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OPTIMAL DESIGNS FOR COMPARING TEST TREATMENTS WITH A CONTROL UTILIZING PRIOR INFORMATION[1]

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Bayes A-optimal designs for the one-way and the two-way elimination of heterogeneity models and optimal Γ-minimax designs for the one-way elimination of heterogeneity model for experiments to compare test treatments with a control are given. Properties such as robustness, of these designs are studied. Optimality results are derived in the *exact theory* setup.

1. Introduction. We consider the problem of comparing a set of v test treatments with a control. Such a problem arises, for example, when a set of new treatments is to be compared with a standard treatment, or one that has been in use for some time. Our objective is to obtain optimal and highly efficient experimental designs utilizing prior information. A major portion of this article is devoted to the Bayesian approach. Γ-minimax strategies are also investigated. Two models are considered: the one-way elimination of heterogeneity model, which is appropriate for block designs, and the two-way elimination of heterogeneity model, which is appropriate for row–column designs.

Owen (1970) was the first to derive Bayes optimal designs for comparing test treatments with a control. He considered the one-way elimination of heterogeneity model. Giovagnoli and Verdinelli (1983, 1985) and Verdinelli (1983) have also contributed significantly to this area. Smith and Verdinelli (1980) investigated Bayes optimal designs for the zero-way elimination of heterogeneity model and Toman and Notz (1991) investigated Bayes optimal designs for the two-way elimination of heterogeneity model.

All of these approaches are in the setup of *approximate design theory*, wherein the discrete optimization problem involved in finding an optimal design is replaced by a continuous version. For example, in the block design setup the nonnegative integers n_{ij} (= number of times treatment i occurs in block j) are allowed to be real numbers. This is done primarily for mathematical tractability. Researchers using this approach have produced strong theorems applicable to large classes of prior distributions.

A characteristic of optimal approximate designs is that these designs may be applicable only after "rounding off" of some quantities to nearby integers, thereby introducing inaccuracies. Moreover, the available theory cannot be

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applied to some types of experiments. For instance, all known optimal block designs satisfy

$$n_{1j} = \cdots = n_{vj} \quad \text{for each block } j.$$

Hence they are not directly applicable to situations where the block size is less than the number of treatments, that is, *incomplete block* experiments, which are often encountered in practice, and consequently constitute a major area of interest for researchers in design theory.

In *exact design theory*, on the other hand, the integer variables are not extended to include real numbers in their domain. The optimization problem is more involved here. It is extremely unlikely that one method which solves the problem for all priors can be developed. In this article we derive exact optimal designs for subclasses of priors which are rich enough to be quite widely applicable.

Exact Bayes optimal block designs for a very special type of prior were given in Majumdar (1988). We give more details on this in Section 3. Recently, Stufken (1991) has identified some elegant families of Bayes optimal designs for this class of priors.

Our study of the properties of exact Bayes optimal designs has produced strong evidence that these designs are quite robust against certain departures from the specified prior distribution. Robustness of designs over priors is certainly a very desirable property. DasGupta and Studden (1991) have investigated robust optimal designs—though not specifically for control–treatment comparisons. The reader may also refer to Chaloner (1984) for general results on Bayes optimal designs and to DasGupta and Studden's paper for more references on Bayes optimal designs.

There is considerable literature on optimal designs for comparing test treatments with a control from a frequentist viewpoint—a brief survey is given in Hedayat, Jacroux and Majumdar (1988).

Notation, definitions and criteria for optimality are established in Section 2. Section 3 gives optimal block designs, including Bayes optimal designs and optimal Γ-minimax designs. Robustness and other properties of these designs are investigated in Section 4. In Section 5 we give Bayes optimal row–column designs. Some concluding remarks are given in Section 6.

2. Preliminaries. We start by giving Owen's (1970) formulation of the optimal design problem for treatment–control comparisons from a Bayesian viewpoint. Let $\mathscr{D}$ be the set of all designs under consideration. For a design d in $\mathscr{D}$, let Y be the vector of observations and suppose the model is

$$Y = X_{1d}\theta + X_2\gamma + \varepsilon, \tag{2.1}$$

where θ is the vector of parameters that we wish to estimate, γ is the vector of nuisance parameters and ε is a random vector of errors. X_2 is a known matrix and X_{1d} is a known matrix which depends on the design. Suppose

$$Y|\theta, \gamma \sim N(X_{1d}\theta + X_{2\gamma}, E),$$

while the prior is given by

$$\begin{pmatrix}\theta\\ \gamma\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_\theta\\ \mu_\gamma\end{pmatrix}, \begin{pmatrix}B^* & 0\\ 0 & B\end{pmatrix}\right).$$

The posterior distribution of θ is

$$\theta | Y \sim N(M_d, D_d),$$

where

$$D_d^{-1} = X_{1d}'(E + X_2BX_2')^{-1}X_{1d} + B^{*-1} \tag{2.2}$$

and

$$D_d^{-1}M_d = X_{1d}'(E + X_2BX_2')^{-1}(Y - X_2\mu_\gamma) + B^{*-1}\mu_\theta.$$

Under squared error loss $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)'(\hat{\theta} - \theta)$, the Bayes estimator of θ is $\hat{\theta} = M_d$ with expected loss $\operatorname{tr} D_d$. Owen's optimal design d^* is given by

$$\operatorname{tr} D_{d^*} = \min_{d \in \mathscr{D}} \operatorname{tr} D_d. \tag{2.3}$$

d^* is called a Bayes A-optimal design, since (2.3) is a natural extension of the definition of A-optimality to the Bayesian setup.

Next, we introduce some notation. $\mathscr{D}_1(v+1, b, k)$ denotes the set of all (exact) designs in b blocks of size k each, based on $v + 1$ treatments. $\mathscr{D}_2(v + 1, c, r)$ denotes the set of all (exact) designs in r rows and c columns, one observation per cell, based on $v + 1$ treatments. The treatments are labeled $0, 1, \ldots, v$, with 0 as the control. For $d \in \mathscr{D}_1(v + 1, b, k)$, let

$$n_{dij} = \text{number of occurrences of treatment } i \text{ in block } j,$$

$$r_{di} = \sum_{j=1}^{b} n_{dij}, \quad \text{the replication of treatment } i,$$

and for $i \neq i'$,

$$\lambda_{dii'} = \sum_{j=1}^{b} n_{dij} n_{di'j}.$$

BIB(v, b, r, k, λ) denotes, as usual, a balanced incomplete block design, that is, a design in $\mathscr{D}(v, b, k)$ with

$$r_{di} = r \quad \text{and} \quad \lambda_{dii'} = \lambda \quad \text{for all } i \neq i'.$$

Pearce (1960) and Bechhofer and Tamhane (1981) noted that a special class of designs—called designs with supplemented balance by Pearce and balanced treatment incomplete block (BTIB) designs by Bechhofer and Tamhane—possesses desirable properties for control–treatment comparisons. These designs can be defined as follows.

Definition 2.1. A design d in $\mathscr{D}_1(v + 1, b, k)$ is a BTIB design if

$$\lambda_{d01} = \cdots = \lambda_{d0v},$$
$$\lambda_{d12} = \cdots = \lambda_{dv-1,v}.$$

The Fisher information matrix for control–test treatment contrasts is completely symmetric if the design is a BTIB design [see Bechhofer and Tamhane (1981)]. A particular type of BTIB designs given below is important for us.

DEFINITION 2.2. A design d in $\mathscr{D}_1(v+1,b,k)$ is a BTIB$(v,b,k;t,s)$ if d is a BTIB design with the property:

$$n_{dij}\in\{0,1\},\qquad i=1,\dots,v,\ j=1,\dots,b,$$
$$n_{d01}=\cdots=n_{d0s}=t,$$
$$n_{d0s+1}=\cdots=n_{d0b}=t+1,\quad \text{respectively}.$$

The notation BTIB$(v,b,k;t,s)$ is due to Stufken (1987).

Several authors have shown that certain BTIB$(v,b,k;t,s)$ designs are highly efficient, and indeed optimal according to criteria that do not involve prior knowledge. Construction of these designs has been studied too, though not as extensively. The reader is referred to Hedayat, Jacroux and Majumdar (1988) for further information and references.

