The Annals of Statistics
1992, Vol. 20, No. 1, 161–194

SAMPLING DESIGNS FOR ESTIMATING INTEGRALS OF STOCHASTIC PROCESSES

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The problem of estimating the integral of a stochastic process from observations at a finite number of sampling points is considered. Sacks and Ylvisaker found a sequence of asymptotically optimal sampling designs for general processes with exactly 0 and 1 quadratic mean (q.m.) derivatives using optimal-coefficient estimators, which depend on the process covariance. These results were extended to a restricted class of processes with exactly K q.m. derivatives, for all $K = 0, 1, 2, \dots,$ by Eubank, Smith and Smith. The asymptotic performance of these optimal-coefficient estimators is determined here for regular sequences of sampling designs and general processes with exactly K q.m. derivatives, $K \geq 0$. More significantly, simple nonparametric estimators based on an adjusted trapezoidal rule using regular sampling designs are introduced whose asymptotic performance is identical to that of the optimal-coefficient estimators for general processes with exactly K q.m. derivatives for all $K = 0, 1, 2, \dots$.

1. Introduction and main results. In addressing problems involving time series, such as prediction, estimation of a weighted average, estimation of regression coefficients and signal detection, one frequently has access only to observations at a finite number of sampling points, rather than over an entire observation interval, and the following questions arise. What is the best design of sampling points and how does its performance compare with other commonly used sampling designs, such as uniform sampling? How should the observations at the sampling points be used to form efficient estimators?

Specifically, we consider the problem of estimating the weighted integral of a stochastic process over a finite interval:

$$I = \int_a^b \phi(t) X(t)\, dt,$$

where $X = \{X(t),\ t \in [a, b]\}$ is a (measurable) process with mean 0 and continuous covariance function $R(s, t) = E[X(s)X(t)]$, and the weight ϕ is a known (nonrandom) function (in $L_2[a, b]$). We want to estimate I linearly, based on observations of the random process X at $n + 1$ sampling points $T_n = \{t_{i,n}\}_{i=0}^n$ over the finite interval $[a, b]$ and using coefficients $C_n = \{c_{i,n}\}_{i=0}^n$,

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Received May 1989; revised January 1991.
[1]Research supported by the Air Force Office of Scientific Research Contract No. F49620 85C 0144.
AMS 1980 *subject classifications*. Primary 60G12, 65D30; secondary 62K05, 62M99, 65B15.
Key words and phrases. Second-order process, integral approximation, regular sampling designs, weighted Euler-MacLaurin and Gregory formulae.

by

$$I_n = \sum_{i=0}^{n} c_{i,n} X(t_{i,n})$$

The performance of I_n is measured through the mean square error $E(I - I_n)^2$.

Our goal is to specify the asymptotic performance of sequences of estimators based on certain sampling designs $\{T_n\}_n$ and coefficients $\{C_n\}_n$. We are also interested in finding asymptotically optimal designs $\{T_n^*\}_n$ and estimator-coefficients $\{C_n^*\}_n$ in the following sense:

$$E(I - I_n^*)^2 \Big/ \inf_{T_n, C_n} E(I - I_n)^2 \to_n 1,$$

where the infimum is taken over all sampling designs T_n with $n + 1$ sample points and all choices of coefficients C_n. Optimal designs for fixed sample size, when they exist, are in general hard to determine.

We consider regular sequences of sampling designs $\{T_n(h) = \{t_{i,n}\}_{i=0}^n, a = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = b\}_{n=1}^\infty$ generated by a *positive* continuous density h via

$$\int_a^{t_{i,n}} h(t)\,dt = \frac{i}{n}, \qquad i = 0, 1, \dots, n, \tag{1.1}$$

that is, the sampling points of $T_n(h)$ are the i/n percentiles of h. When h is uniform over $[a, b]$, regular sampling becomes periodic sampling where the endpoints are included.

The process X is assumed to have exactly K $(= 0, 1, 2, \dots)$ quadratic mean (q.m.) derivatives and to satisfy certain regularity conditions stated and discussed after the main results. We put $\alpha_K(t) = R^{(K,K+1)}(t, t-) - R^{(K,K+1)}(t, t+)$ (≥ 0).

The optimal-coefficient estimators minimize the mean square error $E(I - I_n)^2$ for fixed sampling design T_n. They are of the form

$$\hat{I}_n(h) = f_n' R_n^{-1} X_n, \tag{1.2}$$

where $X_n' = (X(t_{0,n}), \dots, X(t_{n,n}))$, $R_n = \{R(t_{i,n}, t_{j,n})\}_{i,j=0}^n$: $(n+1) \times (n+1)$ covariance matrix assumed nonsingular, $f_n' = (f(t_{0,n}), \dots, f(t_{n,n}))$ and $f(t) = \int_a^b R(s,t)\phi(s)\,ds$. Theorem 1 specifies their asymptotic performance under appropriate regularity conditions. B_m is the mth Bernoulli number.

THEOREM 1. *If Assumption A_K' is satisfied, then*

$$n^{2K+2} E\left[I - \hat{I}_n(h)\right]^2 \to_n \frac{|B_{2K+2}|}{(2K+2)!} \int_a^b \frac{\phi^2(t)\alpha_K(t)}{h^{2K+2}(t)}\,dt. \tag{1.3}$$

In particular, if h^* is proportional to $(\alpha_K \phi^2)^{1/(2K+3)}$, then the sequence of designs $\{T_n(h^*)\}_n$ has asymptotic performance

$$
\begin{aligned}
&n^{2K+2} E\left[I - \hat{I}_n(h^*)\right]^2 \\
(1.4) \qquad &\rightarrow_n \frac{|B_{2K+2}|}{(2K+2)!}\left\{\int_a^b \left[\alpha_K(t)\phi^2(t)\right]^{1/(2K+3)} dt\right\}^{2K+3}
\end{aligned}
$$

with minimal value of the asymptotic constant.

Assumption A'_K used in Theorem 1 (which is slightly stronger than Assumption A_K used in Theorem 2) is satisfied by a large class of processes such as Kth-order iterated integrals of a Wiener process and stationary processes with rational spectral density and exactly K q.m. derivatives.

Theorem 1, together with the asymptotic optimality of the sequence of designs $\{T_n(h^*)\}_n$, was shown for $K = 0$ and 1 by Sacks and Ylvisaker (1966, 1968, 1970a, 1970b) under slightly different regularity conditions, and for general K by Eubank, Smith and Smith (1982) but for a more restrictive class of covariances including the Kth-order iterated integrals of a Wiener process. Theorem 1 provides further support to the conjecture of Eubank, Smith and Smith (1982) that the sequence of designs $\{T_n(h^*)\}_n$ is asymptotically optimal for general K and general processes satisfying conditions such as those in Theorem 1 or in Sacks and Ylvisaker.

These optimal-coefficient estimators require the inversion of the covariance matrix R_n, which, for large sample size, may lead to numerical instabilities. More crucially, they also require precise knowledge of the covariance function R, and hence they are not robust. When the covariance function is not known precisely or when an estimate is used, the performance of the resulting "optimal-coefficient" estimator may be inferior to that in Theorem 1. The work of Stein (1988) suggests that if the covariance used is incorrect (through misspecification or estimation) but is compatible (in an appropriate sense) with the true covariance, then the resulting optimal estimator has the same asymptotic performance as in Theorem 1. However, in most cases, estimated (or misspecified) covariances are not likely to be compatible (in the desired sense) to the correct covariance and the asymptotic performance of the resulting estimators is not currently known.

For processes with exactly K q.m. derivatives ($K = 0, 1, 2, \ldots$) we now introduce simple nonparametric estimators using regular sampling designs generated by a positive density h. These estimators are not generally optimal for fixed sampling design $T_n(h)$ but instead they are based on an adjusted trapezoidal rule for integral approximation (cf. Proposition 3). For $K \geq 1$, they are of the form

$$
\begin{aligned}
I_n(h) = &\frac{1}{n}\left\{\frac{1}{2}\frac{\phi(a)}{h(a)}X(a) + \sum_{i=1}^{n-1}\frac{\phi(t_{i,n})}{h(t_{i,n})}X(t_{i,n}) + \frac{1}{2}\frac{\phi(b)}{h(b)}X(b)\right\} \\
(1.5) \qquad &-\sum_{j=1}^{K}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}\left\{Y_{(j)}(b) - Y_{(j)}(a)\right\},
\end{aligned}
$$

where the first term is the trapezoidal rule for integral approximation and the second, correction, term involves the K existing q.m. derivatives of X at the endpoints a, b via

$$(1.6) \qquad Y_{(1)} = \frac{1}{h}\left(\frac{\phi}{h}X\right)^{(1)}, \qquad Y_{(j)} = \frac{1}{h}Y_{(j-1)}^{(1)}, \qquad j = 2, \dots, K.$$

For $K = 0$, no correction term is necessary and the estimators $I_n(h)$ are given by the trapezoidal term. The asymptotic performance of these simple-coefficient estimators is identical to that of the optimal-coefficient estimators.

THEOREM 2. *Under Assumption A_K, $I_n(h)$ of (1.5) satisfies* (1.3).

In particular, if h^* is proportional to $(\alpha_K \phi^2)^{1/(2K+3)}$, then the asymptotic performance of $I_n(h^*)$ is as in (1.4). When h is uniform over $[a, b]$, then $T_n(h)$ is a periodic sampling design with asymptotic performance

$$(1.7) \quad n^{2K+2}E[I - I_n(\text{unif.})]^2 \to_n \frac{|B_{2K+2}|}{(2K+2)!}(b-a)^{2K+2}\int_a^b \alpha_K(t)\phi^2(t)\,dt$$

and the asymptotic constant is generally larger than the one in (1.4). $I_n(h^*)$ will be asymptotically optimal if the conjectured asymptotic optimality of the optimal-coefficient estimators $\hat{I}_n(h^*)$ is true. Therefore for $K = 0$ and 1 the sequence of estimators $I_n(h^*)$ is asymptotically optimal (under Assumption A_K and those required by Sacks and Ylvisaker).

The plain trapezoidal rule (i.e., without the correction term) is not asymptotically optimal when $K \geq 1$ as follows from Cambanis and Masry (1988), where it is shown that when $K = 1$ it has rate of convergence n^{-4} but larger asymptotic constant. When $K = 0$ (no q.m. derivative) an asymptotic performance as in Theorem 2 was established by Schoenfelder (1978) for the simple-coefficient estimators based on the median rule instead of the trapezoidal rule; see also Cambanis (1985).

These simple-coefficient estimators do not require precise knowledge of the covariance function R, other than the exact number K of its (K, K) mixed partial derivatives, and hence they are fairly robust. They are also numerically stable in view of their simple form. However, they use the q.m. derivatives of X at the endpoints a, b, and these frequently cannot be observed in practice. Therefore, it is necessary to estimate these q.m. derivatives from the samples in such a way as to preserve the rate of convergence of the mean square error and also the asymptotic constant.

The estimators derived from the approximation of the q.m. derivatives based on Newton's finite difference formulae (Proposition 5), that is, from Gregory's formula (Proposition 7), and the regular sampling points of $T_n(h)$

are of the form

$$
(1.8)\quad \begin{aligned}
\bar{I}_n(h) &= \frac{1}{n}\left\{\frac{1}{2}\frac{\phi(a)}{h(a)}X(a) + \sum_{i=1}^{n-1}\frac{\phi(t_{i,n})}{h(t_{i,n})}X(t_{i,n}) + \frac{1}{2}\frac{\phi(b)}{h(b)}X(b)\right\} \\
&\quad - \frac{1}{n}\sum_{j=1}^{K} C_j\left\{\Delta^j\left[\frac{\phi(t_{n-j,n})}{h(t_{n-j,n})}X(t_{n-j,n})\right] + (-1)^j\Delta^j\left[\left(\frac{\phi(a)}{h(a)}X(a)\right]\right]\right\},
\end{aligned}
$$

where Δ^j denotes jth-order difference,

$$
\Delta^j g(t_{i,n}) = \sum_{r=0}^{j}(-1)^{j-r}\binom{j}{r}g(t_{i+r,n}), \qquad 0 \le i+j \le n,
$$

and where the constants C_j are defined in Proposition 7. For $j \ge 1$, they can be written in the form

$$
(1.9)\qquad \bar{I}_n(h) = \frac{1}{n}\sum_{i=0}^{n} a_i\left(\frac{\phi X}{h}\right)(t_{i,n}),
$$

where the coefficients a_i, $i = 0, \ldots, n$, are symmetric and given by

$$
(1.10)\quad a_i = \begin{cases}
\frac{1}{2} - \sum_{j=1}^{K} C_j, & \text{for } i = 0, \\
1 + (-1)^{i+1}\sum_{j=1}^{K}\binom{j}{i}C_j, & \text{for } 1 \le i \le K, \\
1, & \text{for } K+1 \le i \le n-K-1, \\
a_{n-i}, & \text{for } n-K \le i \le n.
\end{cases}
$$

For example, the values of a_i, $i = 0, \ldots, n$, for $K = 0, 1, 2, 3, 4$ (and appropriately large n) are as follows:

$K = 0$	$\frac{1}{2}$, 1, 1, 1, 1, 1, …, 1, 1, 1, 1, 1, $\frac{1}{2}$;
$K = 1$	$\frac{5}{12}$, $\frac{13}{12}$, 1, 1, 1, 1, …, 1, 1, 1, 1, $\frac{13}{12}$, $\frac{5}{12}$;
$K = 2$	$\frac{3}{8}$, $\frac{7}{6}$, $\frac{23}{24}$, 1, 1, 1, …, 1, 1, 1, $\frac{23}{24}$, $\frac{7}{6}$, $\frac{3}{8}$;
$K = 3$	$\frac{251}{720}$, $\frac{299}{240}$, $\frac{211}{240}$, $\frac{739}{720}$, 1, 1, …, 1, 1, $\frac{739}{720}$, $\frac{211}{240}$, $\frac{299}{240}$, $\frac{251}{720}$;
$K = 4$	$\frac{95}{288}$, $\frac{317}{240}$, $\frac{23}{30}$, $\frac{793}{720}$, $\frac{157}{160}$, 1, …, 1, $\frac{157}{160}$, $\frac{793}{720}$, $\frac{23}{30}$, $\frac{317}{240}$, $\frac{95}{288}$.

