

SOME BOOTSTRAP TESTS OF SYMMETRY FOR UNIVARIATE CONTINUOUS DISTRIBUTIONS¹

BY MIGUEL A. ARCONES AND EVARIST GINÉ

*CUNY (Graduate Center, and The College of Staten Island)
and University of Connecticut*

The Kolmogorov distance between the empirical cdf F_n and its symmetrization sF_n with respect to an adequate estimator of the center of symmetry of P is a natural statistic for testing symmetry. However, its limiting distribution depends on P . Using critical values from the symmetrically bootstrapped statistic (where the resampling is made from sF_n) produces tests that can be easily implemented and have asymptotically the correct levels as well as good consistency properties. This article deals with the asymptotic theory that justifies this procedure in particular for a test proposed by Schuster and Barker. Because of lack of smoothness (in some cases implying non-Gaussianness of the limiting processes), these tests do not seem to fall into existing general frameworks.

1. Introduction. Let P be a probability measure on \mathbb{R} . P is symmetric if there exists $\theta \in \mathbb{R}$ such that for all $A \in \mathcal{B}$, $P(A) = P(2\theta - A)$, where $2\theta - A = \{2\theta - x : x \in A\}$. Then θ is the (unique) center of symmetry of P . There are statistical procedures which are sensitive to departures from symmetry, therefore requiring testing for symmetry in advance [for instance testing for symmetry of errors in linear models may be of interest in some situations—see Boos (1982) and Carroll (1979)]. The problem of testing P for symmetry (with or without a specified center) has received considerable attention in the literature: There are tests based on ranks [see, e.g., Shorack and Wellner (1986) and Hájek and Sidak (1967)], tests based on the empirical characteristic function [e.g., Csörgő and Heathcote (1987)] and tests based on the empirical cdf [e.g., Smirnov (1947), Rothman and Woodroffe (1972), Antille, Kersting and Zucchini (1982), Koziol (1983), Schuster and Barker (1987), etc.]. The test proposed by Schuster and Barker (1987) is as follows: Schuster and Narvarte (1973) defined a location parameter $\theta = \theta(P)$, $P \in \mathcal{P}(\mathbb{R})$, as the center of symmetry of a symmetric probability measure closest to P in the Kolmogorov distance. Then this symmetric probability is $s^\theta P$, defined as $s^\theta P(A) = \frac{1}{2}(P(A) + P(2\theta - A))$, $A \in \mathcal{B}$. If P is symmetric the empirical cdf should be close to being symmetric, hence it makes good sense to reject symmetry for large values of the statistic $T_n = n^{1/2} \|F_n - sF_n\|_\infty$, where F_n is the empirical cdf of P and sF_n is the cdf of the symmetrized empirical measure $sP_n = s^{\theta(P_n)}P_n$. The limiting distribution of T_n depends on P , so they propose to take as critical

Received September 1989; revised May 1990.

¹Research partially supported by NSF Grant DMS-86-19411.

AMS 1980 subject classifications. Primary 62G10, 62F10.

Key words and phrases. Tests of symmetry, symmetric bootstrap, bootstrap of location parameters.

numbers the corresponding quantiles of the “symmetric bootstrap” \hat{T}_n of T_n , that is, of $\hat{T}_n = n^{1/2} \|F_{n,sP_n} - sF_{n,sP_n}\|_\infty$, where F_{n,sP_n} is the empirical cdf obtained by resampling from sP_n , and sF_{n,sP_n} is its symmetrization with respect to $\theta(P_{n,sP_n})$ (actually these authors consider a smoothing of F_{n,sP_n}). Schuster and Barker (1987) present an extensive simulation study on this test but do not provide any analytical justification. We prove that T_n and $\hat{T}_n(\omega)$ for almost every ω , all have the same limiting distribution if P is symmetric whereas $T_n \rightarrow \infty$ and $\hat{T}_n(\omega)$ still converges weakly a.s. if P is not symmetric (this is proved in Sections 2 and 4). Moreover, the limiting distribution has a continuous cdf strictly increasing on its convex domain. Therefore this bootstrap test has asymptotically the correct level under the null hypothesis, the bootstrap critical numbers converge a.s. to the asymptotically correct critical number and the test is consistent against any alternatives (we also consider local alternatives varying with n).