3. Optimal block designs. Suppose the experimental units are grouped into b blocks of k homogeneous units each, on which the v test treatments are to be compared with a control. If treatment i is applied to plot p of block j, then the model for the observation y_{ijp} is

$$y_{ijp}=\mu+\tau_i+\beta_j+\varepsilon_{ijp},$$

where μ is a general effect, τ a treatment effect, β a block effect and ε a random error. Let

$$\theta_i=\tau_i-\tau_0,\qquad i=0,1,\dots,v,$$
$$\gamma_j=\mu+\tau_0+\beta_j,\qquad j=1,\dots,b.$$

The parameters $\theta_1,\dots,\theta_v$ measure the performances of the new treatments relative to the standard treatment, and the parameters $\gamma_1,\dots,\gamma_b$ measure the performances of the standard treatment in the b blocks.

Then the model can be written as

$$y_{ijp}=\theta_i+\gamma_j+\varepsilon_{ijp}.$$

Let $Y=(\dots,y_{i1p},\dots,\dots,y_{i2p},\dots,\dots,\dots,y_{ibp},\dots)$ be the $bk\times 1$ vector of observations written block by block. Recall that $\mathscr{D}_1(v+1,b,k)$ denotes the set of block designs. For $d\in\mathscr{D}_1(v+1,b,k)$, let X_{1d} be a $bk\times v$ matrix which has a 1 in cell (l,i) if the observation number l receives treatment number i, $i=1,\dots,v$, or 0 otherwise. Let

$$X_2=\begin{pmatrix}1_k & 0 & \cdots & 0\\ 0 & 1_k & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1_k\end{pmatrix},$$

a $bk \times b$ matrix, where 1_k is a $k \times 1$ vector of 1's. In matrix notation the model has the same form as (2.1), viz.,

$$Y_d = X_{1d}\theta + X_2\gamma + \varepsilon.$$

In subsequent discussions we assume the following properties for the distributions

$$(3.1)\qquad \begin{aligned} &\mathrm{Var}(\varepsilon_{ijp}) = \sigma^2, \quad \mathrm{Cov}(\varepsilon_{ijp}, \varepsilon_{i'jp'}) = \sigma^2\pi_1 \quad \text{for } p \neq p', \\ &\qquad \text{and} \quad \mathrm{Cov}(\varepsilon_{ijp}, \varepsilon_{i'j'p'}) = \sigma^2\pi_2 \quad \text{for } j \neq j', \end{aligned}$$

$$(3.2)\qquad \mathrm{Var}(\gamma_j) = \sigma^2\delta \quad \text{and} \quad \mathrm{Cov}(\gamma_j, \gamma_{j'}) = \sigma^2\delta\rho \quad \text{for } j \neq j',$$

$$(3.3)\qquad \begin{aligned} &\mathrm{Var}(\theta_i) = \sigma^2\xi_1 \quad \text{and} \quad \mathrm{Cov}(\theta_i, \theta_{i'}) = \sigma^2\xi_2 \\ &\qquad\qquad \text{for } i \neq i',\ i, i' \in \{1, 2, \dots, v\}. \end{aligned}$$

We are, in effect, working in a setup that has exchangeability-type conditions on the random variables. Note that for the designing problem we do not need to impose any restrictions on the prior expectations.

The specialization of the model given by (3.1)–(3.3) is rich enough to accommodate many realistic priors. Consider, for instance, the situation where μ, τ_i's and β_j's are all identifiable and independent, $\{\tau_1, \dots, \tau_v\}$ are i.i.d. and $\{\beta_1, \dots, \beta_b\}$ are i.i.d. Here (3.2) and (3.3) hold with $\rho > 0$ and $\xi_2 > 0$. Relaxing conditions (3.1)–(3.3) is not an easy task; this will be attempted in the future.

It is not very difficult to show that, under (3.1) and (3.2) and the regularity conditions,

$$(3.4)\qquad \begin{aligned} &\pi_1 < 1, \quad \delta(1-\rho) + (\pi_1 - \pi_2) \neq 0, \\ &1 - \pi_1 + k(\pi_1 - \pi_2) + k\delta(1-\rho) + bk(\pi_2 + \delta\rho) \neq 0, \end{aligned}$$

the inverse of the posterior covariance matrix D_d^{-1} given by expression (2.2) reduces to

$$(3.5)\qquad \begin{aligned} \sigma^2(1-\pi_1)D_d^{-1} = {}& \mathrm{Diag}(r_{d1}, \dots, r_{dv}) - (k+\alpha)^{-1}\overline{N}_d\overline{N}_d' \\ & - \eta r_d r_d' + \sigma^2(1-\pi_1)B^{*-1}, \end{aligned}$$

where

$$\alpha = (1-\pi_1)[\delta(1-\rho) + (\pi_1 - \pi_2)]^{-1},$$

$$\begin{aligned} \eta = {}& (1-\pi_1)(\pi_2 + \delta\rho)[1 - \pi_1 + k(\pi_1 - \pi_2) + k\delta(1-\rho)]^{-1} \\ & \times [1 - \pi_1 + k(\pi_1 - \pi_2) + k\delta(1-\rho) + bk(\pi_2 + \delta\rho)]^{-1}, \end{aligned}$$

$$\overline{N}_d = (n_{dij})_{i=1,\dots,v,\ j=1,\dots,b}, \quad \text{a } v \times b \text{ matrix,}$$

which is the usual treatment–block incidence matrix without the row corresponding to the control, and

$$r_d = (r_{d1}, \dots, r_{dv})'.$$

REMARK. In the special case where the errors are homoscedastic, $\pi_1 = \pi_2 = 0$, the expression (3.5) reduces to

$$\sigma^2 D_d^{-1} = \text{Diag}(r_{d1}, \dots, r_{dv}) - \left(k + \delta^{-1}(1-\rho)^{-1}\right)^{-1} \overline{N}_d \overline{N}_d'$$
$$- \delta\rho(1 + k\delta(1-\rho))^{-1}(1 + k\delta(1-\rho) + bk\delta\rho)^{-1} r_d r_d' + \sigma^2 B^{*-1}.$$

In addition, if $\rho = 0$,

$$\sigma^2 D_d^{-1} = \text{Diag}(r_{d1}, \dots, r_{dv}) - (k + \delta^{-1})^{-1} \overline{N}_d \overline{N}_d' + \sigma^2 B^{*-1}.$$

If the prior information on θ is vague, $B^{**-1} = O$,

$$\sigma^2 D_d^{-1} = \text{Diag}(r_{d1}, \dots, r_{dv}) - (k + \delta^{-1})^{-1} \overline{N}_d \overline{N}_d'.$$

When the prior information on γ is also vague, $\delta = \infty$,

$$\sigma^2 D_d^{-1} = \text{Diag}(r_{d1}, \dots, r_{dv}) - k^{-1} \overline{N}_d \overline{N}_d',$$

which coincides with the Fisher information matrix for the treatment–control contrasts. The optimality criteria used in Majumdar and Notz (1983) was based on this information matrix.

We assume that

$$2 \le k \le v. \tag{3.6}$$

Thus we are in the *incomplete block* setup. To state the first theorem of this section, we need some more notation. Let

$$e_1 = (1 - \pi_1)(\xi_1 - \xi_2)^{-1}, \qquad e_2 = (1 - \pi_1)(\xi_1 + (v-1)\xi_2)^{-1}.$$

For nonnegative integers q and z, let $T = bq - z$, $S = bq^2 - 2qz + z$,

$$g(q,z) = v(v-1)^2(k+\alpha)\big[((v-1)(k+\alpha) - v)T + S$$
$$+ v(v-1)(k+\alpha)e_1\big]^{-1}$$
$$+ v(k+\alpha)\big[(k+\alpha)T - \eta(k+\alpha)T^2 - S + v(k+\alpha)e_2\big]^{-1},$$

$$\Lambda = \left\{(q,z)\colon q = \text{int}\left[\tfrac{1}{2}(k+\alpha+1)/(b\eta(k+\alpha)+1)\right] + 1, \dots, k;\right.$$
$$\left. z = 0, 1, \dots, b\right\},$$

where int[·] is the greatest integer function. The function $g(q,z)$ equals $\sigma^{-2}(1-\pi_1)^{-1} \operatorname{tr} D_d$ and the quantities T and S equal T_d and S_d (to be defined later) respectively, when d is a BTIB$(v, b, k; k-q, z)$.

THEOREM 3.1. *Suppose the conditions* (3.1), (3.2), (3.3) *and* (3.6) *hold and further*

$$1 > \pi_1 \ge \pi_2, \qquad \pi_2 + \delta\rho \ge 0, \qquad \xi_2 \ge 0. \tag{3.7}$$

Case 1. $\frac{1}{2}(k + \alpha + 1)/(b\eta(k + \alpha) + 1) < k$.

