b, k - t; t)$. Following Das (1958), some authors refer to such designs as reinforced BIB designs.

Let

$$a = (v-1)^2, \quad c = bvk(k-1),$$

$$p = v(k-1) + k,$$

$$\Lambda_1 = \{(0, z): z = 1, \cdots, b\},$$

$$\Lambda_2 = \{(x, z): x = 1, \cdots, [k/2] - 1; z = 0, 1, \cdots, b\},$$

where $[e]$ is the largest integer not exceeding e. Let

$$\Lambda = \Lambda_1 \cup \Lambda_2,$$

and

$$g(x, z) = a/(c - p(bx + z) + (bx^2 + 2xz + z)) + 1/(k(bx + z) - (bx^2 + 2xz + z)).$$

We observe that,

$$g(x, 0) = g(x - 1, b), \quad \text{for all} \quad x \neq 0.$$

THEOREM 2.1 (Majumdar and Notz, 1983). *Let integers t and s be defined by*

$$g(t, s) = \text{Min}_{(x,z)\in\Lambda}\, g(x, z).$$

Then a BTIB *design with the control replicated $(t + 1)$ times in s blocks and t times in $(b - s)$ blocks and binary in the test treatments is A-optimal.*

The notations of Theorem 2.1 are different from that of Theorem 2.2 of Majumdar and Notz (1983). The symbol r of the original version is replaced by

$bx + z$ of this version. Moreover, the set Λ does not cover all points of the original version when k is odd, since in the latter, the points $x = [k/2]$, $s = 1, \cdots, [b/2]$ are also included. This gap does not affect the result, since when k is odd, $g([k/2], s)$ increases with $s \geq 0$. This version is better suited for our purpose.

If $s = 0$ in Theorem 2.1, then an ABIB $(v, b, k - t; t)$ is A-optimal. In this section, we shall obtain in Theorem 2.6, several families of v, b, k for which every ABIB $(v, b, k - t; t)$ is A-optimal. This is done via an analytical solution to the problem of minimizing $g(x, z)$ over Λ for those cases which lead to $s = 0$. This is given in Theorem 2.5. We start by proving some preliminary results. Lemma 2.2 is given on page 91 of Ture's Ph.D. dissertation (1982). For the sake of better availability, we include a proof of the lemma here.

LEMMA 2.2. *For each fixed x in the interval $[0, (k - 2)/2]$, there exists z_0, a function of x, $z_0 \in [0, b]$, such that $g(x, z)$ decreases with z when $z \in [0, z_0]$ and $g(x, z)$ increases with z when $z \in (z_0, b]$. If $z_0 = 0$ then $g(x, z)$ increases with z in $[0, b]$ and if $z_0 = b$, then $g(x, z)$ decreases with z in $[0, b]$.*

PROOF

$$\partial g(x, z)/\partial z = -a(2x + 1 - p)/(c - p(bx + z) + (bx^2 + 2xz + z))^2$$
$$-(k - 2x - 1)/(k(bx + z) - (bx^2 + 2xz + z))^2.$$

The sign of the partial derivative $\partial g(x, z)/\partial z$ is the same as the sign of

$$-a(2x + 1 - p)(k(bx + z) - (bx^2 + 2xz + z))^2$$
$$- (k - 2x - 1)(c - p(bx + z) + (bx^2 + 2xz + z))^2$$
$$= \gamma_2(x)z^2 + 2\gamma_1(x)z - \gamma_0(x)$$

where,

$$\gamma_0(x) = b^2(k - x)^2(-a(p - 2x - 1)x^2 + (k - 2x - 1)(v(k - 1) - x)^2),$$

$$\gamma_1(x) = bv(k - x)(k - 2x - 1)(p - 2x - 1)((v - 2)x + (k - 1))$$

and

$$\gamma_2(x) = v(k - 2x - 1)(p - 2x - 1)((v - 2)(k - 2x - 1) - (k - 1)).$$

Fix an x in $[0, (k - 2)/2]$ and define

$$h(z) = \gamma_2(x)z^2 + 2\gamma_1(x)z - \gamma_0(x).$$

Observe that $1 \leq k - 2x - 1 < p - 2x - 1$, and hence $\gamma_1(x) > 0$. Moreover, the derivative