The Schuster–Narvarte location parameter can be replaced by any bootstrap consistent location parameter in the construction of the tests (see Definition 2.2). In particular we give, with proofs omitted, the asymptotics of the symmetry tests for the median, the Hodges (1967) and the Hodges and Lehmann (1963) location parameters.

The Schuster–Barker test is a forerunner of a general class of bootstrap tests recently proposed and studied by Romano (1988, 1989). However, the mapping $s: P \rightarrow sP$ is far from satisfying Romano’s conditions [i.e., his smoothness condition (2.1)], and in the cases of Schuster and Barker and Hodges the limit of T_n is not even the sup norm of a Gaussian process. So, not only is it interesting by itself to provide analytic justification for these elegant tests, but it is also mathematically interesting as an example of the bootstrap in a nonsmooth non-Gaussian limit situation.

The symmetry tests we study in this article only apply to absolutely continuous probability measures P with uniformly continuous densities. On the other hand, no integrability assumptions on P are required. [For a symmetry test that applies under no continuity on P , but requires some (weak) integrability, see Csörgő and Heathcote (1987).] The assumptions can be somewhat relaxed if “a.s.” is replaced by “in probability” in the bootstrap limit theorem of Section 2.

The main ingredients in the proofs are a general result of Giné and Zinn (1991) on parametric and semiparametric bootstrap of the empirical process (which in the present situation also follows from special constructions), a result of Stute (1982) controlling the a.s. behavior of the uniform empirical process and the usual special constructions that allow us to replace weak convergence of empirical processes by almost sure convergence.

Next we introduce some notation. P will denote a probability measure on \mathbb{R} , F its cdf, f its density (if it exists) and $F^{-1}(t) := \inf\{x: F(x) \geq t\}$, $t \in [0, 1]$. If $\{X_i\}_{i=1}^\infty$ are iid (P), then $P_{n,P}$ or P_n will be the empirical measure

$$(1.1) \quad P_{n,P} = P_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$$

and $F_{n,P}$ or F_n will indicate its cdf. If F is continuous then, by changing the X_i at most in a set of probability 0, we may assume $X_i = F^{-1}(\xi_i)$, where $\{\xi_i\}_{i=1}^\infty$ are iid uniform on $[0, 1]$. Then, if we let $V_n(t) = n^{-1/2} \sum_{i=1}^n (I(\xi_i \leq t) - t)$ we have

$$(1.2) \quad n^{1/2}(F_n - F)(t) = V_n(F(t)).$$

We will assume $X_i = F^{-1}(\xi_i)$ in (1.1) throughout the paper. We will also use another "special construction": It is well known [Shorack (1972); for example Shorack and Wellner (1986), page 93] that as a consequence of a theorem of Skorokhod, there exists a Brownian bridge U on $[0, 1]$ and, for each n , iid uniform on $[0, 1]$ random variables $\xi_{n1}, \dots, \xi_{nn}$, such that if $U_n(t) = n^{-1/2} \sum_{i=1}^n (I(\xi_{ni} \leq t) - t)$, $t \in [0, 1]$, then

$$(1.3) \quad \|U_n - U\|_\infty \rightarrow 0 \quad \text{a.s.}$$

This construction provides an expeditive way of proving joint weak convergence of the empirical process and the estimators; in more general settings it could be replaced by Skorokhod–Dudley–Wichura type representations.

Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures on \mathbb{R} . A *location parameter* $\theta: \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$ is a function satisfying: (1) if P is symmetric then $\theta(P)$ is its center of symmetry and (2) the function of n real variables $\theta_n(x_1, \dots, x_n) = \theta(n^{-1} \sum_{i=1}^n \delta_{x_i})$, $x_i \in \mathbb{R}$, is measurable. Given a location parameter θ we define $s^{\theta(P)}P$ (sP if no confusion may arise) by

$$(1.4) \quad s^{\theta(P)}P(A) = \frac{1}{2}P(A) + \frac{1}{2}P(2\theta(P) - A), \quad A \in \mathcal{B},$$

and $s^\theta F$, $s^\theta f$, denote respectively the cdf and the density (if it exists) of $s^\theta P$. If no confusion is possible we will use θ for $\theta(P)$, θ_n for $\theta(P_n)$ and $\hat{\theta}_n$ for $\theta(P_{n,sP_n})$, where $P_{n,sP_n} = P_{n,sP_n(\omega)}$ is the empirical measure of n iid random variables $X_{n1}^\omega, \dots, X_{nn}^\omega$ with common probability law $sP_n(\omega)$. The variable ω will often be omitted. Finally, $\widehat{\Pr}$, \widehat{E} will denote \Pr , E conditional on $P_n(\omega)$, and $\rightarrow_{\mathcal{L}}$ will mean convergence in law conditionally on $P_n(\omega)$.

2. A symmetric bootstrap central limit theorem. The proof of the limit theorem for $\{n^{1/2}(F_n - sF_n)\}$ and $n^{1/2}(F_{n,sP_n} - sF_{n,sP_n})$ is relatively long and may be better understood if it is decomposed into two parts: the proof of the limit theorem under consistency hypotheses on the location parameter, and the proof of these properties for each individual parameter. In this section we give the limit theorem.

Given a set D , we let $l^\infty(D)$ denote the Banach space of bounded real functions on D with the sup norm. As usual, we say that $l^\infty(D)$ -valued random elements Z_n (i.e., sample bounded processes indexed by D) converges in law in $l^\infty(D)$ to a sample continuous Gaussian process Z on D if $E^*H(Z_n) \rightarrow EH(Z)$ for all $H: l^\infty(D) \rightarrow \mathbb{R}$ bounded and continuous (E^* denotes outer expectation: $H(Z_n)$ is not necessarily measurable). The Kolmogorov theorem for empirical processes asserts that $n^{1/2}(F_n - F) \rightarrow_{\mathcal{L}} U \circ F$ in $l^\infty(\mathbb{R})$ or in $l^\infty(D_F)$, where $D_F = \{x: 0 < F(x) < 1\}$ is the convex support of F . We will require the following lemma, which follows from Giné and Zinn (1991), Corollary 2.7.

LEMMA 2.1. Let $R_n, R, n \in \mathbb{N}$, by probability measures on \mathbb{R} such that $\sup_{t \in \mathbb{R}} |R_n(-\infty, t] - R(-\infty, t]| \rightarrow 0$ as $n \rightarrow \infty$. Then

$$n^{1/2}(P_{n, R_n} - R_n) \rightarrow_{\mathcal{L}} Y$$

in $l^\infty(J)$, where J is the collection of all (finite and infinite) intervals and Y is the centered Gaussian process on J with covariance $EY(B)Y(C) = R(B \cap C) - R(B)R(C)$, $B, C \in J$.

Here is a consequence of Lemma 2.1: If $\theta(P)$, $P \in \mathcal{P}(\mathbb{R})$, is a location parameter and if P has a continuous distribution, then

$$(2.1) \quad \theta(P_n) \rightarrow \theta(P) \quad \text{a.s.} \quad \Rightarrow \quad n^{1/2}(F_{n, sP_n} - sF_n) \rightarrow_{\mathcal{L}} U \circ (sF) \quad \text{in } l^\infty(D_P) \text{ } P\text{-a.s.}$$

The consistency requirements on $\theta_n = \theta(P_n)$ are slightly stronger than usual in the sense that we must impose *joint* convergence of the parameter and the empirical process. Of course, this is automatic if $\theta(P)$ is differentiable in any reasonable sense.