Let $g(q_0, z_0) = \min\{g(q, z)\colon (q, z) \in \Lambda\}$. *Then a* BTIB$(v, b, k; k - q_0, z_0)$ *is Bayes A-optimal in* $\mathscr{D}_1(v + 1, b, k)$.

Case 2. $\frac{1}{2}(k + \alpha + 1)/(b\eta(k + \alpha) + 1) \geq k$.

Then a BIB(v, b, r, k, λ) *in the test treatments is Bayes A-optimal in* $\mathscr{D}_1(v + 1, b, k)$.

REMARK 1. (3.7) is a technical condition which we need to prove the theorem. It is not clear to us whether the results hold even when (3.7) is false. Since the matrix B is positive definite, $\rho < 1$; hence (3.7) ensures (3.4). The condition $\pi_2 + \delta\rho \geq 0$ means that $(\gamma_{ij} + \varepsilon_{ijp})$'s are positively correlated and $\xi_2 \geq 0$ means that θ_i's are positively correlated. The condition (3.7) holds, for instance, when $\{\tau_1, \dots, \tau_v\}$ are i.i.d. and $\{\beta_1, \dots, \beta_b\}$ are i.i.d., μ, τ_i's and β_j's are independent and $1 > \pi_1 \geq \pi_2 \geq 0$.

REMARK 2. In case $\pi_2 + \delta\rho = 0$, the condition for Case 2, viz., $k + \alpha + 1 \geq 2kb\eta(k + \alpha) + 2k$ holds if and only if

$$\delta \leq (1 - k\pi_1)(k - 1)^{-1}.$$

In case $\pi_2 + \delta\rho > 0$, the condition holds if and only if

$$bk\alpha(\pi_2 + \delta\rho)(\alpha + k - 1) \leq (1 - \pi_1)(k + \alpha)(\alpha - k + 1),$$

for which it is necessary that

$$\delta(1 - \rho) \leq (1 - k\pi_1)(k - 1)^{-1} + \pi_2 \quad \text{and} \quad \delta\rho \leq (1 - \pi_1)(bk)^{-1} - \pi_2.$$

These two inequalities imply

$$\delta \leq (1 - k\pi_1)(k - 1)^{-1} + (1 - \pi_1)(bk)^{-1}.$$

It follows, therefore, that we are in Case 2 when the prior information on the performance of the control has very little uncertainty (δ small). In the case $\pi_2 + \delta\rho > 0$, an alternative interpretation is that the available number of experimental units is small (b small). When this is the case, the theorem states that, for the model and the criterion under consideration, a BIB design in the test treatments is optimal; all information on the control is to be derived from the prior. In actual practice, however, there may be other, overriding factors against not allocating any experimental unit to the control.

PROOF OF THEOREM 3.1. For any $d \in \mathscr{D}_1(v + 1, b, k)$, we show that $\sigma^{-2}(1 - \pi_1)^{-1} \operatorname{tr} D_d \geq g(q_0, z_0)$, with equality when d is a BTIB$(v, b, k; k - q_0, z_0)$. Here is an outline of the proof.

The first step is to note that, by "averaging" over all permutations of the test treatments $1, \dots, v$, we get

$$\sigma^2(1 - \pi_1)^{-1} \operatorname{tr} D_d \geq (v - 1)(P + e_1)^{-1} + (Q + e_2)^{-1},$$

where

$$P = \frac{1}{v}T_d - \frac{1}{(v-1)(k+\alpha)}\sum_{j=1}^{b}\sum_{i=1}^{v} n_{dij}^2 + \frac{1}{v(v-1)(k+\alpha)}S_d$$

$$- \frac{\eta}{v-1}\sum_{i=1}^{v} r_{di}^2 + \frac{\eta}{v(v-1)}T_d^2,$$

$$Q = \frac{1}{v}T_d - \frac{1}{v(k+\alpha)}S_d - \frac{\eta}{v}T_d^2,$$

$$T_d = \sum_{j=1}^{b} t_{dj}, \qquad S_d = \sum_{j=1}^{b} t_{dj}^2, \qquad t_{dj} = \sum_{i=1}^{v} n_{dij}, \qquad j = 1, \ldots, b.$$

Next observe that since $k/v \le 1$, $\Sigma_{i=1}^{v} n_{dij}/v \le 1$. Hence

$$\sum_{j=1}^{b}\sum_{i=1}^{v} n_{dij}^2 \ge T_d, \quad \text{equality when } n_{dij} \in \{0,1\}, \text{for } i = 1,\ldots,v,\ j = 1,\ldots,b.$$

Also

$$\sum_{i=1}^{v} r_{di}^2 \ge \frac{T_d^2}{v}, \quad \text{equality when } r_{d1} = \cdots = r_{dv}.$$

Combining these facts with the inequalities $\alpha \ge 0$ and $\eta \ge 0$, which follow from (3.7), we get

$$\sigma^{-2}(1-\pi_1)^{-1}\operatorname{tr} D_d \ge (v-1)(P_1+e_1)^{-1} + (Q+e_2)^{-1},$$

where $P_1 = (v^{-1} - (v-1)^{-1}(k+\alpha)^{-1})T_d + v^{-1}(v-1)^{-1}(k+\alpha)^{-1}S_d$. Let us write

$$(v-1)(P_1+e_1)^{-1} + (Q+e_2)^{-1} = g_1(T_d, S_d),$$

a function of T_d and S_d. It can be shown, using the fact $e_1 \ge e_2$, which is a consequence of (3.7), that

$$\frac{\partial}{\partial S_d} g_1(T_d, S_d) \ge 0.$$

This implies that, for fixed T_d, $g_1(T_d, S_d)$ is smaller when the t_{dj}'s are chosen such that $|t_{dj} - t_{dj'}| \le 1$. Thus $g_1(T_d, S_d) \ge g(q, z)$ with $q = \operatorname{int}[T_d/b] + 1$, $z = bq - T_d$. The rest of the proof follows once it is noted that, for a fixed q in $[0, \frac{1}{2}(k+\alpha+1)/(b\eta(k+\alpha)+1)]$, the quantities $((v-1)(k+\alpha) - v)T + S$ and $(k+\alpha)T - \eta(k+\alpha)T^2 - S$ are each nonincreasing in z and hence $g(q,z)$ is nondecreasing in z, hence the theorem. □

EXAMPLE 3.1. Let $v = 7$, $b = 7$ and $k = 5$. A BTIB(7, 7, 5; 1, 0) is Bayes A-optimal when $\pi_1 = 0.2$, $\pi_2 = 0.1$, $\rho = 0.5$, $\xi_1 = 1.64$ and $\xi_2 = 0.57$. The design BTIB(7, 7, 5; 1, 0) can be constructed by augmenting each block of a BIB(7, 7, 4, 4, 2) in the test treatments by one replication of the control.

Writing columns as blocks, an example of this design is

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 5 & 6^* & 7 & 1 & 2 & 3 & 4 \\ 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}.$$

If we change ξ_1 and ξ_2 to $\xi_1 = 0.806$ and $\xi_2 = 0.706$ but keep the values of all other parameters unchanged, then a BTIB(7, 7, 5; 2, 0) is Bayes A-optimal. This design is a BIB(7, 7, 3, 3, 1) augmented by two replications of the control in each block. It can be written as

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{matrix}.$$

This example serves to illustrate Theorem 3.1. As a matter of practical application, however, it is more useful to identify a design which is highly efficient, if not actually optimal, for a large set of values in the range of the parameters. Based on our experience thus far, we believe that a Bayes optimal design of Theorem 3.1 possesses such a robustness property. More detail will be provided in the next section.