$$dh(z)/dz = 2(\gamma_2(x)z + \gamma_1(x)) > 0$$

whenever $(v - 2)(k - 2x - 1) - (k - 1) \geq 0$, and $z \geq 0$. Even when $(v - 2)(k - 2x - 1) - (k - 1) < 0$ (for example, $v = k$, $x = (k - 2)/2$), and

$0 \le z \le b$,

$$dh(z)/dz \ge 2bv(k - 2x - 1)(p - 2x - 1)$$
$$\cdot ((v - 2)((k - x)x + (k - 2x - 1)) + (k - 1)(k - x - 1)) > 0,$$

using $x \in [0, (k-2)/2]$, (2.1) and (2.2). Hence, for each fixed x in $[0, (k-2)/2]$, one of the three cases may happen:

(i) There exists $z_0 \in [0, b]$ such that $h(z) < 0$ when $0 \le z < z_0$, $h(z_0) = 0$, $h(z) > 0$ when $z_0 < z \le b$.
(ii) $h(z) < 0$ whenever $0 \le z \le b$.
(iii) $h(z) > 0$ whenever $0 \le z \le b$.

Since the signs of $\partial g(x, z)/\partial z$ and $h(z)$ are the same, the lemma follows if we choose $z_0 = b$ in (ii) and $z_0 = 0$ in (iii).

LEMMA 2.3. *Let* $x \in [0, (k-2)/2]$.

(i) *A necessary and sufficient condition for* $g(x, 0) = \text{Min}_{z=0,1,\cdots,b}\, g(x, z)$ *is* $g(x, 0) \le g(x, 1)$.

(ii) *A necessary and sufficient condition for* $g(x, b) = \text{Min}_{z=0,1,\cdots,b}\, g(x, z)$ *is* $g(x, b) \le g(x, b-1)$.

PROOF. (i) Necessity is obvious. To prove sufficiency note that, from the proof of Lemma 2.2, $g(x, 0) \le g(x, 1)$ implies $z_0 < 1$. Hence the result. (ii) is proved similarly.

LEMMA 2.4. (i) *Let* $t \in (0, (k-2)/2]$. *Then* $g(t, 0) \le g(t, 1)$ *implies* $g(x, 0) \le g(x, 1)$, *for all* $x \in [t, (k-2)/2]$.

(ii) *Let* $t \in [0, (k-2)/2]$. *Then* $g(t, b) \le g(t, b-1)$ *implies* $g(x, b) \le g(x, b-1)$ *for all* $x \in [0, t]$.

PROOF. Note that, whenever $0 \le x \le (k-2)/2$ and $0 \le z \le b$, $c - p(bx + z) + (bx^2 + 2xz + z) > 0$ and $k(bx + z) - (bx^2 + 2xz + z) > 0$, unless $x = z = 0$. We first give a brief outline of a proof of (i).

Some algebraic computations show that $g(x, 0) \le g(x, 1)$ holds if and only if $f(x) \le 0$, where

$$f(x) = b(k - x)((k - 2x - 1)(v(k - 1) - x)^2 - ax^2(p - 2x - 1))$$
$$- v(k - 2x - 1)(p - 2x - 1)(k - 1 + (v - 2)x).$$

Observe that $f(x)$ is a polynomial in x of degree 4, with coefficient of x^4 negative. Moreover, $f((k-1)/2) < 0$, $f((p-1)/2) > 0$, $f(0) > 0$ and $f(t) \le 0$ by virtue of the condition $g(t, 0) \le g(t, 1)$, where $t \in (0, (k-2)/2]$. So $f(x)$ has one real root in each of the intervals $(-\infty, 0)$, $(0, t]$, $((k-1)/2, (p-1)/2)$ and $((p-1)/2, \infty)$. Hence $f(x) \le 0$, for each x in $[t, (k-2)/2]$, which establishes part (i) of the lemma.