The asymptotic performance of this sequence of simple-coefficient estimators $\bar{I}_n(h)$ is shown to be identical to that of $I_n(h)$ and therefore also of $\hat{I}_n(h)$.

THEOREM 3. *Under Assumption A_K, $\bar{I}_n(h)$ of* (1.8) *or* (1.9) *satisfies* (1.3).

Just as in Theorem 2, the sequence $\bar{I}_n(h^*)$ will be asymptotically optimal provided the conjectured asymptotic optimality of the optimal-coefficient estimators $\hat{I}_n(h^*)$ is true. We thus have asymptotic optimality for $K = 0$ and 1 for general processes satisfying Assumption A_K and those required by Sacks and Ylvisaker, and for general K for the restricted class of processes considered by Eubank, Smith and Smith.

We now state and discuss the assumptions used in the theorems. The following notation is used: $R^{(p,q)}(s,t) = \partial^{p+q}R(s,t)/\partial s^p\,\partial t^q$, $R^{(p,q)}(t,t-) = \lim_{s\uparrow t} R^{(p,q)}(t,s)$ and $R^{(p,q)}(t,t+) = \lim_{s\downarrow t} R^{(p,q)}(t,s)$.

ASSUMPTION A_K ($K = 0,1,2,\dots$). (i) $Q := R^{(K,K)}$ exists and is continuous on the square $[a,b]\times[a,b]$.

(ii) If p,q are nonnegative integers with $p+q\le 2$, then $Q^{(p,q)}(s,t)$ exists and is continuous off the diagonal of the square $[a,b]\times[a,b]$ (i.e., for $s\ne t$); at the diagonal $s=t$, it has left and right limits, that is, $Q^{(p,q)}(t,t+), Q^{(p,q)}(t,t-)$ all exist and are finite, and $\sup_{s\ne t}|Q^{(p,q)}(s,t)| < \infty$.

(iii) $\alpha_K(t) = R^{(K,K+1)}(t,t-) - R^{(K,K+1)}(t,t+)$ is positive and continuous on $[a,b]$.

(iv) ϕ and h have $K+2$ continuous derivatives on $[a,b]$.

ASSUMPTION A'_K ($K = 0,1,2,\dots$). Assume (i), (iii) and (iv) of Assumption A_K and instead of (ii), the following:

(ii′) Each $R^{(p,q)}(t,s)$ with $p+q = 2K$ exists on the square $[a,b]\times[a,b]$, has continuous mixed partial derivatives up to order 2 off the diagonal ($t\ne s$) and has left and right derivatives at the diagonal ($t=s$), that is $R^{(p,q)}(t+,t)$ and $R^{(p,q)}(t-,t)$ exist and are finite for all $p+q = 2K+1$, and $\sup_{s\ne t}|R^{(0,2K+1)}(s,t)| < \infty$. For each $t\in[a,b]$, $R^{(0,2K+2)}(\cdot,t+)\in H(R)$, the reproducing kernel Hilbert space of the covariance R with norm $\|\cdot\|_R$, and $\sup_t\|R^{(0,2K+2)}(\cdot,t+)\|_R < \infty$.

Part (i) of Assumption A_K is the necessary and sufficient condition for the process X to have K continuous q.m. derivatives. Parts (ii) and (ii′) are smoothness conditions off the diagonal and thus they are weak. Part (iii) guarantees the process X has no more than K q.m. derivatives. Assumptions (i), (ii) [or (ii′)] and (iii) are satisfied by a large class of processes including Kth-order iterated integrals of a Wiener process and stationary processes with rational spectral densities. When X is stationary, then conditions (ii) and (iii) are satisfied if and only if $R^{(2K+2)}(t)$ exists and is continuous for $t\ne 0$ and $R^{(2K+1)}(0\pm)$ exist and are finite and the jump $\alpha_K(t) = R^{(2K+1)}(0-) - R^{(2K+1)}(0+) = \alpha_K$ is positive.

Sacks and Ylvisaker considered also the case where the process $X(t)$ together with its existing q.m. derivatives $X^{(1)}(t),\dots,X^{(K)}(t)$ are used at the sampling points. The corresponding optimal-coefficient estimators have the same rate of convergence but smaller asymptotic constant. Simple-coefficient estimators using the K q.m. derivatives of X at the sampling points can also be constructed based on the trapezoidal rule with a correction term that depends on the values of the q.m. derivatives $X^{(1)}(t),\dots,X^{(K)}(t)$ at all sampling points and have the same asymptotic performance as the optimal-coefficient estimators using all existing q.m. derivatives at the sampling points [see

Benhenni and Cambanis (1990)]. Although these simple estimators have better performance than those in Theorem 3, they are impractical for applications where q.m. derivatives cannot be observed.

In Section 2 we develop versions of classical results on the approximation of integrals of nonrandom functions based on regular sampling points instead of the classical periodic samples. They are used in the proofs of the theorems, which are given in Section 3.

The following example compares, for a stationary second-order Markov process, the finite sample size performance of the optimal-coefficient and the simple-coefficient estimators under both uniform sampling and the sampling designs $T_n(h^*)$, which from now on we refer to for simplicity as asymptotically optimal.

EXAMPLE. We consider the estimation of the integral

$$I = \int_0^1 e^{\beta t} X(t)\, dt,$$

where the stationary process X has covariance $R(t,s) = (1 + \alpha|t-s|)e^{-\alpha|t-s|}$, with $\alpha > 0$, and spectral density $\varphi(\lambda) = (2\alpha^3/\pi)(\alpha^2 + \lambda^2)^{-2}$. The process X has exactly 1 q.m. derivative, that is, $K = 1$. When $\beta \neq 0$, the density that generates the asymptotically optimal sampling design is $h^*(t) = (2\beta/5)(e^{2\beta/5} - 1)^{-1} e^{2\beta t/5}$, $0 \le t \le 1$, and the corresponding sampling points are

$$t^*_{i,n} = (5/2\beta)\ln\left[1 + (e^{2\beta/5} - 1)i/n\right], \qquad i = 0, 1, \dots, n.$$

The simple-coefficient estimators with asymptotically optimal and uniform sampling designs, $\bar{I}_n(h^*)$ and $\bar{I}_n(\text{unif.})$, are given by (1.9), where the coefficients a_i of (1.10) take the values shown on the line $K = 1$ in the list following (1.10). The sample size of each design T_n is $N = n + 1$.

When β is close to zero, then $h^*(t)$ becomes close to the uniform density, and thus no interesting comparison can be made between the uniform and the asymptotically optimal sampling designs. (For $\beta < 0$, the normalized mean square errors are too small for a wide range of values of α even when $n = 2$.) We thus choose a moderate positive value of $\beta = 3$, so as to be able to distinguish between the two sampling designs. The asymptotic constant in (1.3) when $h = h^*$ or $h = \text{unif.}$ takes the following values $[\alpha_1(t) = 4\alpha^3$, $B_4 = -1/30]$:

$$C^* = \alpha^3 \gamma^{-5}/180, \qquad C^u = \alpha^3(e^{2\beta} - 1)\beta^{-1}/360.$$

The improvement provided by using the asymptotically optimal design over the uniform design becomes significant as β increases, since for large sample sizes the number of samples N^* (resp., N^u) required for a given mean square error when using the asymptotic optimal (resp., the uniform) design are

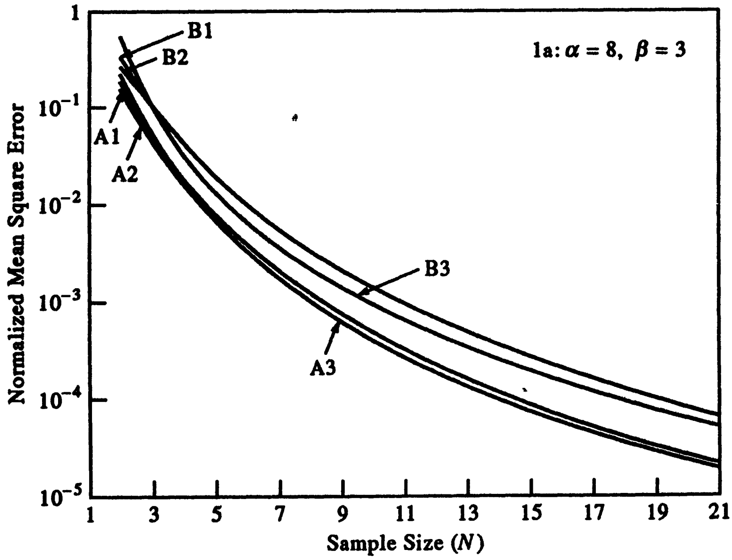

FIG. 1. *Normalized mean square error versus sample size*: A *denotes asymptotically optimal sampling design*; B *denotes uniform sampling design*; 1 *denotes simple-coefficient estimators*; 2 *denotes optimal-coefficient estimators*; 3 *denotes asymptotic expressions*.

related by

$$\frac{N^* - 1}{N^u - 1} = \left\{\frac{C^*}{C^u}\right\}^{1/4} = \left\{\frac{2\beta}{(e^{2\beta} - 1)\gamma^5}\right\}^{1/4} = 0.797, \quad \text{for } \beta = 3$$

(which tends to 1 as $\beta \to 0$ and to 0 as $\beta \to +\infty$).

When α is close to zero (very highly correlated samples), then the covariance matrix $\{R(t_{k,n}, t_{j,n})\}_{k=j=0}^{n}$ becomes nearly singular and the optimal-coefficient estimator is affected by this numerical instability whereas the simple-coefficient estimator is not. For small values of α (highly correlated samples), the normalized mean square errors are very small; for example, when $\alpha = 1$ and $N = 3$ the normalized mean square error is of order 10^{-3}. Whereas for larger values of α (less correlated samples) the normalized mean square errors are significantly higher; for example, when $\alpha = 8$ and $N = 3$ the normalized mean square error is of order 10^{-1}.

