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VINCENTIZATION REVISITED

BY CHRISTIAN GENEST

Université Laval

Vincentization is a convenient method of aggregating a set of $n \geq 2$ probability distributions $F_1, \dots, F_n$ in such a way that their synthesis, $F = T(F_1, \dots, F_n)$, be of the same functional form as the F_i's when the latter are identical up to a location-scale transformation. A characterization of this combination rule is proposed and some of its consequences are outlined.

1. Introduction. Vincentization was originally conceived as a method for combining sets of independent repeated measurements made on individual subjects into a synthetic probability distribution F based on a small number of observations per subject cell. The procedure, which was described by Ratcliff (1979), is named after biologist S. B. Vincent (1912), who used something very similar to it for constructing learning curves at the beginning of this century. It basically consists of averaging $n \geq 2$ subjects' estimated or elicited quantile functions in order to define group quantiles from which F can be constructed. To cast it in its greatest generality, let $F_1, \dots, F_n$ represent arbitrary (empirical or theoretical) distribution functions and define their corresponding quantile functions by

$$F_i^{-1}(\alpha) = \inf\{t \in \mathscr{R}: F_i(t) \geq \alpha\}, \qquad 0 < \alpha \leq 1.$$

The *Vincent average* of the F_i's is then computed as

$$F^{-1}(\alpha) = \sum_{i=1}^{n} w_i F_i^{-1}(\alpha), \qquad 0 < \alpha \leq 1, \tag{1.1}$$

where $w_1, \dots, w_n$ are arbitrarily chosen nonnegative numbers summing up to 1.

This combination rule is an instance of what is known in the literature as a *pooling operator*, that is, a mapping T which extracts a synthetic distribution $F = T(F_1, \dots, F_n)$ from any set of probability measures, expressed here as cumulative distribution functions on $\mathscr{R}$, the real line. A distinctive and appealing feature of Vincent's procedure over other aggregation formulae surveyed by French (1985) and Genest and Zidek (1986) is its compliance with the following axiom.

SHAPE-PRESERVATION PROPERTY. A pooling operator T is said to be *shape-preserving* if cumulative distribution functions $F_1, \dots, F_n$ belonging to the

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same location-scale family are always merged by T into a synthetic probability distribution, $F = T(F_1, \dots, F_n)$, of the same functional form as the F_i's. More precisely, one must have

$$T(F_1, \dots, F_n)^{-1}(\alpha) = \mu^* + \sigma^* L^{-1}(\alpha), \qquad 0 < \alpha \leq 1,$$

for some $\mu^* \in \mathscr{R}$ and $\sigma^* > 0$ whenever there exist a cumulative distribution function L and sets of parameters $\mu_i \in \mathscr{R}$ and $\sigma_i > 0$ such that, for all $1 \leq i \leq n$,

$$F_i^{-1}(\alpha) = \mu_i + \sigma_i L^{-1}(\alpha), \qquad 0 < \alpha \leq 1.$$

The theoretical interest and practical value of shape-preserving pooling operators is manifest; their use may also be critical in certain kinds of applications. In experimental psychology, for example, researchers who have collected a relatively small number of observations on reaction time from a few subjects often wish to combine the data so as to test their hypotheses on a larger sample size. Unfortunately, if raw reaction times from various subjects were simply pooled, the resulting distribution would not necessarily reflect the shape of the individual ones. Similar difficulties might arise if classical aggregation rules, such as the *linear* or the *logarithmic opinion pool* discussed by French (1985) and Genest and Zidek (1986), were used to merge the subjects' personal reaction time distributions. It is well known, for instance, that a weighted arithmetic mean of Cauchy or logistic densities will not itself be Cauchy or logistic unless all but one of the weights are equal to zero. A similar comment applies to the weighted geometric averaging procedure, although it has been reported by Gilardoni (1989) to inherit the structure of density functions belonging to the same exponential family. In contrast, shape-preserving pooling operators are designed to yield synthetic distributions that retain information about the functional form of their constituents when the latter are identical up to a location-scale transformation. In this manner, it is thus possible to apply meaningfully, and with the increased power provided by the accumulation of observations, tests of change in, say, the location, scale or skewness of a reaction time distribution under various experimental conditions. These points, as well as a number of practical considerations surrounding the implementation of Vincentization, are covered by Thomas and Ross (1980).