We turn our attention to optimal Γ-minimax designs. Instead of the exchangeability conditions (3.2) and (3.3), suppose the prior belongs to the wider class

$$\Gamma = \{B, B^*: \sigma^2\delta_m I \le B \le \sigma^2\delta_M I, \sigma^2\xi_m I \le B^* \le \sigma^2\xi_M I\}, \tag{3.8}$$

where " $\le$ " is with respect to the nonnegative definite, or Löwner, ordering. From expressions (2.2) and (2.3) it is clear that, for a design d, the risk of the Γ-minimax rule [cf. Berger (1980), page 134] is given by $\operatorname{tr} D_d(\Gamma)$, where

$$D_d(\Gamma)^{-1} = X_{1d}'(E + \sigma^2\delta_M X_2 X_2')^{-1} X_{1d} + \sigma^{-2}\xi_M^{-1} I. \tag{3.9}$$

An optimal Γ-minimax design can be defined as one that minimizes this risk, that is, d^* given by

$$\operatorname{tr} D_{d^*}(\Gamma) = \operatorname*{Min}_{d \in \mathscr{D}_1(v+1, b, k)} \operatorname{tr} D_d(\Gamma).$$

Priors similar to those given by (3.8) have been considered by DasGupta and Studden (1991). The class Γ is very general since the only restriction in it is a bound on the spectrum of the prior covariance matrices. As a matter of fact, it follows from (3.9) that, for our purpose, the lower bounds δ_m and ξ_m can be taken to be 0 without any loss of generality.

The following theorem is immediate.

THEOREM 3.2. *Suppose the conditions* (3.1) *and* (3.6) *hold and* $1 > \pi_1 \geq \pi_2 \geq 0$. *Suppose* Γ *is given by* (3.8). *An optimal* Γ-*minimax design is given by Theorem* 3.1 *with*

$$\rho = \xi_2 = 0, \qquad \delta = \delta_M \quad \text{and} \quad \xi_1 = \xi_M.$$

The test treatments in some applications are new, untried treatments, while the control is a standard treatment. The prior information on $\theta_i = \tau_i - \tau_0$, $i = 1, \ldots, v$, may be sufficiently vague in these experiments to imply $\xi_M = \infty$. In addition, let us assume, for simplicity, that the errors ε_{ijp} are homoscedastic ($\pi_1 = \pi_2 = 0$). Γ-minimax designs for this special, but nevertheless important, case may be derived from Theorem 3.2. These designs coincide with Bayes A-optimal designs for homoscedastic errors ($\pi_1 = \pi_2 = 0$), vague prior on $\theta(B^{*-1} = O)$ and $B = \sigma^2\delta I$. We studied this special case in Majumdar (1988); indeed, Theorem 3.1 can be looked upon as a generalization of Theorems 2.1 and 2.3 of Majumdar (1988). To aid in the study of robust designs in the next section, we restate the result for this case, using a slightly different notation, in Corollary 3.1. Let

$$\Lambda(\delta) = \{(q,z) : q = \operatorname{int}[(k + 1 + \delta^{-1})/2] + 1, \ldots, k; z = 0, 1, \ldots, b\},$$

$$A = (k + \delta^{-1})^{-1}, \qquad C = v^{-1} - A(v-1)^{-1}, \qquad E = Av^{-1}(v-1)^{-1}.$$

$T = bq - z$ and $S = bq^2 - 2qz + z$, as before, and

$$f(q,z;\delta) = v(T - AS)^{-1} + (v-1)(CT + ES)^{-1}.$$

COROLLARY 3.1. *Let* $v \geq k \geq 2$ *and* $E = \sigma^2 I$. *Define* d_0 *as follows.*

(i) *If* $(k + 1 + \delta^{-1})/2 < k$, *then* d_0 *is a* BTIB$(v, b, k; k - q_0, z_0)$, *where* $f(q_0, z_0; \delta) = \min\{f(q, z; \delta) : (q, z) \in \Lambda(\delta)\}$.

(ii) *If* $(k + 1 + \delta^{-1})/2 \geq k$, *then* d_0 *is a* BIB(v, b, r, k, λ) *based on the test treatments only.*

Then d_0 *is an optimal* Γ-*minimax design for* $\Gamma = \{B, B^* : B \leq \sigma^2\delta I, B^{*-1} = O\}$, *in* $\mathscr{D}_1(v + 1, b, k)$; d_0 *is also Bayes A-optimal for* $B = \sigma^2\delta I$, $B^{*-1} = O$, *in* $\mathscr{D}_1(v + 1, b, k)$.

4. Robustness and approximation. The study of robustness is very important since, as Berger (1980), page 129, puts it, "the main worry is that, in a Bayesian analysis, one could be led, by an inadequate description of prior beliefs, into making a bad decision." Efficient designs that are robust over priors will be very useful; a practitioner may be quite willing to give up some efficiency to ensure against departures from the assumed prior. This is precisely what DasGupta and Studden (1991) do. They determine the most robust design among designs that are at least $100(1 - \varepsilon)\%$ efficient.

Even though our method is different, our goal is the same, that is, to identify designs which are robust and highly efficient. We would like to obtain a finite partition of the parameter space of the prior distributions such that, for each set in the partition, there is one design which, at each point, is optimal or at worst $100(1-\varepsilon)\%$ efficient for a small ε. As a first step we intend to study how much perturbation in the prior parameters can be introduced before an optimal design ceases to be optimal.

There is another problem closely related to robustness. Suppose, for a given set of prior parameters, an optimal design is unknown, that is, it cannot be obtained from Theorem 3.1 or Theorem 3.2. Do any of the known optimal designs, perhaps one with a slightly different set of values of the prior parameters, provide a good approximation? We shall consider both problems simultaneously.

The presence of several parameters and the complicated structure of the functions g in Theorem 3.1 and f in Corollary 3.1 makes the problems of robustness and approximation very difficult; indeed a proper treatment of the problem in its generality is a major research project by itself. We will initiate this research by establishing some interesting properties of Bayes and Γ-minimax optimal designs. This will enable us to partition the range of the prior parameters and establish robust optimal designs for some sets of the partition. For the remaining sets, these results along with the detailed study of an example will indicate what the robust efficient designs are most likely to be. Since it is technically very difficult to tackle this problem when there are several parameters, we shall consider the simplest situation, that is, the setup of Corollary 3.1 which contains only one parameter δ. This will also enable us to interpret the results easily and will serve as a first step toward solving the more general problem. Even in this setup the problem is fairly complicated.

We will examine the robustness and other properties of optimal Bayes and Γ-minimax designs given by Corollary 3.1. *For the rest of this section we adopt the notation of Corollary* 3.1.

LEMMA 4.1. *Given v, b, k and δ, suppose $(k+1+\delta^{-1})/2 < k$ and $q \in \{\mathrm{int}[(k+1+\delta^{-1})/2]+1,\dots,k\}$.*

(i) *For each fixed q, there exists $z^* \in [0,b]$, a function of q, such that $f(q,z;\delta)$ decreases with z when $z \in [0,z^*]$ and $f(q,z;\delta)$ increases with z when $z \in (z^*,b]$. If $z^*=0$, then $f(q,z;\delta)$ increases with z in $[0,b]$ and if $z^*=b$, then $f(q,z;\delta)$ decreases with z in $[0,b]$.*

(ii) (a) *For $q \in \{\mathrm{int}[(k+1+\delta^{-1})/2]+2,\dots,k\}$, $f(q,b-1;\delta) \le f(q,b;\delta)$ implies $f(q-1,0;\delta) \le f(q-1,1;\delta)$.* (b) *For $q \in \{\mathrm{int}[(k+1+\delta^{-1})/2]+1,\dots,k-1\}$; $f(q,1;\delta) \le f(q,0;\delta)$ implies $f(q+1,b;\delta) \le f(q+1,b-1;\delta)$.*

When $\delta=\infty$, Lemma 4.1(i) reduces to Lemma 2.2 of Ture (1982) [also reproduced in Lemma 2.2 of Hedayat and Majumdar (1985)]. Moreover, when $\delta=\infty$, Lemma 4.1(ii) reduces to Lemma A.2 of Cheng, Majumdar, Stufken and Ture (1988). The proof of Lemma 4.1 needs no new techniques beyond those used to prove Ture's and Cheng, Majumdar, Stufken and Ture's results and, indeed, follows along the same lines. We omit the details.

Let $r_0 = bk - bq + z$. As functions of $r_0, q = \text{int}[(bk - r_0)/b] + 1$; $z = bq + r_0 - bk$, $T = bk - r_0$, $S = (b\,\text{int}[T/b] + b - T)(\text{int}[T/b])^2 + (T - b\,\text{int}[T/b])(\text{int}[T/b] + 1)^2$. So we may write

$$f(q, z; \delta) = f^*(r_0; \delta).$$

The range of r_0 as (q, z) varies in $\Lambda(\delta)$ is $\{0, 1, \dots, bk - b\,\text{int}[(k + 1 + \delta^{-1})/2] + 2b\}$. Note that $f(q, b; \delta) = f(q - 1, 0; \delta)$. Using q_0, z_0 from Corollary 3.1, we get

$$f^*(r_0^*, \delta) = \min_{r_0} f^*(r_0; \delta), \quad \text{where } r_0^* = bk - bq_0 + z_0.$$

The following theorem is an immediate consequence of Lemma 4.1.