The normalized mean square errors $E(I - \hat{I}_n)^2/EI^2$ and $E(I - \hat{I}_n)^2/EI^2$ using optimal and uniform sampling designs, along with their asymptotic expressions $n^{-4}C^*/EI^2$ and $n^{-4}C^u/EI^2$, are plotted versus the sample size $N = 2, \dots, 21$ in Figure 1a, 1b, 1c for $\alpha = 8, 14, 20$, respectively. It is seen from these plots that (as expected) for small sample size the optimal-coefficient estimator outperforms the simple-coefficient estimator, but as n increases, the improvement in performance is negligible. Table 1 shows the number of samples N required to achieve a specified performance by the various designs,

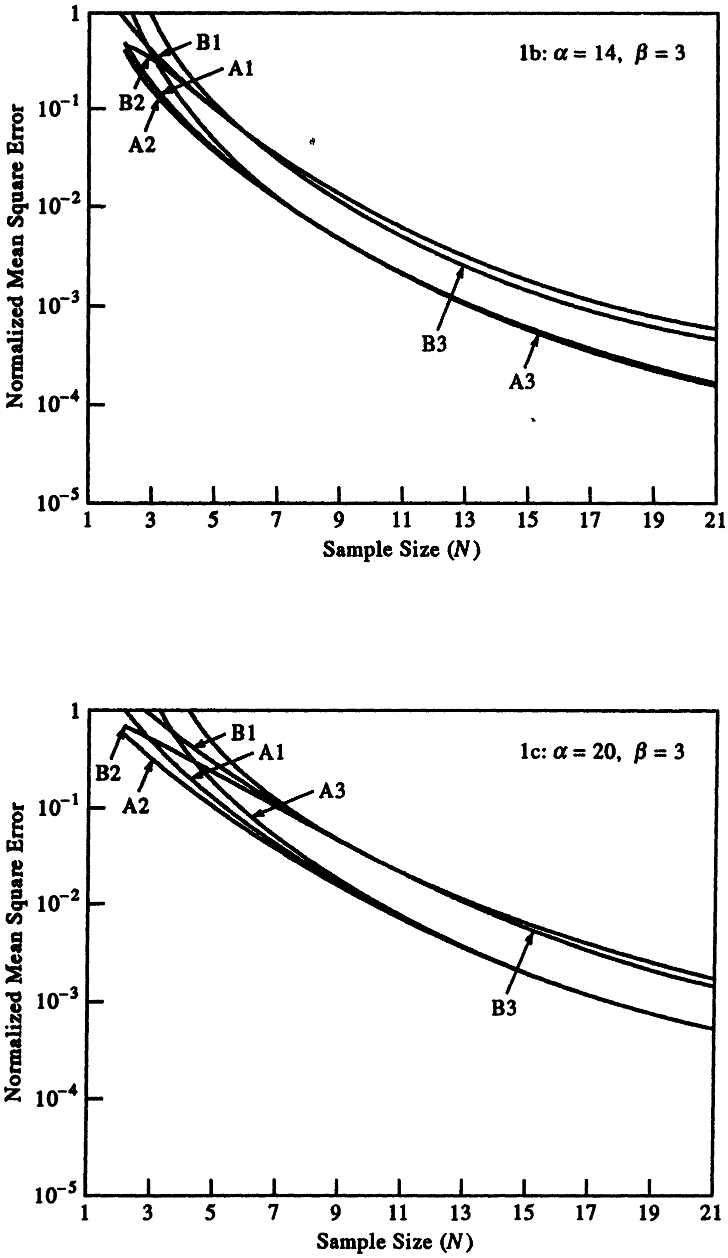

FIG. 1. *Continued.*

TABLE 1
Number of samples N required to achieve a specified normalized mean square error under different designs and estimators for $\alpha = 14$, $\beta = 3$.

Design	Estimator	Normalized mean square error		
		10^{-1}	10^{-2}	10^{-3}
Asymptotically optimal sampling	Optimal-coefficients $\hat{I}_n^*$	5	8	14
	Simple-coefficients $\bar{I}_n^*$	5	8	14
	Asymptotic expression (C^*)	5	8	14
Uniform sampling	Optimal-coefficients $\hat{I}_n$(unif.)	6	10	18
	Simple-coefficients $\bar{I}_n$(unif.)	6	10	18
	Asymptotic expression (C^u)	6	10	17

as well as the number determined by the asymptotic expressions of the mean square error, when $\alpha = 14$ and $\beta = 3$.

2. Approximation of integrals of nonrandom functions using regular sampling. We develop (in Proposition 3) the Euler–MacLaurin form for the improved accuracy trapezoidal rule when, instead of the customary periodic samples [Krylov (1962)], we use the points of a regular sampling design generated by a positive density function h. In order to achieve this, it is necessary to develop "h-weighted" versions of Taylor's expansion (in Proposition 1) and of the expansion in terms of Bernoulli polynomials (in Proposition 2).

We will use the h-weighted derivatives. If the function f and the density h have m continuous derivatives, then the h-weighted derivatives of f are defined by

$$f_{(0)} = f/h, \qquad f_{(j)} = f_{(j-1)}^{(1)}/h, \quad \text{for } j = 1, 2, \dots, m.$$

The probability function $H(x, y)$ is defined by

$$H(x,y) = \begin{cases} \int_x^y h(t)\,dt, & \text{for } a \le x \le y \le b, \\ -H(y,x), & \text{for } a \le y \le x \le b. \end{cases}$$

It follows that, for all $x \neq y$ and $j \geq 0$,

$$\frac{\partial}{\partial x} H(x,y) = -h(x), \qquad \int_x^y H^j(t,y) h(t)\,dt = \frac{1}{j+1} H^{j+1}(x,y).$$

In the particular case where h is the uniform density on $[a, b]$, $h(t) = (b-a)^{-1}$, then $f_{(j)} = (b-a)^{j+1} f^{(j)}$, $j \geq 0$, and $H(x, y) = |y - x|/(b - a)$.

PROPOSITION 1 (Weighted Taylor expansion). *If f and h have m (≥ 1) continuous derivatives on $[a, b]$, then, for all $t, x \in [a, b]$,*

$$f_{(0)}(t) = \sum_{j=0}^{m-1} \frac{1}{j!} H^j(x,t) f_{(j)}(x) + \frac{1}{(m-1)!} \int_x^t H^{m-1}(u,t) f_{(m)}(u) h(u)\, du.$$

PROOF. The result follows by repeated integration by parts of the remainder. □

Proposition 2 gives the h-weighted expansion of a function f in terms of the periodic extensions $B_j^*(x)$, $-\infty < x < \infty$, of the usual Bernoulli polynomials $B_j(x)$, $0 \leq x \leq 1$.

PROPOSITION 2 (Weighted expansion in Bernoulli polynomials). *If f and h have m (≥ 1) continuous derivatives on $[a, b]$, then, for any $z \in [x, y] \subset [a, b]$, we have*

$$\begin{aligned} f_{(0)}(z) = {} & \frac{1}{H(x,y)} \int_x^y f_{(0)}(t) h(t)\, dt \\ & + \sum_{j=1}^{m-1} \frac{1}{j!} H^{j-1}(x,y) B_j\left(\frac{H(x,z)}{H(x,y)} \right) [f_{(j-1)}(y) - f_{(j-1)}(x)] \\ & - \frac{1}{m!} H^{m-1}(x,y) \int_x^y f_{(m)}(t) \left[B_m^*\left(\frac{H(t,z)}{H(x,y)} \right) - B_m^*\left(\frac{H(x,z)}{H(x,y)} \right) \right] h(t)\, dt. \end{aligned}$$

For $m = 1$ the second term is not present.

PROOF. Consider the following (integral) remainder:

$$\begin{aligned} \rho_m(z) & = \frac{1}{m!} H^{m-1}(x,y) \int_x^y f_{(m)}(t) B_m^*\left(\frac{H(t,z)}{H(x,y)} \right) h(t)\, dt \\ & = \frac{1}{m!} H^{m-1}(x,y) \int_x^y B_m^*\left(\frac{H(t,z)}{H(x,y)} \right) f'_{(m-1)}(t)\, dt. \end{aligned}$$

For $m > 1$, integrating by parts $(m + 1)$ times and using $H(y, z) = H(x, z) - H(x, y) \leq 0$, we obtain

$$\begin{aligned} \rho_m(z) = {} & \sum_{j=1}^{m} \frac{1}{j!} H^{j-1}(x,y) B_j\left(\frac{H(x,z)}{H(x,y)} \right) [f_{(j-1)}(y) - f_{(j-1)}(x)] \\ & - f_{(0)}(z) + \frac{1}{H(x,y)} \int_x^y f_{(0)}(t) h(t)\, dt; \end{aligned}$$

writing $f_{(m-1)}(y) - f_{(m-1)}(x) = \int_x^y f'_{(m-1)}(t)\, dt = \int_x^y f_{(m)}(t) h(t)\, dt$ yields the final expression in Proposition 2. □

PROPOSITION 3 (Weighted Euler–MacLaurin formula for regular sampling). *If f and h have m (≥ 1) continuous derivatives on $[a, b]$, and if $\{T_n(h)\}_{n=1}^{\infty}$ is a regular sequence of sampling designs, then for $m \geq 2$ we have*

$$\int_a^b f(t)\,dt = \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_{(0)}(t_{i,n}) + f_{(0)}(t_{i+1,n})\right] - \sum_{j=1}^{m-2}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}\left[f_{(j)}(b) - f_{(j)}(a)\right] + \frac{1}{m!n^m}\sum_{i=0}^{n-1}\int_{t_{i,n}}^{t_{i+1,n}} f_{(m)}(t)\left[B_m(nH(t, t_{i+1,n})) - B_m\right]h(t)\,dt,$$

and for $m = 1$ we have

$$\int_a^b f(t)\,dt = \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_{(0)}(t_{i,n}) + f_{(0)}(t_{i+1,n})\right] + \frac{1}{n}\sum_{i=0}^{n-1}\int_{t_{i,n}}^{t_{i+1,n}} f_{(1)}(t)B_1(nH(t, t_{i+1,n}))h(t)\,dt.$$

PROOF. For simplicity of notation we will write t_i for $t_{i,n}$. We first split the integral into the following sum: $\int_a^b f(t)\,dt = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(t)\,dt$. Applying Proposition 2 in each subinterval $[t_i, t_{i+1}]$ [and using $H(t_i, t_{i+1}) = 1/n$ by (1.1)], first with $z = t_i$ then with $z = t_{i+1}$, and averaging the two expressions, the final result is obtained, since $B_1(0) = -1/2$, $B_1(1) = 1/2$ and $B_j(0) = B_j(1) = B_j$ for $j > 1$ and $B_m^*(nH(t, t_i)) = B_m(nH(t, t_{i+1}))$. □

The rate of convergence of the error in the approximation of the integral $I(f) = \int_a^b f(t)\,dt$ by the corrected trapezoidal rule $I_n(f; h, m)$ of degree m based on a regular sampling design $T_n(h)$ can be derived from Proposition 3, with the following notation:

$$I_n(f; h, m) = \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_{(0)}(t_{i,n}) + f_{(0)}(t_{i+1,n})\right] - \sum_{j=1}^{m-2}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}\left[f_{(j)}(b) - f_{(j)}(a)\right],$$

for $m \geq 2$; when $m = 0, 1$ no correction is necessary and $I_n(f; h, m)$ is given by the trapezoidal term. Then

$$n^m\left[I(f) - I_n(f; h, m)\right] \to_n \begin{cases} -\dfrac{B_m}{m!}\left[f_{(m-1)}(b) - f_{(m-1)}(a)\right], & \text{for } m \geq 2, \\ 0, & \text{for } m = 0, 1. \end{cases}$$

Approximation of the h-weighted derivatives. The Euler–MacLaurin formula of Proposition 3 uses h-weighted derivatives of f at the endpoints a, b.

In order to approximate these derivatives by values of f at the sampling points, we develop a generalized version of the forward and backward Newton's finite difference formulae based on regular sampling, rather than the customary periodic sampling (Propositions 4 and 5). This gives rise to Gregory's formula for a regular sampling design (Proposition 6).

We will use the h-weighted divided differences of a function f on $[a, b]$, which are defined for any $x_0, x_1, \dots, x_i$ in $[a, b]$ as follows:

$$f_{(0)}(x_0, x_1, \dots, x_i) = \frac{f_{(0)}(x_1, \dots, x_i) - f_{(0)}(x_0, \dots, x_{i-1})}{H(x_0, x_i)}, \qquad i \geq 1.$$

When $(x_0, \dots, x_i) = (t_{0,n}, \dots, t_{i,n})$, $0 \leq i \leq n$, where $\{T_n(h) = (t_{0,n}, \dots, t_{n,n})\}_{n=1}^{\infty}$, since $H(t_{j,n}, t_{k,n}) = (k - j)/n$ for $0 \leq j \leq k \leq n$, we have

$$f_{(0)}(t_{0,n}, t_{1,n}, \dots, t_{i,n}) = \frac{n^i}{i!} \Delta^i f_{(0)}(t_{0,n}),$$

where $\Delta^i f_{(0)}(t_{0,n})$ are the usual finite differences. We will also use the "classical" polynomial $W_i(\cdot)$ in Newton's interpolation formula when using a periodic sampling, where $W_0(u) = 1$ and, for $i \geq 1$,

$$W_i(u) = u(u-1) \cdots (u - i + 1).$$

PROPOSITION 4 (Newton's interpolation formulae for regular sampling). *If $\{T_n(h)\}_{n=1}^{\infty}$ is a regular sequence of sampling designs and f is a continuous function on $[a, b]$, then, for $0 \leq m \leq n$ and $t \in [a, b]$, we have the following interpolation formulae*:

(*Forward*)

$$\begin{aligned} f_{(0)}(t) = {} & \sum_{i=0}^{m} \frac{1}{i!n^i} W_i[nH(t_{0,n}, t)]\, \Delta^i f_{(0)}(t_{0,n}) \\ & + \frac{1}{n^{m+1}} W_{m+1}[nH(t_{0,n}, t)]\, f_{(0)}(t, t_{0,n}, \dots, t_{m,n}); \end{aligned}$$

(*Backward*)

$$\begin{aligned} f_{(0)}(t) = {} & \sum_{i=0}^{m} \frac{(-1)^i}{i!n^i} W_i[-nH(t_{n,n}, t)]\, \Delta^i f_{(0)}(t_{n-i,n}) \\ & + \frac{(-1)^{m+1}}{n^{m+1}} W_{m+1}[nH(t, t_{n,n})]\, f_{(0)}(t, t_{n,n}, \dots, t_{n-m+1,n}). \end{aligned}$$