The purpose of the present note is to suggest a characterization of Vincent's method within the class of shape-preserving pooling operators. It will be shown, in Section 2, that up to a scale factor $\sigma > 0$, formulae displayed in (1.1) are the only ones which possess that property among those that can be expressed in the form

$$F^{-1}(\alpha) = H\{F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)\}, \qquad 0 < \alpha \leq 1, \tag{1.2}$$

for some arbitrary function $H: \mathscr{R}^n \to \mathscr{R}$. This regularity condition is akin to the *locality assumption* of Genest (1984a) and the *strong setwise function property* of McConway (1981), interpreted by Wagner (1982) as an independence of irrelevant alternatives hypothesis. Its meaning will be briefly examined in Section 3, along with the implications of a somewhat weaker version of (1.2) wherein the function H itself could vary with α.

2. Characterization. The following result provides a characterization of Vincent's averaging procedure, up to a multiplicative factor.

PROPOSITION 2.1. *Let Δ be the class of cumulative distribution functions on $\mathscr{R}$ and let $T\colon \Delta^n \to \Delta$ be a shape-preserving pooling operator. If there exists a mapping $H\colon \mathscr{R}^n \to \mathscr{R}$ such that*

$$T(F_1,\dots,F_n)^{-1}(\alpha) = H\{F_1^{-1}(\alpha),\dots,F_n^{-1}(\alpha)\}, \qquad 0 < \alpha \le 1,$$

is a quantile function for all $F_1,\dots,F_n$ in Δ, then there must also exist a scalar $\sigma > 0$ and nonnegative numbers $w_1,\dots,w_n$ adding up to 1 for which the identity

$$T(F_1,\dots,F_n)^{-1}(\alpha) = \sigma \sum_{i=1}^{n} w_i F_i^{-1}(\alpha) \tag{2.1}$$

holds true for all $F_1,\dots,F_n \in \Delta$ and all $0 < \alpha \le 1$.

PROOF. Let L denote a continuous, strictly increasing cumulative distribution function on $\mathscr{R}$ and consider quantile functions defined for $1 \le i \le n$ by

$$F_i^{-1}(\alpha) = \mu_i + \sigma_i L^{-1}(\alpha), \qquad 0 < \alpha \le 1,$$

with arbitrary location and scale parameters $\mu_i \in \mathscr{R}$ and $\sigma_i > 0$, respectively. Under the hypotheses of the proposition, one must have

$$H\{\mu_1 + \sigma_1 L^{-1}(\alpha),\dots,\mu_n + \sigma_n L^{-1}(\alpha)\} = \mu^* + \sigma^* L^{-1}(\alpha),\ 0 < \alpha \le 1, \tag{2.2}$$

for some $\mu^* \in \mathscr{R}$ and $\sigma^* > 0$ whose values may depend on the choice of the μ_i's and the σ_i's but not on the value of $L^{-1}(\alpha)$. When the latter vanishes, it is found that $\mu^* = H(\mu_1,\dots,\mu_n)$; likewise, setting $L^{-1}(\alpha) = 1$ yields

$$\sigma^* = H(\mu_1 + \sigma_1,\dots,\mu_n + \sigma_n) - H(\mu_1,\dots,\mu_n).$$

Substituting in (2.2) and switching to vector notation for convenience, it can be seen at once that

$$H(\mu + \lambda\sigma) = H(\mu) + \lambda\{H(\mu + \sigma) - H(\mu)\},$$

where $\lambda = L^{-1}(\alpha)$ varies freely in $\mathscr{R}$ and, by convention, addition is performed componentwise. Setting $\lambda = 1/2$ and $\nu = \mu + \sigma$, it follows that H satisfies Jensen's functional equation on a restricted domain, namely,

$$\{H(\mu) + H(\nu)\}/2 = H\{(\mu + \nu)/2\},$$

with $\mu, \nu \in \mathscr{R}^n$ and $\nu_i > \mu_i$, $1 \le i \le n$. In view of Theorem 1 in Radó and Baker (1987), it is then plain that H obeys Cauchy's functional equation

$$H(\mu) + H(\nu) = H(\mu + \nu), \qquad \mu, \nu \in \mathscr{R}^n,$$

on its entire domain. Furthermore, H must be nondecreasing in each of its coordinates, since $F^{-1}(\alpha) = H\{F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)\}$ is required to be a quantile function for all choices of $F_1, \dots, F_n$. This allows one to conclude that

$$H(\mu_1, \dots, \mu_n) = \sigma \sum_{i=1}^{n} w_i \mu_i$$

for some $\sigma > 0$ and a collection of nonnegative weights $w_1, \dots, w_n$ satisfying $\sum_{i=1}^{n} w_i = 1$. □