Part (ii) is similarly proved. It can be shown that $g(x, b) \leq g(x, b-1)$ holds if and only if $f_1(x) \leq 0$, where

$$f_1(x) = b(k-(x+1))(a(x+1)^2(p-2x-1) - (k-2x-1) \cdot (v(k-1)-(x+1))^2) - v(k-2x-1)(p-2x-1)(k-1+(v-2)(x+1)).$$

$f_1(x)$ is a polynomial of degree 4 with a positive coefficient of x^4, $f_1(-1) < 0$, $f_1((k-1)/2) > 0$, $f_1((p-1)/2) < 0$ and $f_1(t) \leq 0$ since $g(t, b) \leq g(t, b-1)$. Hence $f_1(x) \leq 0$ for all x in $[0, t]$. This establishes Lemma 2.4.

THEOREM 2.5. (i) *Let $t \in [1, [k/2]-1]$ be an integer. Then*

$$g(t, 0) = \mathrm{Min}_{(x,z)\in\Lambda}\, g(x, z)$$

if and only if $$g(t, 0) \leq g(t, 1)$$

and $$g(t-1, b) \leq g(t-1, b-1).$$

(ii) $$g([k/2], 0) = \mathrm{Min}_{(x,z)\in\Lambda}\, g(x, z)$$

if and only if $g([k/2]-1, b) \leq g([k/2]-1, b-1)$.

PROOF. We outline a proof of (i) only; (ii) can be similarly proved. Since $g(x+1, 0) = g(x, b)$, necessity is obvious. For sufficiency, observe that $g(t, 0) \leq g(t, 1)$ implies, by Lemma 2.3(i),

$$g(t, 0) \leq \mathrm{Min}_{z=1,\cdots,b}\, g(t, z) < g(t, b) \leq g(t+1, 0) \leq g(t+1, 1)$$

where Lemma 2.4(i) is used for the last inequality. Continuing this chain of arguments, $g(t, 0)$ is seen to be the minimum of $g(x, z)$ over points (x, z), $x = t, \cdots, [k/2]-1$; $z = 0, 1, \cdots, b$. Similarly using the condition $g(t-1, b) \leq g(t-1, b-1)$, Lemma 2.3(ii) and Lemma 2.4(ii), $g(t, 0)$ is the minimum of $g(x, z)$ over points (x, z), $x = 0, 1, \cdots, t-1$; $z = 0, 1, \cdots, b$. Hence the theorem.

The experimental design version of Theorem 2.5 is given in Theorem 2.6, where we write out expressions $f(t) \leq 0$ and $f_1(t-1) \leq 0$ in condition (2.3) and (2.4) respectively.

THEOREM 2.6. *Let $t \in [1, [k/2]]$ be an integer. An ABIB $(v, b, k-t; t)$ is A-optimal if v, b, k and t satisfy*

$$b(k-t)((k-2t-1)(v(k-1)-t)^2 - at^2(p-2t-1)) \leq v(k-2t-1)(p-2t-1)(k-1+(v-2)t) \tag{2.3}$$

and

$$b(k-t)(at^2(p-2t+1) - (k-2t+1)[v(k-1)-t]^2 \leq v(k-2t+1)(p-2t+1)(k-1+(v-2)t). \tag{2.4}$$

When $t = [k/2]$, only condition (2.4) has to be satisfied.

REMARK. In Theorem 2.6 we have extracted all the ABIB designs which can be proved to be A-optimal if we limit outselves to the technique of Theorem 2.1.