PROOF. The definition of the h-weighted divided differences yields the expression of $f_{(0)}(t)$ in terms of the successive divided differences at the sampling points

$$f_{(0)}(t) = \sum_{i=0}^{m} P_{i,n}(t)\, f_{(0)}(t_{0,n}, \dots, t_{i,n}) + P_{m+1,n}(t)\, f_{(0)}(t, t_{0,n}, \dots, t_{m,n}),$$

where $P_{i,n}(t) = H(t_{0,n}, t) \cdots H(t_{i-1,n}, t) = n^{-i} W_i[nH(t_{0,n}, t)]$, $1 \leq i \leq n + 1$,

$P_{0,n}(t) = 1$. The forward formula follows from the expression of h-weighted divided differences in terms of finite differences. The same procedure can be used to derive the backward formula. □

PROPOSITION 5 (Approximation of weighted derivatives for regular sampling). *Let $\{T_n(h)\}_{n=1}^{\infty}$ be a regular sequence of sampling designs, and f, h have m continuous derivatives.*

(*Forward formula.*) *For $t \notin (t_{0,n}, t_{m-1,n})$ and $0 \le j \le m-1 \le n$ we have for some $\xi_t = \xi_t(j,m) \in (t_{0,n}, \max(t, t_{m-1,n}))$,*

$$f_{(j)}(t) = n^j \sum_{i=j}^{m-1} \frac{1}{i!} W_i^{(j)}[nH(t_{0,n}, t)]\, \Delta^i f_{(0)}(t_{0,n}) + \frac{1}{m!\,n^{m-j}} W_m^{(j)}[nH(t_{0,n}, t)]\, f_{(m)}(\xi_t).$$

(*Backward formula.*) *For $t \notin (t_{n-m+1,n}, t_{n,n})$ and $0 \le j \le m-1 \le n$ we have for some $\eta_t = \eta_t(j,m) \in (\min(t, t_{n-m+1,n}), t_{n,n})$,*

$$f_{(j)}(t) = n^j \sum_{i=j}^{m-1} \frac{(-1)^{i+j}}{i!} W_i^{(j)}[nH(t, t_{n,n})]\, \Delta^i f_{(0)}(t_{n-i,n}) + \frac{(-1)^{m-j}}{m!\,n^{m-j}} W_m^{(j)}[nH(t, t_{n,n})]\, f_{(m)}(\eta_t).$$

PROOF. For simplicity we will write t_i for $t_{i,n}$ and P_i for $P_{i,n}$, and we denote the h-weighted differential operator by d_j: $d_j f = f_{(j)}$. Taking the jth h-weighted derivative in Newton's forward formula of Proposition 4, we obtain

$$f_{(j)}(x) = n^j \sum_{i=j}^{m-1} \frac{1}{i!} W_i^{(j)}[nH(t_0, t)]\, \Delta^i f_{(0)}(t_0) + d_j\{P_m(x) f_{(0)}(x, t_0, \ldots, t_{m-1})\}.$$

The function $g(x, C) = P_m(x) f_{(0)}(x, t_0, \ldots, t_{m-1}) - C P_m(x)/m!$, where C is a constant, vanishes at least $m+1$ times in $[t_0, t_{m-1}]$, since $P_m(\cdot)$ has exactly m roots $t_0, t_1, \ldots, t_{m-1}$. Applying Rolle's theorem j times, $1 \le j \le m-1$, we obtain that each $g_{(j)}(x, C)$ vanishes at least $m-j$ times in (t_0, t_{m-1}). Now, fix $t \notin (t_0, t_{m-1})$ (but in $[a, b]$). Then $d_j P_m(t) \ne 0$, and thus there exists a value $C_t = C_t(j, m)$ such that $g_{(j)}(t, C_t) = 0$. Let $g(x) = g(x, C_t)$. Then $g_{(j)}(\cdot)$ has at least $m - j + 1$ roots in $(t_0, \max(t, t_{m-1}))$ and we have $g_{(m)}(x) = d_{m-j}\{g_{(j)}(x)\} = d_m\{P_m(x) f_{(0)}(x, t_0, \ldots, t_{m-1})\} - C_t$, since $d_m P_m(x) = W_m^{(m)}[nH(t_0, x)] = m!$. Taking the mth h-weighted derivative in the forward formula of Proposition 4 and since $d_m P_i(x) = 0$ for $0 \le i \le m-1$, we have $f_{(m)}(x) = d_m f_{(0)}(x) = d_m\{P_m(x) f_{(0)}(x, t_0, \ldots, t_{m-1})\}$. Since $g_{(j)}(\cdot)$ has at least $m - j + 1$ roots in $(t_0, \max(t, t_{m-1}))$, repeated use of Rolle's theorem ($m-j$ times) implies that $g_{(m)}(\cdot)$ vanishes at least once in $(t_0, \max(t, t_{m-1}))$. Thus, there exists $\xi_t = \xi_t(j, m) \in (t_0, \max(t, t_{m-1}))$ such that $g(m)(\xi_t) = f_{(m)}(\xi_t) - C_t = 0$, and therefore

$$d_j\{P_m(t) f_{(0)}(t, t_0, \ldots, t_{m-1})\} = f_{(m)}(\xi_t) \frac{1}{m!} d_j P_m(t).$$

The same technique can be used to derive the backward formula. □

Alternative expressions of the remainder in the forward and backward formulas in integral form are as follows:

(Forward formula)

$$-\frac{n^j}{(m-1)!}\sum_{i=j}^{m-1}\frac{1}{i!}W_i^{(j)}[nH(t_{0,n},t)]$$
$$\times\sum_{r=0}^{i}(-1)^{i-r}\binom{i}{r}\int_t^{t_{r,n}}H^{m-1}(x,t_{r,n})\,f_{(m)}(x)h(x)\,dx;$$

(Backward formula)

$$\frac{(-1)^{m-j}+1}{(m-1)!}\sum_{i=j}^{m-1}\frac{1}{i!}W_i^{(j)}[nH(t,t_{n,n})]$$
$$\times\sum_{r=0}^{i}(-1)^{i-r}\binom{i}{r}\int_{t_{n-r,n}}^{t}H^{m-1}(x,t_{n-r,n})\,f_{(m)}(x)h(x)\,dx.$$

These are obtained by expanding $\Delta^i f_{(0)}(t_{0,n})$ in Taylor series around $t_{0,n}$ as in Proposition 1 and using Lemma 4 (in Section 3 prior to the proof of Theorem 3).

PROPOSITION 6 (Weighted Gregory's formula for regular sampling). *If f and h have m continuous derivatives on $[a,b]$ and $\{T_n(h)\}_{n=1}^{\infty}$ is a regular sequence of sampling designs, then*

$$\begin{aligned}\int_a^b f(t)\,dt &= \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_{(0)}(t_{i,n})+f_{(0)}(t_{i+1,n})\right]\\
&-\frac{1}{n}\sum_{j=1}^{m-2}C_j\left[\Delta^j f_{(0)}(t_{n-j,n})+(-1)^j\Delta^j f_{(0)}(a)\right]\\
&-\frac{1}{(m-1)!n^m}\left[C_{m-1}(-1)^m-\frac{B_m}{m}\right]\\
&\quad\times\left[(-1)^m f_{(m-1)}(b)-f_{(m-1)}(a)\right]\\
&-\frac{1}{(m-1)!m!n^{m+1}}W_m^{(m-1)}(0)\left[C_{m-1}(-1)^m-\frac{B_m}{m}\right]\\
&\quad\times\left[(-1)^m f_{(m)}(\xi_{b,m})+f_{(m)}(\xi_{a,m})\right]\\
&-\frac{1}{m!n^{m+1}}\sum_{j=1}^{m-2}\frac{B_{j+1}}{(j+1)!}W_m^{(j)}(0)\left[(-1)^{m-j}f_{(m)}(\xi_{b,j})-f_{(m)}(\xi_{a,j})\right]\\
&+\frac{1}{m!n^m}\sum_{i=0}^{n-1}\int_{t_{i,n}}^{t_{i+1,n}}f_{(m)}(t)\{B_m[nH(t,t_{i+1,n})]-B_m\}h(t)\,dt,\end{aligned}$$

where $\xi_{a,j} \in (a, t_{m-1,n})$, $\xi_{b,j} \in (t_{n-m+1,n}, b)$ *and* $C_0 = 1/2$,

$$C_i := \frac{(-1)^i}{(i+1)!}\int_0^1 W_{i+1}(t)\,dt = \frac{(-1)^{i+1}}{i!}\sum_{j=1}^{i}\frac{B_{j+1}}{(j+1)!}W_i^{(j)}(0), \quad \textit{for } i \ge 1.$$

PROOF. Denote by $D(a,b)$ the second term in the Euler–MacLaurin expression of Proposition 3 (without the minus sign). Applying the formulae of Proposition 5 at the endpoints a, b, and since the B_{j+1}'s with j odd are nonzero, we obtain

$$D(a,b) = \sum_{j=1}^{m-2}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}$$
$$\times\left\{n^j\sum_{i=j}^{m-1}W_i^{(j)}(0)\frac{1}{i!}\left[(-1)^{i+1}\Delta^i f_{(0)}(t_{n-i,n}) - \Delta^i f_{(0)}(a)\right]\right.$$
$$\left.+\frac{1}{n^{m-j}}W_m^{(j)}(0)\frac{1}{m!}\left[(-1)^{m-j}f_{(m)}(\xi_{b,j}) - f_{(m)}(\xi_{a,j})\right]\right\}$$

where $\xi_{a,j} \in (a, t_{m-1})$ and $\xi_{b,j} \in (t_{n-m+1}, t_n)$. Inverting the order of summation and applying the formulae of Proposition 5 to $f_{(m-1)}(a)$ and $f_{(m-1)}(b)$, we obtain the final expression of $D(a,b)$ by showing that

$$\frac{1}{i!}\sum_{j=1}^{i}\frac{B_{j+1}}{(j+1)!}W_i^{(j)}(0) = (-1)^{i+1}C_i.$$

The latter follows by expanding W_{i+1} in terms of Bernoulli numbers (i.e., the standard version of Proposition 2 with $x = 0$, $y = 1$ and h uniform over [0, 1]), and using $W_{i+1}^{(j)}(1) - W_{i+1}^{(j)}(0) = (i+1)W_i^{(j)}(0)$. □

Integral estimators are constructed from the weighted Gregory's formula for regular sampling:

$$\bar{I}_n(f; h, m) = \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_{(0)}(t_{i,n}) + f_{(0)}(t_{i+1,n})\right]$$
$$-\frac{1}{n}\sum_{j=1}^{m-2}C_j\left[\Delta^j f_{(0)}(t_{n-j,n}) + (-1)^j\Delta^j f_{(0)}(a)\right],$$

for $m \ge 3$; for $m = 0, 1, 2$ no correction is necessary and $\bar{I}_n(f; h, m)$ is given by the trapezoidal term. Their rate of convergence is

$$n^m\left[I(f) - \bar{I}_n(f; h, m)\right]$$
$$\to_n -\frac{1}{(m-1)!}C_{m-1}\left[f_{(m-1)}(b) + (-1)^{m-1}f_{(m-1)}(a)\right].$$

3. Proofs of theorems. The most important properties of Bernoulli polynomials have been used in Propositions 2 and 3. The expressions of the following integrals are established for use in the proof of Theorem 2: with $\overline{B}_K(x) = B_K(x) - B_K$,

$$\beta_{i,K} = \int_0^1 (1-x)^i \overline{B}_K(x)\,dx,$$

$$\gamma_{1,K} = \iint_{0<x<y<1} \overline{B}_K(x)\overline{B}_{K+1}(x)\,dx\,dy,$$

$$\gamma_{2,K} = \iint_{0<x<y<1} \overline{B}_{K+1}(x)\overline{B}_K(y)\,dx\,dy.$$

LEMMA 1. *For $K \geq 1$, we have the following*:

(i) $$\beta_{0,K} = -B_K, \qquad \beta_{1,K} = -\frac{B_{K+1}}{K+1} - \frac{1}{2}B_K.$$

(ii) $$\frac{1}{(K+1)!^2}\int_0^1 B_{K+1}^2(y)\,dy = (-1)^K \frac{B_{2K+2}}{(2K+2)!}.$$

(iii) $$\gamma_{1,K} = (-1)^K K!(K+1)!\frac{B_{2K+2}}{(2K+2)!} + \frac{B_{K+1}^2}{K+1} + \frac{1}{2}B_K B_{K+1} - B_K\frac{B_{K+2}}{K+2}.$$

$$\gamma_{2,K} = -\gamma_{1,K} + \beta_{0,K}\beta_{0,K+1}.$$

PROOF. (i) Follows from integration by parts using $B_{K+1}(1) = B_{K+1}(0) = B_{K+1}$, $K \geq 1$.