THEOREM 4.1. *Suppose* $(k + 1 + \delta^{-1})/2 < k$. *Then* $f^*(r_0; \delta) \le f^*(r_0 - 1; \delta)$ *for* $r_0 \le r_0^*$ *and* $f^*(r_0; \delta) \le f^*(r_0 + 1; \delta)$ *for* $r_0 \ge r_0^*$.

COROLLARY 4.1. *Suppose* $(k + 1 + \delta^{-1})/2 < k$. *Then a* BTIB$(v, b, k; k - q_0, z_0)$ *with* $bq_0 - z_0 = bk - r_0^*$ *is optimal Bayes and optimal* Γ-*minimax design in* $\mathscr{D}_1(v + 1, b, k)$ *if* $f^*(r_0^*; \delta) \le \min\{f^*(r_0^* - 1; \delta), f^*(r_0^* + 1; \delta)\}$.

Corollary 4.1 gives a simple way of identifying optimal designs. In many situations the experimenter may be able to arrive at an approximate value of r_0^*. A quick search in the neighborhood of this value, with the help of Corollary 4.1, will lead the experimenter to the correct r_0^*. We shall soon see that Theorem 4.1 also plays a prominent role in studying approximations and robustness.

Let us turn our attention to the behavior of the optimal number of replications of the control (r_0^*) as δ varies.

First, observe that Lemma 4.1 guarantees that for each δ, r_0^* is unique, or it is either one or two consecutive integers. We shall henceforth replace r_0^* by the notation $r_0^*(\delta)$, which, for each $\delta \in [0, \infty]$, is

$$r_0^*(\delta) = \begin{cases} r_0^*, & \text{if } f^*(r_0; \delta) \text{ has a unique minimum } r_0^*, \\ r_0^* + 1, & \text{if } f^*(r_0^*; \delta) = f^*(r_0^* + 1, \delta) = \min_{r_0} f^*(r_0; \delta). \end{cases}$$

THEOREM 4.2. *Let* δ_1 *and* δ_2 *be any two nonnegative real numbers satisfying* $\delta_1 < \delta_2$. *Then*

$$r_0^*(\delta_1) \le r_0^*(\delta_2). \tag{4.1}$$

PROOF. It follows from Theorem 4.1 and the fact $f(q, b; \delta) = f(q - 1, 0; \delta)$, that (4.1) is equivalent to the following statement for each (q, z) and $\delta_1 < \delta_2$:

$$\text{If } f(q, z; \delta_2) < f(q, z + 1; \delta_2), \quad \text{then } f(q, z; \delta_1) < f(q, z + 1; \delta_1). \tag{4.2}$$

Since $f(q, z; \delta)$ is increasing in z for each $q \le \text{int}[(k + 1)/2]$, it is enough to prove (4.2) for an arbitrary (q, z) with $q \in \{\text{int}[(k + 1)/2] + 1, \dots, k\}$, $z \in \{0, 1, \dots, b - 1\}$.

Recall that $T = bq - z$, $S = bq^2 - 2qz + z$. Let $T_1 = bq - (z+1)$, $S_1 = bq^2 - 2q(z+1) + (z+1)$, $a = (v-1)^2$ and $B = (v-1) - Av$. Then we can write

$$v^{-1}f(q, z; \delta) = (T - AS)^{-1} + a(BT + AS)^{-1}.$$

As δ increases from 0 to ∞, $A = (k + \delta^{-1})^{-1}$ increases monotonically from 0 to k^{-1}. For A in $[0, k^{-1}]$, it is not difficult to see that $f(q, z; \delta)$ is $>$, $=$ or $< f(q, z+1; \delta)$ if and only if $f_1(A)$ is $<$, $=$ or $>$ 0, respectively, where

$$f_1(A) = f_1(A; r_0) = a(T - AS)(T_1 - AS_1)((v-1) + A(2q - 1 - v)) \\ -(BT + AS)(BT_1 + AS_1)(A(2q-1) - 1),$$

where $r_0 = bk - bq + z$.

In order to study the behavior of $f_1(A)$ for $A \in [0, k^{-1}]$, it will be useful to consider $f_1(A)$ as a function of A in $(-\infty, \infty)$. f_1 is a cubic in A, and

$$f_1(0) > 0, \tag{4.3}$$

$$f_1(1) \geq 0, \tag{4.4}$$

with equality in (4.4) only when $k = 2$, $q = 2$ and $z = b - 1$. It can be shown that

$$k^{-1} \leq T/S < T_1/S_1 \leq 1.$$

If we choose an arbitrary real number A_0 from the open interval $(T/S, T_1/S_1)$, then it is easy to see that, with (q, z) as chosen before and $B_0 = (v-1) - A_0 v$, $T - A_0 S < 0$, $T_1 - A_0 S_1 > 0$, $(v-1) + A_0(2q - 1 - v) > 0$, $B_0 T + A_0 S > 0$, $B_0 T_1 + A_0 S_1 > 0$ and $A_0(2q-1) - 1 > 0$. Hence

$$f_1(A_0) < 0 \tag{4.5}$$

for an arbitrary A_0 in $(T/S, T_1/S_1)$. It follows from (4.4) and (4.5) that $f_1(A)$ has at least one root in (A_0, ∞). Since $f_1(A)$ is cubic in A, it follows from (4.3) and (4.5) that $f_1(A)$ has exactly one root in $(0, A_0)$. Let $A_i = (k + \delta_i^{-1})^{-1}$, $i = 1, 2$. Clearly, $0 < A_i \leq k^{-1}$. From this analysis of $f_1(A)$ it follows that

$$f_1(A_2) \geq 0 \quad \text{implies } f_1(A) > 0 \text{ for all } A \in [0, A_2).$$

Thus we get (4.2). This establishes Theorem 4.2. □

In other words, Theorem 4.2 says that as the prior knowledge on the control increases, its replication in the optimal design decreases. This is a very appealing property.

For a pair (q, z), for which $f_1(k^{-1}) \leq 0$, there exists exactly one δ for which $f_1(A) = 0$ with $A = (k + \delta^{-1})^{-1}$. For this δ, both (q, z) and $(q, z+1)$ minimize the function f. For all other δ's, f is minimized at a single point (q, z). From these observations, we can make the following statements:

$$f^*(r_0; \delta_2) \leq f^*(r_0 + 1; \delta_2) \quad \text{implies} \quad f^*(r_0; \delta) < f^*(r_0 + 1; \delta) \\ \text{for all } \delta \in [0, \delta_2), \tag{4.6}$$

$$f^*(r_0;\delta_1) \geq f^*(r_0+1;\delta_1) \quad \text{implies} \quad f^*(r_0;\delta) > f^*(r_0+1;\delta) \quad \text{for all } \delta \in (\delta_1, \infty). \tag{4.7}$$

COROLLARY 4.2. *Let* $\delta_1 < \delta_2$ *be such that* $r_0^*(\delta_1) = r_0^*(\delta_2)$. *Then* $r_0^*(\delta) = r_0^*(\delta_1)$ *for all* δ *in* $[\delta_1, \delta_2]$.

PROOF. Let $r_0^* = r_0^*(\delta_1) = r_0^*(\delta_2)$, and let $\delta \in (\delta_1, \delta_2)$. If $f^*(r_0^*;\delta) > f^*(r_0^*+1;\delta)$, then, using (4.7), we arrive at a contradiction to the fact that $f^*(r_0^*;\delta_2) \leq f^*(r_0^*+1;\delta_2)$. Thus $f^*(r_0^*;\delta) \leq f^*(r_0^*+1;\delta)$. Similarly, it can be shown that $f^*(r_0^*;\delta) \leq f^*(r_0^*-1;\delta)$. This establishes the corollary. □

The value of $r_0^*(\delta)$, in the case of a vague prior on $\dot{\gamma}$ ($\delta = \infty$), is denoted by $r_0^*(\infty)$. Let us define a nonnegative real number δ_u by the relation

$$A_u = \left(k + \delta_u^{-1}\right)^{-1} \quad \text{with} \quad f_1(A_u; r_0^*(\infty) - 1) = 0,$$

with f_1 as in the proof of Theorem 4.2. Clearly,

$$r_0^*(\delta) = r_0^*(\delta_u) = r_0^*(\infty) \quad \text{for all } \delta \in [\delta_u, \infty],$$

but

$$r_0^*(\delta) < r_0^*(\infty) \quad \text{for all } \delta < \delta_u.$$

COROLLARY 4.3. *Let* r_0 *be an integer in the interval* $[0, r_0^*(\delta_u)]$. *Then there exists a* δ, δ_0 *say, such that* $r_0^*(\delta_0) = r_0$.