Before studying the special cases $t = 1$ and $t = 2$, we dispose of the other extreme, $t = [k/2]$. When $k = 3$, we shall see in Example 3.2(i) that an ABIB (5, 10, 2; 1) is optimal in $\mathscr{D}(5, 10, 3)$. However, when k is even the technique of Theorem 2.1 cannot be used to determine whether or not an ABIB $(v, b, k/2; k/2)$ is A-optimal. To establish this we can use Fisher's inequality $b \geq v$ of BIB $(v, b, k/2)$ to show that $f_1(k/2 - 1) > 0$, where $f_1(x)$ is defined in the proof of Lemma 2.3. So, condition (2.4) is violated.

We may note that Theorem 2.6 can be used to expand the catalog of A-optimal designs given in Hedayat and Majumdar (1984).

3. Optimality of ABIB (v, b, $k - 1$; 1).

Some statistical aspects of BIB designs whose blocks are augmented by a single replication of a control have been studied by several authors including Pesěk (1974) and Constantine (1983). In this section we shall identify cases for which we can guarantee the A-optimality of an ABIB $(v, b, k - 1; 1)$. Putting $t = 1$ in Theorem 2.6 and simplifying, we may conclude that an ABIB $(v, b, k - 1; 1)$ is A-optimal if

$$\begin{aligned} b(k-1)((k-2)(k-3)(vk-2) - (k-1)(v-1)^2) \\ \leq (k-3)(v+k-3)(v(k-1)+k-3) \end{aligned} \tag{3.1}$$

and

$$b((v+1)(v-3+2k) - vk^2) \leq (v+1)(v+k-3). \tag{3.2}$$

Clearly, an ABIB $(v, b, k - 1; 1)$ is A-optimal in $\mathscr{D}(v, b, k)$ if its parameters v and k make the left sides of inequalities (3.1) and (3.2) nonpositive. This happens whenever

$$(k-2)(k-3)(vk-2) \leq (k-1)(v-1)^2 \tag{3.3}$$

$$(v+1)(v-3+2k) \leq vk^2. \tag{3.4}$$

In the next theorem, we give a simple expression equivalent to (3.3) and (3.4).

THEOREM 3.1. *Positive integers v, k satisfy inequalities* (2.2), (3.3) *and* (3.4) *if and only if they satisfy* (2.2) *and*

$$(k-2)^2 + 1 \leq v \leq (k-1)^2. \tag{3.5}$$

Consequently, an ABIB $(v, b, k - 1; 1)$ is A-optimal in $\mathscr{D}(v, b, k)$ whenever v and k satisfy (3.5).

PROOF. When $k = 3$, the intersection of (2.2) and (3.5) is $v = 3, 4$. Moreover since $k = 3$, (3.3) is always satisfied, and it is not difficult to see that the intersection of (2.2) and (3.4) is $v = 3, 4$. Henceforth, we shall deal only with $k \geq 4$. Fix such a k arbitrarily and define

$$q_1(v) = v^2 - ((k-1)^2 + 1)v + (2k-3),$$

and

$$q_2(v) = (k-1)v^2 - (2(k-1) + k(k-2)(k-3))v$$
$$+ ((k-1) + 2(k-2)(k-3)).$$

Clearly, (3.4) is equivalent to $q_1(v) \le 0$, and (3.3) is equivalent to $q_2(v) \ge 0$. Moreover, it may be verified that $q_1(v)$ and $q_2(v)$, for each fixed k, have the following properties:

$$q_1((k-1)^2) < 0 \tag{3.6}$$

$$q_1((k-1)^2 + 1) > 0 \tag{3.7}$$

$$q_1((k-2)^2 + 1) < 0 \tag{3.8}$$

$$q_2((k-2)^2 + 1) > 0 \tag{3.9}$$

$$q_2((k-2)^2) < 0 \tag{3.10}$$

$$q_2(1) < 0. \tag{3.11}$$

Suppose, v_1, v_2 are the real roots of $q_1(v) = 0$ and v_3, v_4 are the real roots of $q_2(v) = 0$ (all v_i's exist). Since q_1 and q_2 are convex in v (we have fixed k arbitrarily), $q_1(v) \le 0$ if and only if $v \in [v_1, v_2]$ and $q_2(v) \ge 0$ if and only if $v \in (-\infty, v_3]$ or $v \in [v_4, \infty)$. To prove the theorem, we have to identify the integers v in the set

$$S = [v_1, v_2] \cap (-\infty, v_3] \cap [v_4, \infty) \cap \{\text{positive integers}\}.$$