(ii) For any $n > 1$, $m \geq 1$, integration by parts yields

$$\int_0^1 B_n(x)B_m(x)\,dx = \frac{-n}{m+1}\int_0^1 B_{n-1}(x)B_{m+1}(x)\,dx,$$

and (ii) is obtained by applying this relation repeatedly and using $B_1(1) = 1/2$, $B_1(0) = -1/2$.

(iii) The second part follows from $\gamma_{1,K} + \gamma_{2,K} = \beta_{0,K}\beta_{0,K+1}$. The first part follows from (i) and (ii) along with

$$\gamma_{1,K} = \int_0^1 dy\,\overline{B}_{K+1}(y)\int_0^y dx\,\overline{B}_K(x)$$
$$= \frac{1}{K+1}\int_0^1 \overline{B}_{K+1}^2(y)\,dy - B_K\int_0^1 y\overline{B}_{K+1}(y)\,dy. \qquad \square$$

We will use the following h-weighted derivatives of the covariance R:

$$R_{(p,q)}(t,s) = E\{Y_{(p)}(t)Y_{(q)}(s)\}, \quad \text{for } 0 \leq p,q \leq K \text{ and } t,s \in [a,b].$$

We also define recursively $R_{(p,q)}(t,s)$, for $\{K < p$ or $K < q$ and $0 \le p+q \le 2K$, $t, s \in [a,b]\}$ and for $\{2K+1 \le p+q \le 2K+2$, $t \ne s$ in $[a,b]\}$, as follows:

$$R_{(p,q)}(t,s) = \begin{cases} \dfrac{1}{h(s)} R^{(0,1)}_{(p,q-1)}(t,s), & K < q, \\ \dfrac{1}{h(t)} R^{(1,0)}_{(p-1,q)}(t,s), & K < p. \end{cases}$$

Lemmas 2 and 3 express the jump along the diagonal of the h-weighted derivatives $R_{(K,K+1)}$ and $R_{(0,2K+1)}$ in terms of the jump α_K of the derivative $R^{(K,K+1)}$ and will be used for the expression of the asymptotic constant in Theorem 2.

LEMMA 2. *Under Assumption A_K, we have*

$$R_{(K,K+1)}(t,t-) - R_{(K,K+1)}(t,t+) = \phi^2(t)\alpha_K(t)/h^{2K+3}(t).$$

PROOF. We have $R_{(K,K+1)}(t,t+) = \lim_{u\downarrow 0} R^{(0,1)}_{(K,K)}(t,t+u)/h(t+u)$. We can write $Y_{(i)} = h^{-i}Y^{(i)}_{(0)} + Z_{i-1}$, where the stochastic process Z_{i-1} is a linear combination of the processes $Y^{(j)}_{(0)}$, $1 \le j \le i-1$, with coefficients depending on powers and derivatives of the density h, and thus has $K-(i-1)$ q.m. derivatives. Thus, for any $t, s \in [a,b]$,

$$\begin{aligned} R_{(K,K)}(t,s) &= R^{(K,K)}_{(0)}(t,s)h^{-K}(t)h^{-K}(s) \\ &\quad + \big\{h^{-K}(t)EY^{(K)}_{(0)}(t)Z_{K-1}(s) \\ &\quad + h^{-K}(s)EY^{(K)}_{(0)}(s)Z_{K-1}(t) + EZ_{K-1}(t)Z_{K-1}(s)\big\} \end{aligned}$$

and

$$R^{(0,1)}_{(K,K)}(t,s) = R^{(K,K+1)}_{(0)}(t,s)h^{-K}(t)h^{-K}(s) + W(t,s),$$

where $W(t,s)$ is a linear combination of $R^{(i,j)}(t,s)$, $0 \le i, j \le K$, with coefficients depending on powers and derivatives of h at the points t, s, and thus is continuous on $[a,b] \times [a,b]$. It follows that

$$R_{(K,K+1)}(t,t\pm) = R^{(K,K)}_{(0)}(t,t\pm)h^{-2K-1}(t) + W(t,t)h^{-1}(t).$$

Therefore, we obtain

$$\begin{aligned} &R_{(K,K+1)}(t,t-) - R_{(K,K+1)}(t,t+) \\ &\quad = h^{-2K-1}(t)\big[R^{(K,K+1)}_{(0)}(t,t-) - R^{(K,K+1)}_{(0)}(t,t+)\big]. \end{aligned}$$

But, $R_{(0)}(t,s) = (\phi/h)(t)(\phi/h)(s)R(t,s)$ and the continuity of ϕ, h and

$R^{(i,j)}(t,s)$, $0 \le i, j \le K$, imply that

$$\begin{aligned} R_{(0)}^{(K,K+1)}(t,t-) - R_{(0)}^{(K,K+1)}(t,t+) &= \left(\frac{\phi}{h}\right)^2(t)\left[R^{(K,K+1)}(t,t-) - R^{(K,K+1)}(t,t+)\right] \\ &= \left(\frac{\phi}{h}\right)^2(t)\alpha_K(t) \end{aligned}$$

from which the final expression follows. □

LEMMA 3. *Under Assumption A'_K, we have*

$$\int_a^b\left[R_{(0,2K+1)}(t,t-) - R_{(0,2K+1)}(t,t+)\right]h(t)\,dt = (-1)^K\int_a^b \frac{\phi^2(t)\alpha_K(t)}{h^{2K+2}(t)}\,dt.$$

PROOF. For small $u > 0$, we have

$$\frac{d}{dt}R_{(0,2K)}(t,t\pm u) = h(t)R_{(1,2K)}(t,t\pm u) + h(t\pm u)R_{(0,2K+1)}(t,t\pm u),$$

and, integrating, we obtain

$$\begin{aligned} &\int_{a+u}^{b-u}\left[R_{(0,2K+1)}(t,t-u)h(t-u) - R_{(0,2K+1)}(t,t+u)h(t+u)\right]dt \\ &\quad = \sum_{j=0}^{K-1}(-1)^j\left[R_{(j,2K-j)}(t,t-u) - R_{(j,2K-j)}(t,t+u)\right]_{a+u}^{b-u} \\ &\qquad +(-1)^K\int_{a+u}^{b-u}\left[R_{(K,K+1)}(t,t-u) - R_{(K,K+1)}(t,t+u)\right]h(t)\,dt. \end{aligned}$$

Then by the dominated convergence theorem, since $\sup_{t\neq s}|R^{(K,K+1)}(t,s)| < \infty$ and $\sup_{t\neq s}|R^{(0,2K+1)}(t,s)| < \infty$, and by the continuity of $R_{(p,q)}(t,s)$ with $p+q \le 2K$, letting $u \to 0$, the result follows from Lemma 2. □

PROOF OF THEOREM 1. Let $\mathscr{R}$ be the reproducing kernel Hilbert space (RKHS) generated by the covariance kernel $R_{(0)}(t,s)$, $t, s \in [a,b]$, with inner product $\langle f(\cdot), R_{(0)}(\cdot,t)\rangle = f(t)$, $t \in [a,b]$, for $f \in \mathscr{R}$, and let $\mathscr{H}$ be the L_2-closure of the linear span of the random variables $\{Y_{(0)}(t),\ t \in [a,b]\}$. Then every $f \in \mathscr{R}$ is of the form $f(t) = E[Y_{(0)}(t)Z]$ for some $Z \in \mathscr{H}$ and the correspondence $f \leftrightarrow Z$ is an isomorphism between $\mathscr{R}$ and $\mathscr{H}$, so that if $f_i \leftrightarrow Z_i$, then $\langle f_1, f_2\rangle = E[Z_1Z_2]$ and in particular $\langle f(\cdot), R_{(0)}(\cdot,t)\rangle = E[Y_{(0)}(t)Z] = f(t)$, $t \in [a,b]$. It follows that if $f \in \mathscr{R}$ with $f(t) = E[Y_{(0)}(t)Z]$, $Z \in \mathscr{H}$, then we have

$$\left\langle f(\cdot), \int_a^b R_{(0)}(\cdot,s)h(s)\,ds\right\rangle = E\left[Z\int_a^b Y_{(0)}(s)h(s)\,ds\right] = \int_a^b f(s)h(s)\,ds,$$

$$\langle f(\cdot), R_{(0,j)}(\cdot,t)\rangle = E\left[Y_{(j)}(t)Z\right] = f_{(j)}(t), \qquad t \in [a,b], 0 \le j \le K.$$

where the last equality is proven by induction, using the definitions of the weighted derivative $f_{(j)}(t)$ and weighted q.m. derivative $Y_{(j)}(t)$.

Let P_n be the projection from the Hilbert space $\mathscr{R}$ onto $\overline{\text{sp}}\{R_{(0)}(\cdot, t), t \in T_n\}$. In view of the isomorphism between $\mathscr{R}$ and $\mathscr{H}$, with $f(t) = \int_a^b R_{(0)}(t, s)h(s)\,ds$, we have

$$E(I - \hat{I}_n)^2 = \| f - P_n f \|^2 = \langle f - P_n f, f \rangle = \int_a^b (f - P_n f)(s) h(s)\, ds.$$

Let $g = f - P_n f$. Note that $g(t_{i,n}) = 0$ and, by Assumption A'_K, g has $2K$ continuous derivatives on (a, b) and $2K + 2$ continuous derivatives on each subinterval $(t_{i,n}, t_{i+1,n})$. An inspection of the proof of Proposition 3 makes clear that the weighted Euler–MacLaurin formula with $m = 2K + 2$ is applicable [as only the everywhere continuous derivatives of order up to $2K$ are involved in the expression of each integral over $(t_{i,n}, t_{i+1,n})$ and therefore cancel upon summation]. Using the expression of its second term $D(a, b)$ given in the proof of Proposition 6, gives

$$\begin{aligned} E(I - \hat{I}_n)^2 &= \int_a^b g(s) h(s)\, ds \\ &= D(a,b) + \frac{1}{(2K+2)!\,n^{2K+2}} \\ &\quad \times \sum_{i=0}^{n-1} \int_{t_{i,n}}^{t_{i+1,n}} g_{(2K+2)}(t) \{ B_{2K+2}[nH(t, t_{i+1,n})] - B_{2K+2} \} h(t)\, dt, \end{aligned} \tag{3.1}$$

where, in view of the remark following Proposition 5,

$$\begin{aligned} D(a,b) = \frac{1}{(2K+1)!\,n} \sum_{j=1}^{2K} \frac{B_{j+1}}{(j+1)!} \sum_{i=j}^{2K+1} \frac{1}{i!} W_i^{(j)}(0) \sum_{r=0}^{i} (-1)^{i-r} \binom{i}{r} \\ \times \Bigg\{ (-1)^j \int_{t_{n-r,n}}^{b} H^{2K+1}(x, t_{n-r,n}) g_{(2K+2)}(x) h(x)\, dx \\ - \int_a^{t_{r,n}} H^{2K+1}(x, t_{r,n}) g_{(2K+2)}(x) h(x) \Bigg\}. \end{aligned}$$

For the *first term* $D(a, b)$, since $(P_n f)_{(2K+2)}(t)$ [and thus $g_{(2K+2)}(t)$] does not exist at $t \in T_n$, we write the integrals as sums of integrals in between successive points of T_n and applying the mean value theorem we obtain

$$\begin{aligned} D(a,b) = \frac{1}{(2K+2)!\,n^{2K+3}} \sum_{j=1}^{2K} \frac{B_{j+1}}{(j+1)!} \sum_{i=j}^{2K+1} \frac{1}{i!} W_i^{(j)}(0) \\ \times \sum_{r=1}^{i} (-1)^{i-r} \binom{i}{r} \sum_{l=1}^{r} \left[l^{2K+2} - (l-1)^{2K+2} \right] \\ \times \left\{ (-1)^j g_{(2K+2)}(\xi'_l) - g_{(2K+2)}(\xi_l) \right\}, \end{aligned}$$

where $t_{l-1,n} < \xi_l < t_{l,n}$ and $t_{n-r+l-1,n} < \xi'_l < t_{n-r+l,n}$. From Assumption A'_K, we have

$$f_{(2K+1)}(t) = \int_a^t R_{(0,2K+1)}(s,t+)h(s)\,ds + \int_t^b R_{(0,2K+1)}(s,t-)h(s)\,ds$$

and, differentiating, we obtain

$$f_{(2K+2)}(t) = -\beta_K(t) + \int_a^b R_{0,2K+2)}(s,t+)h(s)\,ds,$$

where

$$\beta_K(t) = R_{(0,2K+1)}(t,t-) - R_{(0,2K+1)}(t,t+).$$

From (ii′) of Assumption A'_K, $R_{(0,2K+2)}(\cdot,t+) \in \mathscr{R}$ and, since $P_n f \in \operatorname{sp}\{R_{(0)}(\cdot,t),\ t \in T_n\}$,

$$\langle P_n f, R_{(0,2K+2)}(\cdot,t+)\rangle = (P_n f)_{(2K+2)}(t+) = (P_n f)_{(2K+2)}(t), \quad \text{for } t \notin T_n.$$

Writing $R_{(0,2K+2)}(\cdot,t+) = EY_{(0)}\xi_t$, $\xi_t \in \mathscr{H}$, we have

$$\begin{aligned}\langle f, R_{(0,2K+2)}(\cdot,t+)\rangle &= \int_a^b E[Y_{(0)}(s)\xi_t]h(s)\,dx\\ &= \int_a^b R_{(0,2K+2)}(s,t+)h(s)\,ds.\end{aligned}$$

It then follows that, for $t \notin T_n$,

$$g_{(2K+2)}(t) = -\beta_K(t) + \langle f - P_n f, R_{(0,2K+2)}(\cdot,t+)\rangle.$$

Since $|\beta_K(t)| \le c_1 < \infty$ by the continuity of $\beta_K(t)$ on $[a,b]$,

$$\sup_t \| R_{(0,2K+2)}(\cdot,t+)\| =: M_{0,2K+2} < \infty$$

and $\|f - P_n f\| \le \|f\|$, we have, for $t \notin T_n$, $|g_{(2K+2)}(t)| \le c_1 + \|f\| M_{0,2K+2}$. Therefore, there exists a constant c_2 such that $|D(a,b)| \le c_2/n^{2K+3}$ so that $D(a,b) = o(n^{-(2K+2)})$.