PROOF. Let us define

$$\delta_1 = \operatorname{Inf}\{\delta\colon f^*(r_0;\delta) \leq f^*(r_0-1;\delta)\},$$

$$\delta_2 = \operatorname{Sup}\{\delta\colon f^*(r_0;\delta) \leq f^*(r_0+1;\delta)\}.$$

First, observe that $\delta_1 \leq \delta_2$, since if $\delta_1 > \delta_2$, then for any $\delta \in (\delta_2, \delta_1)$, $f^*(r_0;\delta) > f^*(r_0+1;\delta)$ and $f^*(r_0;\delta) < f^*(r_0-1;\delta)$; this contradicts Theorem 4.1. To complete the proof, now observe that in view of Theorem 4.2, any $\delta \in [\delta_1, \delta_2]$ can serve as δ_0. Hence the corollary. □

These results give us a good insight into the nature of the optimal designs. We summarize this in the following theorem.

THEOREM 4.3. (i) *For* $i = 1, \dots, r_0^*(\delta_u) - 1$, *define* δ_{1i} *by* $f_1((k+\delta_{1i}^{-1})^{-1}; i-1) = 0$. *For* $i = 0, 1, \dots, r_0^*(\delta_u) - 2$, *define* δ_{2i} *by* $f_1((k+\delta_{2i}^{-1})^{-1}; i) = 0$. *Let* $\delta_{10} = 0$, $\delta_{1r_0^*(\delta_u)} = \delta_{2,r_0^*(\delta_u)-1} = \delta_u$, $\delta_{2r_0^*(\delta_u)} = \infty$. *Then, for* $i = 0, 1, \dots, r_0^*(\delta_u) - 1$,

$$\delta_{1i} < \delta_{2i} = \delta_{1i+1}.$$

(ii) *For* $i = 0, 1, \ldots, r_0^*(\delta_u)$, *let the interval* $[\delta_{1i}, \delta_{2i}]$ *be denoted by* E_i. *Then* $\bigcup E_i = [0, \infty]$, *the range of* δ.

(iii) *For* $i = 0, 1, \ldots, r_0^*(\delta_u)$, *a* BTIB$(v, b, k; t, s)$ *with* $bt + s = i$ *is optimal for each* δ *in* E_i.

For intervals E_i which are such that a BTIB$(v, b, k; t, s)$ with $bt + s = i$ exists, the BTIB$(v, b, k; t, s)$ is optimal and robust for all δ in E_i. The problem, therefore, reduces to finding robust optimal (or highly efficient) designs for the remaining intervals E_i, that is, those for which no BTIB$(v, b, k; t, s)$ with $bt + s = i$ exist. Let us begin by studying an example in detail.

EXAMPLE 4.1. Let $v = 3$, $k = 2$ and $b = 24$. It is easy to see that $q_0 = 2$ and z_0 is given by

$$f(2, z_0; \delta) = \min\{f(2, b(\text{int}[w]); \delta), f(2, b(\text{int}[w]) + 1; \delta)\},$$

where [recall $A = (2 + \delta^{-1})^{-1}$],

$$w = \left[(3A - 1)^{1/2}(2 - A) - \sqrt{2}(2 - 4A)\right]\left[\sqrt{2}(3A - 1) + (3A - 1)^{1/2}\right]^{-1}.$$

The optimal replication of the control is $r_0^*(\delta) = z_0$. When $\delta \le 1.58$, it can be seen that $r_0^*(\delta) = 0$; in these situations, a BIB(3, 24, 16, 2, 8) based on the test treatments only is optimal. When $\delta > 1.58$, an optimal design is BTIB$(3, 24, 2; 0, r_0^*(\delta))$. This design can be obtained as the union of $r_0^*(\delta)/3$ copies of d_1 and $(24 - r_0^*(\delta))/3$ copies of d_2, where d_1 and d_2 are (columns are blocks)

$$d_1: \begin{matrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{matrix} \qquad d_2: \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}.$$

Clearly, a BTIB$(3, 24, 2; 0, r_0^*(\delta))$ exists if and only if $r_0^*(\delta) \equiv 0 \pmod 3$.

In Table 1 we consider 22 different values of δ. The largest $r_0^*(\delta)$ is $r_0^*(\delta_u) = 18$ which corresponds to $\delta = \infty$ (vague prior on γ). If $0 \le r_0^*(\delta) \le 18$, then a BTIB$(3, 24, 2; 0, r_0^*(\delta))$ exists only for $r_0^*(\delta) = 0$, 3, 6, 9, 12, 15 and 18. For each δ, there are two possibilities. (1) $r_0^*(\delta) \equiv 0 \pmod 3$, in which case an optimal design can be constructed as mentioned above. (2) $r_0^*(\delta) \not\equiv 0 \pmod 3$, in which case we look for an approximately optimal (highly efficient) design. Following Cheng, Majumdar, Stufken and Ture (1988), we seek a BTIB$(v, b, k; t, s)$ with $r_0 = bt + s$ "close" to the "optimal" r_0, that is, r_0^*.

Suppose $\delta = \delta_0$ is such that the corresponding $r_0^*(\delta) \not\equiv 0 \pmod 3$. Let r_{01} and r_{02} be such that $r_{01} < r_0^*(\delta) < r_{02}$, $r_{01} \equiv 0 \pmod 3$, $r_{02} - r_{01} = 3$. Theorem 4.1 tells us that the "best" BTIB design for $\delta = \delta_0$ is one of BTIB$(3, 24, 2; 0, r_{0i})$, $i = 1, 2$. The efficiency of BTIB$(3, 24, 2; 0, r_{0i})$ will be measured by

$$W_i = f(2, r_0^*(\delta_0); \delta_0)/f(2, r_{0i}; \delta_0).$$

TABLE 1
Efficiency of designs for $v = 3$, $k = 2$, $b = 24$
(see Example 4.1)

(1) δ	(2) $r_0^*(\delta)$	(3) r_0, Efficiency	(4) r_0, Efficiency	(5) Efficiency of $r_0 = 18$	(6) Efficiency of non-Bayes strategy
1.70	0	0, 1	—	0.8476	0.6450
1.75	1	0, 0.9995	3, 0.9984	0.8556	0.6544
1.80	2	3, 0.9996	0, 0.9982	0.8630	0.6632
1.81	3	3, 1	—	0.8647	0.6652
2.00	5	6, 0.9993	3, 0.9979	0.8874	0.6940
2.10	6	6, 1	—	0.8969	0.7070
2.20	7	6, 0.9994	9, 0.9970	0.9054	0.7191
2.30	8	9, 0.9988	6, 0.9979	0.9129	0.7300
2.50	9	9, 1	—	0.9252	0.7495
3.00	11	12, 0.9992	9, 0.9952	0.9464	0.7870
3.15	12	12, 1	—	0.9510	0.7961
3.50	12	12, 1	—	0.9598	0.8145
4.00	13	12, 0.9976	15, 0.9963	0.9684	0.8352
5.00	14	15, 0.9999	12, 0.9902	0.9794	0.8655
5.50	15	15, 1	—	0.9824	0.8765
6.00	15	15, 1	—	0.9850	0.8859
6.50	15	15, 1	—	0.9875	0.8946
7.00	16	15, 0.9994	18, 0.9891	0.9891	0.9016
10.00	16	15, 0.9958	18, 0.9947	0.9947	0.9297
25.00	17	18, 0.9996	15, 0.9844	0.9996	0.9732
30.00	18	18, 1	—	1	0.9762
∞	18	18, 1	—	1	1

This gives a lower bound to the actual efficiency, since

$$f(2, r_0^*(\delta_0); \delta_0) \le \min\{\sigma^{-2} \operatorname{tr} D_d \colon d \in \mathscr{D}(v+1, b, k),\ \delta = \delta_0\}.$$

In Table 1, column (2) gives $r_0^*(\delta)$ and columns (3) and (4) gives the values of r_{0i}, W_i for $i = 1, 2$. Column (3) gives the better approximation. Both approximations are seen to be excellent for this example. When $r_0^*(\delta) \equiv 0 \pmod 3$ column (3) shows an efficiency of 1—signaling the fact that a BTIB$(3, 24, 2; 0, r_0^*(\delta))$ exists and it is optimal.