But, $(k-1)^2 < v_2 < (k-1)^2 + 1$, from (3.6) and (3.7), and the integer $(k-2)^2 + 1 \in [v_1, v_2]$. On the other hand, (3.9) and (3.10) imply $(k-2)^2 < v_4 < (k-2)^2 + 1$, and (3.11) implies $v_3 < 1$. Clearly

$$S = [(k-2)^2 + 1, (k-1)^2] \cap \{\text{positive integers}\}.$$

This completes the proof of Theorem 3.1.

For each v, k satisfying (3.5), there are plenty of integers b for which ABIB $(v, b, k-1; 1)$ exists. We thus get many families of optimal designs. In particular, let $v = v_k = (k-2)^2 + (k-2) + 1$. v_k and k satisfy (3.5), and choosing $b = v_k$, a BIB $(v_k, v_k, k-1)$ may be constructed from a finite projective geometry. Clearly every ABIB $(v_k, v_k, k-1; 1)$ obtained from this BIB design is A-optimal. Similarly, if $v = v_k' = (k-1)^2$ and $b = (k-1)^2 + (k-1)$ then a BIB $(v_k', v_k' + k - 1, k-1)$ may be constructed from an Euclidean geometry. The corresponding ABIB $(v_k', v_k' + k - 1, k-1; 1)$ is A-optimal. Unions of copies of such designs are also optimal.

EXAMPLES 3.1. (i) $k = 3$. Any ABIB $(3, 3m, 2; 1)$ and any ABIB $(4, 6m, 2; 1)$ is A-optimal, $m = 1, 2, 3, \cdots$.

(ii) $k = 4$. Any ABIB $(v, b, 3; 1)$ with $5 \le v \le 9$ is A-optimal. ABIB $(7, 7, 3; 1)$ can be constructed from a finite projective geometry. ABIB $(9, 12, 3; 1)$ can be constructed from an Euclidean geometry.

Let us summarize our findings. If we denote by $\mathscr{S}$ the set of all (v, b, k) which satisfy (3.1) and (3.2) then we know that any ABIB $(v, b, k-1; 1)$ is A-optimal as long as its parameters (v, b, k) are in $\mathscr{S}$. Condition (3.5) picked up a good chunk of $\mathscr{S}$. Denote this part by $\mathscr{S}_1$. A question of interest is this: How many more (v, b, k) are in $\mathscr{S}$ which are not in $\mathscr{S}_1$? Surprisingly, not much. The precise answer is this:

COROLLARY 3.2. *Consider those ABIB $(v, b, k-1; 1)$ for which (v, b, k) is in $\mathscr{S}$ but not in $\mathscr{S}_1$. Then these (v, b, k) have to be of the form either*

$$(v, b, k) = (5, b, 3)$$

or

$$= ((k-2)^2, b, k), \quad k \geq 4$$

or

$$= ((k-1)^2 + 1, b, k), \quad k \geq 4.$$

PROOF. First suppose that (v, b, k) satisfies (3.1), (3.2) but $v \in [k, (k-2)^2]$. Clearly $k \geq 4$, since $(k-2)^2 < k$ when $k = 3$. We shall use the functions $q_1(v)$ and $q_2(v)$ defined in the proof of Theorem 3.1. We may point out that any v in $[k, (k-2)^2]$ satisfies (3.2). But for such a v, $q_2(v) < 0$. Therefore (3.3) is violated, and using the Fisher's inequality $b \geq v$ for a BIB $(v, b, k-1)$, we obtain from condition (3.1) that $h_1(v) \geq 0$, where

$$\begin{aligned} h_1(v) = {} & (k-3)(v+k-3)(v(k-1)+k-3) \\ & - v(k-1)((k-2)(k-3)(vk-2) - (k-1)(v-1)^2). \end{aligned}$$