The *second term* can be written as

$$\frac{1}{(2K+2)!\,n^{2K+2}} \sum_{i=0}^{n-1} \int_{t_{i,n}}^{t_{i+1,n}} \{-\beta_K(t) + \langle f - P_n f, R_{(0,2K+2)}(\cdot,t)\rangle\} \times \{B_{2K+2}[nH(t,t_{i+1,n})] - B_{2K+2}\}h(t)\,dt.$$

The sum of second terms involving $\langle\cdot,\cdot\rangle$ is bounded in absolute value by

$$\frac{M_{0,2K+2}}{(2K+2)!\,n^{2K+2}} \|f - P_n f\| \int_0^1 |B_{2K+2}(x) - B_{2K+2}|\,dx.$$

Since $g_{2K+2}(t)$ is bounded we have, from (3.1),

$$\|f - P_n f\|^2 = E(I - \hat{I}_n)^2 \le c_2 n^{-(2K+3)} + c_2 n^{-(2K+2)}.$$

It follows that the sum of the second terms is $o(n^{-(3K+2)})$. Since the Bernoulli polynomials $B_{2K+2}(x) - B_{2K+2}$ have constant sign on $[0,1]$, the mean value theorem can be applied to the sum of the first terms involving β_K to obtain

$$-\frac{1}{(2K+2)!\,n^{2K+2}} \sum_{i=0}^{n-1} \beta_K(\xi_{i,n}) \frac{1}{n} \int_0^1 [B_{2K+2}(x) - B_{2K+2}]\,dx,$$

where $t_{i,n} < \xi_{i,n} < t_{i+1,n}$. Using $1/n = h(\zeta_{i,n})(t_{i+1,n} - t_{i,n})$, $t_{i,n} < \zeta_{i,n} < t_{i+1,n}$, Lemma 1(i) and the above results, we obtain, by Riemann integrability,

$$n^{2K+2}E\left(I - \hat{I}_n\right)^2 \to_n \frac{B_{2K+2}}{(2K+2)!}\int_a^b \beta_K(t)h(t)\,dt$$

and the final expression of the asymptotic constant follows from Lemma 3. □

PROOF OF THEOREM 2. For simplicity we write I_n for $I_n(h)$. In view of the isomorphism, we have

$$E(I - I_n)^2 = \| f_n \|^2,$$

where

$$\begin{aligned} f_n(t) &= E\left[Y_{(0)}(t)(I - I_n)\right] \\ &= \int_a^b R_{(0)}(t,s)h(s)\,ds - \frac{1}{2n}\sum_{i=0}^{n-1}\left[R_{(0)}(t,t_i) + R_{(0)}(t,t_{i+1})\right] \\ &\quad + \sum_{j=1}^{K}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}\left[R_{(0,j)}(t,b) - R_{(0,j)}(t,a)\right], \end{aligned}$$

for $K \geq 1$; for $K = 0$ the last sum is not present. Thus, for $K \geq 1$,

$$\begin{aligned} E(I - I_n)^2 = \langle f_n, f_n \rangle &= \int_a^b f_n(t)h(t)\,dt - \frac{1}{2n}\sum_{i=0}^{n-1}\left[f_n(t_i) + f_n(t_{i+1})\right] \\ &\quad + \sum_{j=1}^{K}\frac{1}{n^{j+1}}\frac{B_{j+1}}{(j+1)!}\left[(f_n)_{(j)}(b) - (f_n)_{(j)}(a)\right]; \end{aligned}$$

again, for $K = 0$, the last sum is not present. The case $K = 0$ will be treated separately since the Euler–MacLaurin formula can only be used for $K \geq 1$.

CASE $K \geq 1$. When $K \geq 1$, applying the weighted Euler–MacLaurin formula for regular sampling (Proposition 3) to f_n, we obtain

$$\begin{aligned} E(I - I_n)^2 &= \frac{1}{n^{K+1}}\frac{B_{K+1}}{(K+1)!}\left[(f_n)_{(K)}(b) - (f_n)_{(K)}(a)\right] \\ &\quad + \frac{1}{(K+1)!\,n^{K+1}} \\ &\quad \times \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(f_n)_{(K+1)}(t)\left[B_{K+1}(nH(t,t_{i+1})) - B_{K+1}\right]h(t)\,dt \\ &= \frac{1}{(K+1)!\,n^{K+1}}\sum_{i=0}^{n-1}\left\{B_{K+1}\left[(f_n)_{(K)}(t_{i+1}) - (f_n)_{(K)}(t_i)\right]\right. \\ &\quad \left. + \int_{t_i}^{t_{i+1}}(f_n)_{(K+1)}(t)\left[B_{K+1}(nH(t,t_{i+1})) - B_{K+1}\right]h(t)\,dt\right\}. \end{aligned}$$

Applying Proposition 3 to the integral of $R_{(0)}(t,\cdot)$ in the expression of f_n and then taking the lth weighted derivative ($l \le K+1$), we obtain

$$(f_n)_{(l)}(t) = \frac{1}{K!n^K}\sum_{j=0}^{n-1}\Bigg\{B_K\big[R_{(l,K-1)}(t,t_{j+1}) - R_{(l,K-1)}(t,t_j)\big] + \frac{1}{n}\frac{B_{K+1}}{K+1}\big[R_{(l,K)}(t,t_{j+1}) - R_{(l,K)}(t,t_j)\big] + \int_{t_j}^{t_{j+1}} R_{(l,K)}(t,s)\big[B_K\big(nH(s,t_{j+1})\big) - B_K\big]h(s)\,ds\Bigg\}.$$

Therefore for $K \ge 1$, the mean square error can be written as

$$E(I-I_n)^2 = \frac{1}{K!(K+1)!n^{2K+1}}\sum_{i,j=0}^{n-1} M_{i,j},$$

where

$$\begin{aligned}
M_{i,j} &= B_K B_{K+1}\big[R_{(K,K-1)}(t_{i+1},t_{j+1}) + R_{(K,K-1)}(t_i,t_j) \\
&\qquad - R_{(K,K-1)}(t_{i+1},t_j) - R_{(K,K-1)}(t_i,t_{j+1})\big] \\
&\quad + \frac{1}{n}\frac{B_{K+1}^2}{K+1}\big[R_{(K,K)}(t_{i+1},t_{j+1}) + R_{(K,K)}(t_i,t_j) \\
&\qquad - R_{(K,K)}(t_{i+1},t_j) - R_{(K,K)}(t_i,t_{j+1})\big] \\
&\quad + \int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} R_{(K+1,K)}(t,s)\,\overline{B}_{K+1}\big(nH(t,t_{i+1})\big) \\
&\qquad \times \overline{B}_K\big(nH(s,t_{j+1})\big)h(t)h(s)\,dt\,ds \\
&\quad + B_{K+1}\int_{t_j}^{t_{j+1}}\big[R_{(K,K)}(t_{i+1},s) - R_{(K,K)}(t_i,s)\big] \\
&\qquad \times \overline{B}_K\big(nH(s,t_{j+1})\big)h(s)\,ds \\
&\quad + B_K\int_{t_i}^{t_{i+1}}\big[R_{(K+1,K-1)}(t,t_{j+1}) - R_{(K+1,K-1)}(t,t_j)\big] \\
&\qquad \times \overline{B}_{K+1}\big(nH(t,t_{i+1})\big)h(t)\,dt \\
&\quad + \frac{1}{n}\frac{B_{K+1}}{K+1}\int_{t_i}^{t_{i+1}}\big[R_{(K+1,K)}(t,t_{j+1}) - R_{(K+1,K)}(t,t_j)\big] \\
&\qquad \times \overline{B}_{K+1}\big(nH(t,t_{i+1})\big)h(t)\,dt \\
&:= \sum_{l=1}^{6} E_{l,i,j},
\end{aligned}$$

where, for each $l = 1,\dots,6$, $E_{l,i,j}$ denotes the term on the lth line.

For *diagonal terms* $M_{i,i}$, we use Proposition 1 to do the h-weighted Taylor expansion of $R_{(i,j)}$, $0 \le i, j \le K$, around (t_i, t_i). For the *first term* $E_{1,i,i}$, we find

$$E_{1,i,i} = B_{K+1}B_K\left\{\int_{t_i}^{t_{i+1}} R_{(K+1,K)}(x, t_i)h(x)\,dx + \int_{t_i}^{t_{i+1}}\left[R_{(K,K+1)}(t_{i+1}, x) - R_{(K,K+1)}(t_i, x)\right] \times H(x, t_{i+1})h(x)\,dx\right\}.$$

By the mean value theorem, since $H(x, t_{i+1})h(x)$ is of constant sign on $t_i < x < t_{i+1}$ and $H(t_i, t_{i+1}) = n^{-1}$ (from the definition of h), we have

$$E_{1,i,i} = \frac{1}{n^2}B_{K+1}B_K\left\{R_{(K+1,K)}(a_i, t_i) + \frac{1}{2}R_{(K,K+1)}(t_{i+1}, b_i) - \frac{1}{2}R_{(K,K+1)}(t_i, b_i)\right\},$$

where the intermediate points a_i, b_i are in (t_i, t_{i+1}). Likewise the *second term* $E_{2,i,i}$ is

$$E_{2,i,i} = \frac{1}{n^2}\frac{B_{K+1}^2}{K+1}\left[R_{(K,K+1)}(t_{i+1}, a'_i) - R_{(K+1,K)}(b'_i, t_i)\right],$$

where the intermediate points a'_i, b'_i are in (t_i, t_{i+1}).

For the evaluation of the *third term* $E_{3,i,i}$, the double integral has to be evaluated separately above and below the diagonal, since $R_{(K+1,K)}(t, s)$ has a jump at the diagonal $t = s$. We write

$$E_{3,i,i} = \iint_{t_i<t<s<t_{i+1}} \Big\{R_{(K+1,K)}(t,s)\overline{B}_K(nH(s,t_{i+1}))\overline{B}_{K+1}(nH(t,t_{i+1})) + R_{(K+1,K)}(s,t)\overline{B}_{K+1}(nH(s,t_{i+1})) \times \overline{B}_K(nH(t,t_{i+1}))\Big\}h(t)h(s)\,dt\,ds.$$

The Bernoulli polynomials $\overline{B}_K(x)$ have constant sign on $[0, 1]$ when K is even, but change sign at $x = 1/2$ when K is odd. In order to apply the mean value theorem in the second term of $E_{3,i,i}$ we decompose the range of integration $\{t_i < t < s < t_{i+1}\}$ in areas where the sign of the integrand remains constant. We will use the median m_i of the subinterval (t_i, t_{i+1}) defined by $\int_{t_i}^{m_i} h(t)\,dt = 1/2n$. The range $\{t_i < t < s < t_{t+1}\}$ is then decomposed as in Figure 2a. We change variables as follows: $x = nH(s, t_{i+1})$, $y = nH(t, t_{i+1})$. Applying the