If no prior information is used to design the experiment, then the optimal replication of the control r_0 is 18. For each δ, column (5) shows the efficiency of BTIB$(3, 24, 2; 0, 18)$, that is, it gives $f(2, r_0^*(\delta); \delta)/f(2, 18; \delta)$. This gives the loss of efficiency suffered by the Bayes, or Γ-minimax, estimator upon using an optimal "non-Bayes" design.

Column (6) gives $f(2, r_0^*(\delta); \delta)/f(2, 18; \infty)$ for each δ. This essentially gives the efficiency of the non-Bayes strategy (one that uses an optimal non-Bayes design and the least squares estimator) with respect to the optimal Bayes, or Γ-minimax, strategy at δ. Not surprisingly, the loss of efficiency in ignoring prior information is quite significant when the prior information is accurate (δ small).

The optimal designs are seen to be quite robust for small and moderate departures from δ. For example, a BTIB(3, 24, 2; 0, 15) is optimal for $\delta = 5.5, 6, 6.5$, and hence for all $\delta \in [5.5, 6.5]$, and highly efficient (perhaps even optimal) for δ as small as 5 and as large as 10.

Given v, b and k, suppose $\{e_1, \dots, e_L\} \subset [0, r_0^*(\delta_u)]$ are integers such that a BTIB$(v, b, k; t_i, s_i)$ exists with $e_i = bt_i + s_i$, for each $i = 1, \dots, L$. We proved earlier in Theorem 4.3 that there are intervals E_i, for each $i = 1, \dots, L$, such that BTIB$(v, b, k; t_i, s_i)$ is optimal for all $\delta \in E_i$. Example 4.1 seems to indicate that there will be intervals $I_1, \dots, I_L$, satisfying

$$I_i \supset E_i, \qquad i = 1, \dots, L; \qquad \bigcup I_i = (0, \infty),$$

such that a BTIB$(v, b, k; t_i, s_i)$ is either optimal or highly efficient for all $\delta \in I_i$. For instance, in Example 4.1, if $e_i = 15$, then I_i can be [5, 10] or wider, with $E_i \supset [5.5, 6.5]$. In other words, a BTIB$(v, b, k; t_i, s_i)$ is expected to be efficient and robust for all δ in I_i. The precise identification of each I_i and a study of the efficiency of BTIB$(v, b, k; t_i, s_i)$ for δ's in I_i will be the subject of future research. We feel that the techniques of Stufken (1988) will prove to be useful in this venture.

5. Bayes *A*-optimal row–column designs. Consider an experiment to compare v test treatments with a control using bk experimental units. Suppose that these units can be arranged, at least conceptually, in a rectangle with k rows and b columns. The set of all designs is denoted by $\mathscr{D}_2(v + 1, b, k)$. The model is

$$y_{ipj} = \mu + \tau_i + \chi_p + \beta_j + \varepsilon_{ipj},$$

where μ is a general effect, τ_i treatment effect, χ_p row effect, β_j column effect and ε_{ipj} the experimental error. Let $\theta_i = \tau_i - \tau_0$ for $i = 0, 1, \dots, v$ and $\gamma_{pj} = \mu + \tau_0 + \chi_p + \beta_j$ for $p = 1, \dots, k$, $j = 1, \dots, b$. The quantity γ_{pj} measures the performance of the control in cell (p, j). Then the model can be written as

$$y_{ipj} = \theta_i + \gamma_{pj} + \varepsilon_{ipj}.$$

Let $d \in \mathscr{D}_2(v + 1, b, k)$, Y and γ be the $bk \times 1$ vectors:

$$Y = (\dots, y_{ip1}, \dots, \dots, y_{ip2}, \dots, \dots, \dots, y_{ipb}, \dots)',$$

$$\gamma = (\dots, \gamma_{p1}, \dots, \dots, \gamma_{p2}, \dots, \dots, \dots, \gamma_{pb}, \dots)'.$$

Let ε be the $bk \times 1$ vector of ε_{ipj}'s and $\theta = (\theta_1, \dots, \theta_v)'$. In matrix notation the model has the same form as (2.1), viz.,

$$Y = X_{1d}\theta + X_2\gamma + \varepsilon,$$

with $X_2 = I_{bk}$ and X_{1d}, a $bk \times v$ matrix with entries 0 and 1. Given a design d, for a particular row, the entry in column i of X_{1d} is 1 if the experimental unit corresponding to this row receives the test treatment i, $i = 1, \dots, v$; it is

0 otherwise. In particular, if the experimental unit receives the control, then the corresponding row has all entries 0.

In addition to the distributions of Y, θ and γ specified in Section 2, we make some additional assumptions. With $L = (1-\rho_1)I_k + \rho_1 1_k 1_k'$ and $L^* = (\rho_2 - \rho_3)I_k + \rho_3 1_k 1_k'$, we demand

$$B + E = \sigma_0^2 \begin{pmatrix} L & L^* & \cdots & L^* \\ L^* & L & \cdots & L^* \\ \vdots & \vdots & \vdots & \vdots \\ L^* & L^* & \cdots & L \end{pmatrix}, \tag{5.1}$$

$$\operatorname{Var}(\theta_i) = \sigma_0^2 \xi_1^*, \qquad \operatorname{Cov}(\theta_i, \theta_{i'}) = \sigma_0^2 \xi_2^* \quad \text{for } i \neq i'. \tag{5.2}$$

Note that the comments which were made regarding assumptions (3.1)–(3.3) hold true here as well. The subscript in σ_0^2 is used to avoid confusion with $\operatorname{Var}(\varepsilon_{ij})$ in Section 3.

For $d \in \mathscr{D}_2(v+1, b, k)$, we denote by m_{dip} the number of times treatment i occurs in row p, n_{dij} the number of times treatment i occurs in column j and r_{di} the $\sum_{j=1}^{b} n_{dij} = \sum_{p=1}^{k} m_{dip}$. We shall use the matrices

$$\overline{M}_d = (m_{dip})_{i=1,\dots,v,\, p=1,\dots,k},$$

$$\overline{N}_d = (n_{dij})_{i=1,\dots,v,\, j=1,\dots,b}$$

and

$$r_d = (r_{d1}, \dots, r_{dv})'.$$

It is not too difficult to show that under (5.1) and the conditions

$$\begin{gathered} \rho_3 < \min(\rho_1, \rho_2), \\ k(\rho_1 - \rho_3) + (\rho_2 - \rho_3) + (1 - \rho_1 - \rho_2 + \rho_3) + \rho_3 bk \neq 0, \end{gathered} \tag{5.3}$$

the inverse of the posterior covariance matrix D_d^{-1} given by expression (2.2) reduces to

$$\begin{aligned} \sigma_0^2(1 - \rho_1 - \rho_2 + \rho_3) D_d^{-1} = {} & \operatorname{Diag}(r_{d1}, \dots, r_{dv}) - (k + \alpha_0)^{-1} \overline{N}_d \overline{N}_d' \\ & - (b + \lambda)^{-1} \overline{M}_d \overline{M}_d' + \psi r_d r_d' \\ & + \sigma_0^2 (1 - \rho_1 - \rho_2 + \rho_3) B^{*-1}, \end{aligned} \tag{5.4}$$

where

$$\alpha_0 = (1 - \rho_1 - \rho_2 + \rho_3)(\rho_1 - \rho_3)^{-1}, \qquad \lambda = (1 - \rho_1 - \rho_2 + \rho_3)(\rho_2 - \rho_3)^{-1},$$

$$\begin{aligned} \psi = {} & [k(\rho_1 - \rho_3) + b(\rho_2 - \rho_3) + 2(1 - \rho_1 - \rho_2 + \rho_3)] \\ & \times [(b + \lambda)(k + \alpha_0)[k(\rho_1 - \rho_3) + b(\rho_2 - \rho_3) + (1 - \rho_1 - \rho_2 + \rho_3)]]^{-1} \\ & - \rho_3(1 - \rho_1 - \rho_2 + \rho_3) \\ & \quad \times [[k(\rho_1 - \rho_3) + b(\rho_2 - \rho_3) + (1 - \rho_1 - \rho_2 + \rho_3)] \\ & \qquad \times [k(\rho_1 - \rho_3) + b(\rho_2 - \rho_3) + (1 - \rho_1 - \rho_2 + \rho_3) + bk\rho_3]]^{-1}. \end{aligned}$$