It can be shown that $h_1(v)$ is a cubic in v for each arbitrarily fixed k, with at most two positive roots. Moreover, $h_1(0) > 0$, $h_1(1) < 0$. Some computations show that (using the relation $(k-2)^2 - 1 = (k-1)(k-3)$),

$$\begin{aligned} h_1((k-2)^2 - 1) = {} & (k-3)(k(k-3)^2 - (k-1)^2((2k^2 - 6k + 6)(k-3) \\ & - (2k-1)(k-3)^2 - (k-1))) \\ < {} & (k-3)((k-1)^2(k-3) - (k-1)^2((2k^2 - 6k + 6)(k-3) \\ & - (2k-1)(k-3)^2 - (k-1))) < 0 \end{aligned}$$

for each $k \geq 4$, our region of interest here. Hence for each $v \in [1, (k-2)^2 - 1]$, $h_1(v) < 0$, or in other words, (3.1) is violated. Therefore, any v in $[k, (k-2)^2]$ satisyfing (3.1) and (3.2), can only be $(k-2)^2$. We shall give some examples of A-optimal ABIB $((k-2)^2, b, k-1; 1)$ in Examples 3.2.

Now consider a (v, b, k) satisfying (3.1) and (3.2), but $v > (k-1)^2$. The proof here is similar to the previous case. Any $v > (k-1)^2$ satisfies (3.1), but violates (3.4). Using Fisher's inequality, (3.2) yields $h_2(v) \leq 0$, where

$$h_2(v) = v\{(v+1)(v-3+2k-\ vk^2)\} - (v+1)(v+k-3).$$

If $k = 3$, the roots of $h_2(v) = 0$ are 0, .385 and 5.615, while $(k-1)^2 + 2 = 6$.

For any fixed $k \geq 4$, $h_2(0) > 0$, $h_2(1) < 0$ and $h_2((k-1)^2 + 2) > 0$. Since $h_2(v)$ is a cubic in v with a positive coefficient of v^3, we get $h_2(v) > 0$ for every $v \geq (k-1)^2 + 2$. This is true for each $k \geq 3$. Hence v can only be $(k-1)^2 + 1$. Optimal designs with such a v are given in Examples 3.2. This establishes Corollary 3.2.

EXAMPLES 3.2. (i) The only A-optimal ABIB $((k-1)^2 + 1, b, k-1; 1)$ we could find has parameters $k = 3$, $b = 10$; $v = (k-1)^2 + 1 = 5$.

(ii) Some examples of A-optimal ABIB $((k-2)^2, b, k-1; 1)$ are given in the following three series:

Series 1. $k = 4$, $v = 4$ and $b = 4, 8, 12, 16$ or 20.

Series 2. $k = 5$, $v = 9$ and $b = 18, 36, 54, 72$ or 90.

Series 3. $k = 6$, $v = 16$ and $b = 48, 96, 144, 192, 240$ or 288.

Finally, we give some examples of A-optimal ABIB $(v, b, k-2; 2)$. These are obtained from Theorem 2.6 by substituting $t = 2$ in (2.3) and (2.4).

EXAMPLES 3.3. (i) $k = 8$, $v = 8$. Any ABIB $(8, b, 6; 2)$ is A-optimal. For example ABIB $(8, 28, 6; 2)$.

(ii) $k = 8$, $v = 9$. Any ABIB $(9, b, 6; 2)$ is A-optimal. For example ABIB $(9, 12, 6; 2)$.

(iii) $k = 8$, $v = 10$. Theorem 2.6 yields no optimal ABIB $(10, b, 6; 2)$ since (2.4) gives $b \leq 12$.

(iv) $k = 9$, $v = 10$. Any ABIB $(10, b, 7; 2)$ is A-optimal. For example ABIB $(10, 120, 7; 2)$.

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