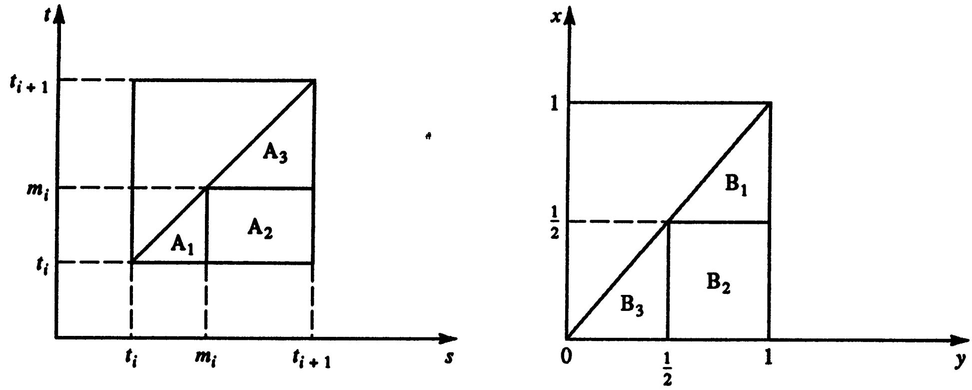

FIG. 2a. *Decomposition of the square* $(t_i, t_{i+1}) \times (t_i, t_{i+1})$ *by its diagonal and the h-median* m_i *of the interval* (t_i, t_{i+1}), *and its standardized form.*

mean value theorem, we obtain

$$E_{3,i,i} = \frac{1}{n^2} \sum_{l=1}^{3} \Big[R_{(K+1,K)}(\xi_{i,l}, \eta_{i,l}) \overline{B}_K(x) \overline{B}_{K+1}(y) + R_{(K+1,K)}(\xi'_{i,l}, \eta'_{i,l}) \overline{B}_{K+1}(x) \overline{B}_K(y) \Big] \, dx \, dy,$$

where $t_i < \xi_{i,l} < \eta_{i,l} < t_{i+1}$, $t_i < \eta'_{i,l} < \xi'_{i,l} < t_{i+1}$, $l = 1, 2, 3$. In the evaluation of the *fourth term* $E_{4,i,i}$, one step h-weighted expansion gives

$$R_{(K,K)}(t, t_{i+1}) - R_{(K,K)}(t, t_i) = R_{(K,K+1)}(t, \xi_t) H(t_i, t) + R_{(K,K+1)}(t, \eta_t) H(t, t_{i+1}),$$

where $t_i < \xi_t < t < \eta_t < t_{i+1}$. As in the evaluation of the term $E_{3,i,i}$ we decompose the range of integration (t_i, t_{i+1}) into (t_i, m_i) and (m_i, t_{i+1}), in which the mean value theorem can be applied to obtain

$$E_{4,i,i} = \frac{1}{n^2} B_{K+1} \left\{ \sum_{l=1}^{2} \int_{D_l} \big[(1-x) R_{(K,K+1)}(a_{i,l}, \xi_{i,l}) + x R_{(K,K+1)}(b_{i,l}, \eta_{i,l}) \big] \overline{B}_K(x) \, dx \right\},$$

where $D_1 = \{0 < x < 1/2\}$, $D_2 = \{1/2 < x < 1\}$ and $t_i < \xi_{i,l} < a_{i,l} < t_{i+1}$, $t_i < b_{i,l} < \eta_{i,l} < t_{i+1}$. Likewise, for the *fifth term* we obtain

$$E_{5,i,i} = \frac{1}{n^2} B_K \left\{ \sum_{l=1}^{2} \int_{D_l} \big[(1-x) R_{(K+1,K)}(a_{i,l}, \xi_{i,l}) + x R_{(K+1,K)}(b_{i,l}, \eta_{i,l}) \big] \overline{B}_{K+1}(x) \, dx \right\},$$

where $t_i < \xi_{i,l} < a_{i,l} < t_{i+1}$, $t_i < b_{i,l} < \eta_{i,l} < t_{i+1}$, and for the *sixth term*,

$$E_{6,i,i} = \frac{1}{n^2}\frac{B_{K+1}}{K+1}\left\{\sum_{l=1}^{2}\int_{D_l}\left[R_{(K+1,K)}(a'_{i,l},t_{i+1})\right.\right.$$
$$\left.\left.-R_{(K+1,K)}(b'_{i,l},t_i)\right]\overline{B}_{K+1}(x)\,dx\right\},$$

where $t_i < a'_{i,l} < t_{i+1}$ and $t_i < b'_{i,l} < t_{i+1}$.

The final expression for the diagonal terms is obtained by summing all six terms. Using

$$\frac{1}{n} = \int_{t_i}^{t_{i+1}} h(t)\,dt = h(\zeta_i)(t_{i+1}-t_i), \quad t_i < \zeta_i < t_{i+1},$$

and observing that the first term vanishes by Lemma 1(i), we obtain, by Riemann integrability,

$$n^2\sum_{i=1}^{n} M_{i,i} \to_n \int_a^b dt\,h(t)\left\{B_K B_{K+1}\left[\frac{1}{2}R_{(K,K+1)}(t,t+) + \frac{1}{2}R_{(K,K+1)}(t,t-)\right]\right.$$
$$+\frac{B_{K+1}^2}{K+1}\left[R_{(K,K+1)}(t,t-) - R_{(K,K+1)}(t,t+)\right]$$
$$+R_{(K,K+1)}(t,t-)\gamma_{1,K} + R_{(K,K+1)}(t,t+)\gamma_{2,K}$$
$$+B_{K+1}\left[R_{(K,K+1)}(t,t-)\beta_{1,K}\right.$$
$$\left.+R_{(K,K+1)}(t,t+)(\beta_{0,K}-\beta_{1,K})\right]$$
$$+B_K\left[R_{(K,K+1)}(t,t+)\beta_{1,K+1}\right.$$
$$\left.+R_{(K,K+1)}(t,t-)(\beta_{0,K+1}-\beta_{1,K+1})\right]$$
$$\left.+\frac{B_{K+1}}{K+1}\left[R_{(K,K+1)}(t,t-) - R_{(K,K+1)}(t,t+)\right]\beta_{0,K+1}\right\}$$

and replacing $\beta_{i,K}, \gamma_{i,K}$ by their values in Lemma 1, we obtain

$$n^2\frac{1}{K!^2}\sum_{i=1}^{n-1} M_{i,i}$$
$$\to_n (-1)^K\frac{B_{2K+2}}{(2K+2)!}\int_a^b\left[R_{(K,K+1)}(t,t-) - R_{(K,K+1)}(t,t+)\right]h(t)\,dt$$
$$= (-1)^K\frac{B_{2K+2}}{(2K+2)!}\int_a^b\frac{\phi^2(t)\alpha_K(t)}{h^{2K+2}(t)}\,dt \quad \text{(by Lemma 2)}.$$

For the *off-diagonal terms* $M_{i,j}$, Proposition 1 allows us to Taylor-expand the h-weighted cross-covariances $R_{(p,q)}$ around (t_i,t_j) for $i \neq j$ up to order $p = K+2$ or $q = K+2$, since by assumption $R^{(K,K)}(t,s)$ has continuous mixed partial derivatives up to order 2 for $t \neq s$. In what follows, the intermediate points with index i all will be in (t_i, t_{i+1}) and those with index j in (t_j, t_{j+1}). Thus, for the *first term* $E_{1,i,j}$ and the *second term* $E_{2,i,j}$ we obtain

$$E_{1,i,j} = B_K B_{K+1} \frac{1}{n^2} \Big\{ R_{(K+1,K)}(t_i, t_j) + \frac{1}{2n} R_{(K+2,K)}(\xi_{i,1}, t_j) + \frac{1}{2n} R_{(K+1,K+1)}(\xi_{i,2}, t_j) + \frac{1}{3!n} R_{(K,K+2)}(t_{i+1}, \eta_{j,1}) - \frac{1}{3!n} R_{(K,K+2)}(t_i, \eta_{j,2}) \Big\},$$

$$E_{2,i,j} = \frac{B_{K+1}^2}{K+1} \frac{1}{n^2} \Big\{ R_{(K+1,K+1)}(t_i, b_j) + \frac{1}{2} R_{(K+2,K)}(a_{i,1}, t_{j+1}) - \frac{1}{2} R_{(K+2,K)}(a_{i,2}, t_j) \Big\}.$$

For the evaluation of the *third term* $E_{3,i,j}$, the h-weighted Taylor expansion up to order 2 gives

$$R_{(K+1,K)}(t,s) = R_{(K+1,K)}(t_i, t_j) + R_{(K+2,K)}(\xi_{t,s}, t_j) H(t_i, t) + R_{(K+1,K+1)}(t, \eta_s) H(t_j, s),$$

where $t_i < \xi_{t,s} < t$ and $t_j < \eta_s < s$. The range of integration $[t_i, t_{i+1}] \times [t_j, t_{j+1}]$ is decomposed in four different sets as shown on Figure 2b, where m_i is the median of the subinterval $[t_i, t_{i+1}]$. The mean value theorem can be applied in each of the four regions A_1, A_2, A_3, A_4, where the Bernoulli polynomial $\overline{B}_K(x)$ has constant sign. Putting $x = nH(t, t_{i+1})$, $y = nH(s, t_{j+1})$, the

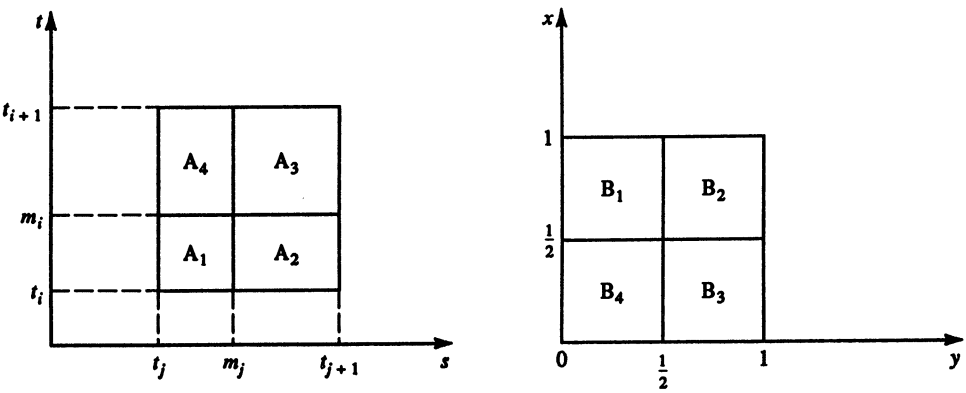

FIG. 2b. *Decomposition of the rectangle* $(t_i, t_{t+1}) \times (t_j, t_{j+1})$ *by the h-medians* m_i *of* (t_i, t_{i+1}) *and* m_j *of* (t_j, t_{j+1}), *and its standardized form.*

four regions are transformed into B_1, B_2, B_3, B_4, and thus we obtain

$$E_{3,i,j} = \frac{1}{n^2}\Bigg\{R_{(K+1,K)}(t_i,t_j)\beta_{0,K}\beta_{0,K+1} + \frac{1}{n}\sum_{l=1}^{4}\int_{B_l}\big[R_{(K+2,K)}(\xi_{i,l},t_j)(1-x) + R_{(K+1,K+1)}(\xi'_{i,l},\eta_{j,l})(1-y)\big]\,\overline{B}_{K+1}(x)\,\overline{B}_K(y)\big]\,dx\,dy\Bigg\}.$$

For the remaining terms of $M_{i,j}$ we have

$$E_{4,i,j} = \frac{1}{n^2}B_{K+1}\Bigg\{R_{(K+1,K)}(t_i,t_j)\beta_{0,K} + \frac{1}{n}\sum_{l=1}^{2}R_{(K+1,K+1)}(t_i,\eta_{j,l})\int_{D_l}(1-x)\,\overline{B}_K(x)\,dx + \frac{1}{2n}R_{(K+2,K)}(a_i,b_j)\beta_{0,K}\Bigg\},$$

$$E_{5,i,j} = \frac{1}{n^2}B_K\Bigg\{R_{(K+1,K)}(t_i,t_j)\beta_{0,K+1} + \frac{1}{n}\sum_{l=1}^{2}R_{(K+2,K)}(\xi_{i,l},t_j)\int_{D_l}(1-x)\,\overline{B}_{K+1}(x)\,dx + \frac{1}{2n}R_{(K+1,K+1)}(\xi'_{i,l},\eta_j)\beta_{0,K+1}\Bigg\},$$

$$E_{6,i,j} = \frac{1}{n^3}\frac{B_{K+1}}{K+1}\sum_{l=1}^{2}R_{(K+1,K+1)}(\xi_{i,l},\eta_j)\int_{D_j}\overline{B}_{K+1}(x)\,dx.$$

Collecting all six terms, and replacing $\beta_{i,K}$ by their values in Lemma 1, the terms involving $R_{(K,K+1)}$ and $R_{(K+1,K)}$ all vanish, and the final expression for the off-diagonal terms is obtained. Then from the mean value theorem, we have $n^{-2} = h(\zeta_i)h(\zeta_j)(t_{i+1}-t_i)(t_{j+1}-t_j)$, where $\zeta_i \in (t_i, t_{i+1})$, and by Riemann integrability, we obtain

$$n^2\sum_{i\neq j} M_{i,j} \to_n \iint_{t\neq s} dt\,ds\,h(t)h(s)\big\{R_{(K+1,K+1)}(t,s)A_{K,1} + R_{(K,K+2)}(t,s)A_{K,2} + R_{(K+2,K)}(t,s)A_{K,3}\big\},$$

where, using the notation of Lemma 1, we have

$$A_{K,1} = \frac{1}{2}B_K B_{K+1} + \frac{B_{K+1}^2}{K+1} + \beta_{0,K+1}\beta_{1,K} + B_{K+1}\beta_{1,K} + \frac{1}{2}B_K\beta_{0,K+1} + \frac{B_{K+1}}{K+1}\beta_{0,K+1} = 0,$$

$$A_{K,2} = B_K B_{K+1}\left(\frac{1}{3!} - \frac{1}{3!}\right) = 0,$$

$$A_{K,3} = \frac{1}{2}B_K B_{K+1} + \frac{B_{K+1}^2}{K+1}\left(\frac{1}{2} - \frac{1}{2}\right) + \beta_{1,K+1}\beta_{0,K} + \frac{1}{2}\beta_{0,K}B_{K+1} + B_K\beta_{1,K+1} = 0.$$

Thus, we obtain

$$n^2 \sum_{i \neq j} M_{i,j} \to_n 0.$$

The result follows by combining the diagonal and off-diagonal terms.