The optimality results of this section relate to experiments where $\min(b, k)$ is "small." Stating it more precisely, we assume, as in (3.6),

$$2 \le k \le v.$$

To state the main theorem of this section, we need some more notation. Let

$$F = k^{-1}(b+\lambda)^{-1} - \psi. \tag{5.5}$$

For integers q and z, denote $T = bq - z$ and $S = bq^2 - 2qz + z$.

$$\Pi = \{(q,z): q = \operatorname{int}[\tfrac{1}{2}(k+\alpha_0+1)/(Fb(k+\alpha_0)+1)] + 1, \dots, k, z = 0, \dots, b\},$$

$$h(q,z) = v(v-1)^2(k+\alpha_0)[((v-1)(k+\alpha_0)-v) + T + S + v(v-1)(k+\alpha_0)\mu_1]^{-1} + v(k+\alpha_0)[(k+\alpha_0)T - (k+\alpha_0)FT^2 - S + v(k+\alpha_0)\mu_2]^{-1},$$

where $\mu_1 = (1-\rho_1-\rho_2+\rho_3)/(\xi_1^* - \xi_2^*)$, and $\mu_2 = (1-\rho_1-\rho_2+\rho_3)/(\xi_1^* + (v-1)\xi_2^*)$.

THEOREM 5.1. *Suppose conditions* (5.1), (5.2) *and* (3.6) *hold and further*

$$0 \le \rho_3 < \min(\rho_1, \rho_2), \qquad \xi_2^* \ge 0. \tag{5.6}$$

Case 1. $\frac{1}{2}(k+\alpha_0+1)/(Fb(k+\alpha_0)+1) < k$. *Let*

$$h(q_0, z_0) = \min\{h(q,z): (q,z) \in \Pi\}.$$

Suppose $d^* \in \mathscr{D}_2(v+1, b, k)$ *is such that it is a* BTIB$(v, b, k; k - q_0, z_0)$ *design in* $\mathscr{D}_1(v+1, b, k)$ *with columns as blocks and*

$$m_{d^*i1} = \cdots = m_{d^*ik} \quad \text{for } i = 0, 1, \dots, v.$$

Then d^* *is Bayes A-optimal in* $\mathscr{D}_2(v+1, b, k)$.

Case 2. $\frac{1}{2}(k+\alpha_0+1)/(Fb(k+\alpha_0)+1) \ge k$. *Then a Youden design* d^* *in* $\mathscr{D}_2(v, b, k)$ *based on the test treatments only is Bayes A-optimal in* $\mathscr{D}_2(v+1, b, k)$.

PROOF. As in the case of Theorem 3.1, we shall show that $\sigma_0^{-2}(1-\rho_1-\rho_2+\rho_3)^{-1} \operatorname{tr} D_d \ge h(q_0, z_0)$, with equality when d is d^*. By "averaging" over all permutations of test treatments $1, \dots, v$, we get

$$\sigma^{-2}(1-\rho_1-\rho_2+\rho_3)^{-1} \operatorname{tr} D_d \ge (v-1)\left(G(v-1)^{-1} - H(v^2-v)^{-1} + \mu_1\right)^{-1} + \left(Hv^{-1} + \mu_2\right)^{-1},$$

where

$$G = T_d - \frac{1}{k+\alpha_0}\sum_{i=1}^{v}\sum_{j=1}^{b} n_{dij}^2 - \frac{1}{b+\lambda}\sum_{i=1}^{v}\sum_{p=1}^{k} m_{dip}^2 + \psi\sum_{i=1}^{v} r_{di}^2,$$

$$H = T_d - \frac{1}{k+\alpha_0}S_d - \frac{1}{b+\lambda}\sum_{p=1}^{k} s_{dp}^2 + \psi T_d^2,$$

with

$$T_d = \sum_{j=1}^{b} t_{dj}, \qquad S_d = \sum_{j=1}^{b} t_{dj}^2, \qquad t_{dj} = \sum_{i=1}^{v} n_{dij}$$

for $j = 1,\dots,b$, and

$$s_{dp} = \sum_{i=1}^{v} m_{dip}$$

for $p = 1,\dots,k$.

Since $k \le v$, $\sum_{i=1}^{v} n_{dij}/v \le 1$. Hence

$$\sum_{j=1}^{b}\sum_{i=1}^{v} n_{dij}^2 \ge T_d$$

with equality whenever $n_{dij} \in \{0,1\}$ for $i = 1,\dots,v$, $j = 1,\dots,b$. Also

$$\sum_{i=1}^{v}\sum_{p=1}^{k} m_{dip}^2 \ge \sum_{i=1}^{v} r_{di}^2/k$$

with equality whenever $m_{di1} = \cdots = m_{dik}$ for $i = 1,\dots,v$. Thus $G \le G^* = (1-(k+\alpha_0)^{-1})T_d - F\sum_{i=1}^{v} r_{di}^2$, with F as in (5.5). From (5.6) it follows that $F \ge 0$. Moreover,

$$\sum_{i=1}^{v} r_{di}^2 \ge \left(\sum_{i=1}^{v} r_{di}\right)^2 \Big/ v$$

with equality whenever $r_{d1} = \cdots = r_{dv}$. Hence

$$G \le G^* \le G^{**} = \left(1-(k+\alpha_0)^{-1}\right)T_d - FT_d^2/v.$$

It follows, therefore,

$$\sigma^{-2}(1-\rho_1-\rho_2+\rho_3)^{-1}\operatorname{tr} D_d$$
$$\ge (v-1)\left(G^{**}(v-1)^{-1} - H(v^2-v)^{-1} + \mu_1\right)^{-1} + \left(Hv^{-1}+\mu_2\right)^{-1}.$$

Let us call the right side of this inequality $h_1(G^{**}, H)$. Using (5.6), it can be shown that

$$\frac{\partial}{\partial H} h_1(G^{**}, H) \le 0.$$

This implies that for fixed T_d, $h_1(G^{**}, H)$ is smaller when the t_{dj}'s are chosen such that $|t_{dj} - t_{dj'}| \leq 1$ and the s_{dp}'s are chosen such that $s_{d1} = \cdots = s_{dk}$. Thus $h_1(G^{**}, H) \geq h(q, z)$ with $q = \text{int}[T_d/b] + 1$, $z = bq - T_d$.

Now observe that $((v-1)(k+\alpha_0) - v)T + S$ is decreasing in z for each q in $\{1, \ldots, k\}$, while $(k+\alpha_0)T - (k+\alpha_0)FT^2 - S$ is decreasing in z for all $q \leq \frac{1}{2}(k+\alpha_0+1)/(Fb(k+\alpha_0)+1)$. This completes the proof of the theorem. □

We give two examples to illustrate the theorem.

EXAMPLE 5.1. Let $v = 9$, $b = 12$ and $k = 4$. The following design is Bayes A-optimal in $\mathscr{D}_2(9, 12, 4)$ when $\rho_1 = 0.2$, $\rho_2 = 0.3$, $\rho_3 = 0.1$, $\xi_1^* = 0.84$ and $\xi_2^* = 0.58$:

0	1	2	3	4	5	0	7	8	9	0	6
1	2	3	4	5	0	7	8	9	0	6	0
3	4	5	0	7	8	9	0	0	6	1	2
9	0	0	6	0	1	2	3	4	5	7	8

EXAMPLE 5.2. Let $v = 3$, $b = 24$, $k = 2$. The following design is Bayes A-optimal in $\mathscr{D}_2(3, 24, 2)$ when $\rho_1 = 0.48$, $\rho_2 = 0.04$, $\rho_3 = 0$ and $B^{*-1} = O$ (vague prior on θ):

0	0	0	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	2	3	0	0	0	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1

6. Concluding remarks. In this initial study to explore the nature of exact optimal designs utilizing prior information, we obtained Bayes A-optimal designs for the one- and two-way elimination of heterogeneity models and Γ-minimax designs for the one-way elimination of heterogeneity model, for a class of distributions of the random variables. It is virtually impossible that one method can be used to obtain the optimal designs for all prior distributions. A priority for future research should be, therefore, to develop tools beyond those used in this article, that can handle other classes of prior distributions.

One attractive feature of the optimal designs of this article is the robustness against misspecifications of the prior.

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