It should be observed that the above result is in fact stronger than stated. Indeed, looking at the proof of Proposition 2.1, Vincent's averaging procedure is seen to obtain, up to a scale factor $\sigma > 0$, as soon as the pooling operator considered is taken to be local in the sense of (1.2) and shape-preserving for a *single* location-scale family with continuous, strictly increasing cumulative distribution function L. However, in circumstances such as described by Ratcliff (1979), where the synthetic probability distribution should be thought of as representing the behavior of an average subject, the presence of the multiplicative factor σ in formula (2.1) might be perceived as undesirable. The situation can then be remedied easily enough, by imposing an additional assumption on the average quantile function. One could demand, for example, that the inequality

$$\min\{F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)\} \le F^{-1}(\alpha) \le \max\{F_1^{-1}(\alpha), \dots, F_n^{-1}(\alpha)\}$$

be checked for all $0 < \alpha \le 1$, which is a basic expectation on any notion of a mean. Alternatively, one could require a form of *unanimity preservation*, namely,

$$F_1^{-1}(\alpha) = \cdots = F_n^{-1}(\alpha) = t \Rightarrow F^{-1}(\alpha) = t.$$

3. Extension and discussion. It is clear that a characterization of Vincent's averaging procedure could not be achieved without some kind of regularity condition such as (1.2). For, the shape-preservation property formulated in the introduction does not restrict the behavior of an abstract pooling operator beyond location-scale families. Naturally, the regularity condition used here is verified by many combination rules besides (1.1), including the *generalized Vincentizing method* of Thomas and Ross (1980), in which

$$F^{-1}(\alpha) = \varphi^{-1}\left[\sum_{i=1}^{n} w_i \varphi\{F_i^{-1}(\alpha)\}\right], \qquad 0 < \alpha \le 1,$$

for some strictly monotone, continuous mapping φ. In that sense, assumption (1.2) is much weaker than the locality conditions of the kind used, *inter alia*, by McConway (1981), Wagner (1982) and Genest (1984a). In the latter paper, for instance, it had been noted that a combination rule T essentially reduced to a linear opinion pool, namely,

$$T(f_1, \dots, f_n) = \sum_{i=1}^{n} w_i f_i, \qquad \mu\text{-a.e.},$$

if a measurable function $h\colon \mathscr{R}^n \to \mathscr{R}$ could be found such that

$$T(f_1,\dots,f_n)(\theta) = h\{f_1(\theta),\dots,f_n(\theta)\}, \qquad \mu\text{-a.e.}$$

for all $f_1,\dots,f_n$ and $T(f_1,\dots,f_n)$, *density functions* with respect to some dominating measure μ on an abstract space $(\Theta, \mathscr{B})$.

What Vincentization shares with other pooling recipes satisfying condition (1.2) is the assumption that *all* group quantiles should be computed in the same way, that is, from the values of the $F_i^{-1}(\alpha)$'s alone, irrespective of the identity of $0 < \alpha \le 1$ itself. This, of course, is a very strong requirement. For, in addition to the fact that individual probability distributions may be based on varying amounts of knowledge and/or data, it is generally admitted that certain quantiles are more difficult to assess than others. While the former is recognized by Vincent's procedure through the introduction of weights w_i, the latter is not. In an attempt to escape this limitation, one might consider relaxing condition (1.2) by assuming that

$$F^{-1}(\alpha) = H_\alpha\{F_1^{-1}(\alpha),\dots,F_n^{-1}(\alpha)\}, \qquad 0 < \alpha \le 1,$$

for some family H_α of functions from $\mathscr{R}^n$ to $\mathscr{R}$. The introduction of such quantile dependent aggregation rules is much in the spirit of the *weak setwise function property* of McConway (1981), also considered by Aczél, Ng and Wagner (1984) and Genest (1984b). Unfortunately, a direct application of the shape-preservation property with different choices of L readily implies that $H_\alpha = H_\beta$ for all $0 < \alpha, \beta \le 1$, so that Vincent averages are again found to be the only shape-preserving pooling operators of this sort. In particular, any recourse to quantile dependent weights $w_i(\alpha)$, to account for biases in the estimation of extreme quantiles, is strictly ruled out. Consequently, if the elicitation or estimation of latent quantiles is suspected to suffer biases, the subjects' individual distributions should most certainly be corrected *before* a Vincentized average is computed. In such cases, some might prefer to abandon all axiomatic considerations based on shape-preservation or any other requirement and resolve to develop a full Bayesian model, perhaps along the lines sketched by Good (1979) or West (1988).

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DÉPARTEMENT DE MATHÉMATIQUES
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UNIVERSITÉ LAVAL
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