Case $K = 0$. Replacing f_n by its expression we obtain

$$E(I - I_n)^2 = \sum_{i=0}^{n-1}\left\{\int_{t_i}^{t_{i+1}} f_n(t)h(t)\,dt - \frac{1}{2n}\left[f_n(t_i) + f_n(t_{i+1})\right]\right\}$$

$$=: \sum_{i,j=0}^{n-1} M_{i,j},$$

where

$$M_{i,j} = \int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} R_{(0)}(t,s)h(t)h(s)\,dt\,ds$$

$$= \frac{1}{2n}\int_{t_i}^{t_{i+1}}\left[R_{(0)}(t,t_j) + R_{(0)}(t,t_{j+1})\right]h(t)\,dt$$

$$- \frac{1}{2n}\int_{t_j}^{t_{j+1}}\left[R_{(0)}(t,t_i) + R_{(0)}(t,t_{i+1})\right]h(t)\,dt$$

$$+ \frac{1}{4n^2}\left[R_{(0)}(t_i,t_j) + R_{(0)}(t_i,t_{j+1}) + R_{(0)}(t_{i+1},t_j) + R_{(0)}(t_{i+1},t_{j+1})\right].$$

As in the case $K \geq 1$, the Taylor expansion of Proposition 1 is used to expand $R_{(0)}(\cdot,\cdot)$ around (t_i, t_i) up to order 1 for the diagonal terms $M_{i,i}$ and around

(t_i, t_j) for $i \neq j$ up to order 2 for the off-diagonal terms $M_{i,j}$ to obtain

$$n^2 \sum_{i=0}^{n-1} M_{i,i} \to_n \frac{1}{12} \int_a^b \frac{\phi^2(t)}{h_*^2(t)} \alpha_0(t)\, dt \quad \text{and} \quad n^2 \sum_{i \neq j} M_{i,j} \to_n 0. \qquad \square$$

The following Lemma will be used in the proof of Theorem 3.

LEMMA 4. *For $1 < l, j \leq m - 1$ and $t \notin (t_{0,n}, t_{m-1,n})$, $s \notin (t_{n-m+1,n}, t_{n,n})$, we have*

$$n^j \sum_{i=j}^{m-1} \frac{1}{i!} W_i^{(j)}[nH(t_{0,n}, t)] \sum_{r=0}^{i} (-1)^{i-r} \binom{i}{r} \frac{1}{l!} H^l(t, t_{r,n})$$

$$= \begin{cases} 0, & \text{for } j \neq l, \\ 1, & \text{for } j = l, \end{cases}$$

$$n^j \sum_{i=j}^{m-1} \frac{(-1)^{i+j}}{i!} W_i^{(j)}[nH(s, t_{n,n})] \sum_{r=0}^{i} (-1)^{i-r} \binom{i}{r} \frac{1}{l!} H^l(s, t_{n-i+r,n})$$

$$= \begin{cases} 0, & \text{for } j \neq l, \\ 1, & \text{for } j = l. \end{cases}$$

PROOF. For $t \notin (t_{0,n}, t_{m-1,n})$, let $f(x) = H^l(t, x)/l!$ for $x \in [a, b]$. The forward formula for $f_{(j)}(t)$ (see Proposition 5) gives

$$f_{(j)}(t) = n^j \sum_{i=j}^{m-1} \frac{1}{j!} W_i^{(j)}[nH(t_{0,n}, t)] \Delta^i f_{(0)}(t_{0,n}),$$

where the remainder vanishes since $f_{(m)} \equiv 0$ for $m > l$. Since $f_{(j)}(t) = 0$ for $j \neq l$ and $f_{(j)}(t) = 1$ for $j = l$, the result follows by using the $\Delta^i f_{(0)}(t_{0,n})$. Likewise, the second result follows from the backward formula (see Proposition 5). $\square$

PROOF OF THEOREM 3. The mean square error resulting from using the estimator $\bar{I}_n(h)$, denoted here by $\bar{I}_n$, of the integral I can be expressed in terms of the mean square error based on the estimator I_n with a residual term which is shown to be of order $o(n^{-(2K+2)})$. We have

$$E(I - \bar{I}_n)^2 = E[(I - I_n) + (I_n - \bar{I}_n)]^2$$

$$= E(I - I_n)^2 + E(I_n - \bar{I}_n)^2 + 2E(I - I_n)(I_n - \bar{I}_n).$$

For the *approximation of quadratic mean derivatives at the endpoints a, b*, consider the error in the approximation of $Y_{(j)}(a)$, by the first term of the forward formula in Proposition 5. The form of the remainder given for the deterministic case in Proposition 5 cannot be used here since the $(K + 1)$th q.m. derivative of the process X does not exist. Hence we proceed as follows.

For $0 \le j \le K$, we denote the remainder by

$$D_j(a) = Y_{(j)}(a) - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0)\, \Delta^i Y_{(0)}(a).$$

We write $\Delta^{0,i}$ for finite differences with respect to the second variable. Let

$$g(t) = E\big[Y_{(0)}(t) D_j(a)\big] = R_{(0,j)}(t,a) - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0)\, \Delta^{0,i} R_{(0)}(t,a).$$

By the assumptions of Theorem 1, g has $K+1$ continuous derivatives. Then from the isomorphism between $\mathscr{R}$ and $\mathscr{H}$ we have

$$\begin{aligned} ED_j(a)^2 &= \|g\|^2 \\ &= \langle g(\cdot), R_{(0,j)}(\cdot,a)\rangle - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0) \langle g(\cdot), \Delta^{0,i} R_{(0)}(\cdot,a)\rangle \\ &= g_{(j)}(a) - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0)\, \Delta^i g(a) \\ &= \frac{1}{n^{K+1-j}} W_{K+1}^{(j)}(0) \frac{1}{(K+1)!} g_{(K+1)}(\xi_{a,j}) \quad \text{(by Proposition 6)}, \end{aligned}$$

where $\xi_{a,j}$ is in (a, t_{K+1}). The $(K+1)$th weighted derivative of g is given by

$$g_{(K+1)}(\xi) = R_{(K+1,j)}(\xi,a) - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0) \sum_{r=0}^{i} (-1)^{i-r} \binom{i}{r} R_{(K+1,0)}(\xi, t_r).$$

Using the h-weighted Taylor expansion of $R_{(K+1,0)}(\xi, t_r)$ for the second argument around a up to order $K+1$, and applying Lemma 4 with $m = K+1$, we obtain

$$\begin{aligned} g_{(K+1)}(\xi) = -n^j &\sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0) \\ &\times \sum_{r=0}^{i} (-1)^{i-r} \binom{i}{r} \frac{1}{K!} \int_a^{t_r} R_{(K+1,K+1)}(\xi,x) H^K(x,t_r) h(x)\, dx, \end{aligned}$$

and since

$$\int_a^{t_r} H^K(x,t_r) h(x)\, dx = \frac{1}{K+1} H^{K+1}(a,t_r) = \frac{1}{K+1}\left(\frac{r}{n}\right)^{K+1},$$

we have

$$\begin{aligned} |g_{(K+1)}(\xi)| &\le \frac{1}{n^{K+1-j}} M_{K+1} \left\{ \sum_{i=j}^{K} \frac{1}{i!} |W_i^{(j)}(0)| \sum_{r=0}^{i} \binom{i}{r} \frac{r^{K+1}}{(K+1)!} \right\} \\ &=: \frac{1}{n^{K+1-j}} a_{K,j}. \end{aligned}$$

Therefore,

$$ED_j(a)^2 \leq \frac{1}{n^{2(K+1-j)}} \left| W_{K+1}^{(j)}(0) \right| \frac{1}{(K+1)!} a_{K,j} =: \frac{1}{n^{2(K+1-j)}} A_{K,j}.$$

The same procedure, using the backward formula for the jth h-weighted derivative of g at the point b, gives for some finite constant $B_{K,j}$,

$$ED_j(b)^2 \leq \frac{1}{n^{2(K+1-j)}} B_{K,j}.$$

We now express $I_n - \bar{I}_n$ linearly in terms of $D_j(a)$ and $D_j(b)$ by using the expression for C_i from Proposition 6, as follows:

$$I_n - \bar{I}_n = \sum_{j=1}^{K} \frac{1}{n^{j+1}} \frac{B_{j+1}}{(j+1)!} \left[D_j(a) - D_j(b) \right]$$

and, taking the mean square error, we have

$$E\left(I_n - \bar{I}_n\right)^2 \leq 4 \sum_{j=1}^{K} \frac{1}{n^{2(j+1)}} \frac{B_{j+1}^2}{(j+1)!^2} E\left[D_j(a)^2 + D_j(b)^2 \right]$$

$$\leq \frac{4}{n^{2K+4}} \sum_{j=1}^{K} \frac{B_{j+1}^2}{(j+1)!^2} \left[A_{K,j} + B_{K,j} \right].$$

It follows that

$$E\left(I_n - \bar{I}_n\right)^2 = o\left(n^{-(2K+3)}\right).$$

For *cross-correlation between quadratic mean derivative and integral approximation*, using the expression of $I_n - \bar{I}_n$, we can write

$$E(I - I_n)\left(I_n - \bar{I}_n\right) = \sum_{j=1}^{K} \frac{1}{n^{j+1}} \frac{B_{j+1}}{(j+1)!} E(I - I_n)\left(D_j(a) - D_j(b)\right).$$

Recall from the proof of Theorem 2 that $f_n(t) = E[Y_{(0)}(t)(I - I_n)]$; since $g(t) = E[Y_{(0)}(t) D_j(a)]$, then in view of the isomorphism between $\mathscr{R}$ and $\mathscr{H}$, we have

$$E(I - I_n) D_j(a) = \langle f_n, g \rangle = (f_n)_{(j)}(a) - n^j \sum_{i=j}^{K} \frac{1}{i!} W_i^{(j)}(0)\, \Delta^i f_n(a)$$

$$= \frac{1}{n^{K+1-j}} W_{K+1}^{(j)}(0) \frac{1}{(K+1)!} (f_n)_{(K+1)}(\xi_{a,j})$$

(by Proposition 5), where $\xi_{a,j}$ is in (a, t_{K+1}). Applying Proposition 3 to

$R_{(0)}(t, \cdot)$, we have

$$f_n(t) = \frac{1}{n^{K+1}} \frac{B_{K+1}}{(K+1)!} \left[R_{(0,K)}(t,b) - R_{(0,K)}(t,a) \right] + \frac{1}{(K+1)!\,n^{K+1}} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} R_{(0,K+1)}(t,x) \bar{B}_{K+1}(nH(x,t_{i+1})) h(x)\,dx.$$

Taking the $(K+1)$th h-weighted derivative, we obtain

$$|(f_n)_{(K+1)}(\xi)| \le \frac{1}{(K+1)!\,n^{K+1}} M_{K+1} \left\{ |B_{K+1}| + \int_0^1 |\bar{B}_{K+1}(x)|\,dx \right\} =: \frac{1}{n^{K+1}} b_K.$$

It follows that

$$|E(I - I_n) D_j(a)| \le \frac{1}{n^{2K+2-j}} |W_{K+1}^{(j)}(0)| \frac{1}{(K+1)!} b_K =: \frac{1}{n^{2K+2-j}} A_{K,j}.$$

Likewise, for some finite constant $B_{K,j}$, we have

$$|E(I - I_n) D_j(b)| \le \frac{1}{n^{2K+2-j}} B_{K,j}.$$

Therefore, we have

$$|E(I - I_n)(I_n - \bar{I}_n)| \le \frac{1}{n^{2K+3}} \sum_{j=1}^{K} \frac{|B_{j+1}|}{(j+1)!} (A_{K,j} + B_{K,j})$$

so that

$$E(I - I_n)(I_n - \bar{I}_n) = o(n^{-(2K+2)}).$$

Thus, the result follows from Theorem 2 and

$$E(I - \bar{I}_n)^2 = E(I - I_n)^2 + o(n^{-(2K+2)}). \qquad \square$$

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