DEFINITION 2.2. Let $\theta = \theta(P)$, $P \in \mathcal{P}(\mathbb{R})$, be a location parameter and let Π be a class of probability measures such that if $P \in \Pi$ then $s^\theta P \in \Pi$. θ is $n^{1/2}$ -bootstrap consistent for the class Π if

$$(a) \quad \theta(P_n) \rightarrow \theta(P) \quad \text{a.s.}$$

(b) For all $P \in \Pi$, P symmetric, there is a random variable $\tilde{\theta}(P)$ such that

$$n^{1/2}(\theta_n - \theta) \rightarrow_{\mathcal{L}} \tilde{\theta}(P)$$

jointly with the empirical process $n^{1/2}(F_n - F)$.

(c) For all $P \in \Pi$,

$$n^{1/2}(\theta(P_{n, sP_n}) - \theta(P_n)) \rightarrow_{\mathcal{L}} \tilde{\theta}(sP)$$

jointly with $n^{1/2}(F_{n, sP_n} - sF_n)$, P -a.s. [these processes converge by (a) and (2.1)].

THEOREM 2.3. Let Π be a set of probability measures P on \mathbb{R} which are absolutely continuous, whose distribution functions F satisfy the Hölder type condition $|F(x) - F(y)| \leq c/|\log|\log|x - y|||^{1+\varepsilon}$ for some $c > 0$, $\varepsilon > 0$ and all $|x - y| \leq 1/4$, and whose densities f are uniformly continuous on D_P . Let θ be a $n^{1/2}$ -bootstrap consistent location parameter for the class Π and let Z_P be the process

$$Z_P(t) = \frac{1}{2}[U(F(t)) + U(1 - F(t))] + \tilde{\theta}(P)f(t), \quad t \in D_P.$$

Then the following limits hold:

(i) For all $P \in \Pi$, for almost every ω ,

$$\lim_{n \rightarrow \infty} n^{1/2}(F_{n, sP_n(\omega)} - sF_{n, sP_n(\omega)})(\cdot) = Z_{sP}(\cdot)$$

in law in $l^\infty(D_{sP})$.

(ii) For all symmetric $P \in \Pi$,

$$\lim_{n \rightarrow \infty} n^{1/2}(F_n - sF_n)(\cdot) = Z_P(\cdot)$$

in law in $l^\infty(D_P)$.

PROOF. We first prove part (ii). Since P is symmetric and the variables X_i are a.s. all different we have

$$\begin{aligned} n^{1/2}(F_n - sF_n)(t) &= \frac{1}{2}n^{1/2}(F_n - F)(t) + \frac{1}{2}n^{1/2}(F_n - F)(2\theta - t) \\ (2.2) \quad &+ \frac{1}{2}n^{1/2}((F_n - F)(2\theta_n - t) - (F_n - F)(2\theta - t)) \\ &+ \frac{1}{2}n^{1/2}(F(2\theta_n - t) - F(2\theta - t)) + O(n^{-1/2}) \quad \text{a.s.} \end{aligned}$$

Since $\theta_n \rightarrow \theta$ a.s.,

$$(2.3) \quad n^{1/2} \sup_{t \in \mathbb{R}} |(F_n - F)(2\theta_n - t) - (F_n - F)(2\theta - t)| \rightarrow 0$$

in probability by Kolmogorov's theorem [which since F is uniformly continuous, implies $n^{1/2} \sup_{|t-s| < \delta_n} |(P_n - P)(s, t)| \rightarrow 0$ in probability as $\delta_n \rightarrow 0$]. By differentiability of F , since $f(2\theta - t) = f(t)$,

$$\begin{aligned} &\frac{1}{2}n^{1/2}(F(2\theta_n - t) - F(2\theta - t)) \\ &= n^{1/2}f(t)(\theta_n - \theta) + n^{1/2}(f(\xi) - f(t))(\theta_n - \theta), \end{aligned}$$

where $|\xi - t| \leq 2|\theta_n - \theta|$; therefore, since $n^{1/2}(\theta_n - \theta)$ converges in law and f is uniformly continuous,

$$(2.4) \quad \sup_{t \in D_P} \left| \frac{1}{2}n^{1/2}(F(2\theta_n - t) - F(2\theta - t)) - n^{1/2}f(t)(\theta_n - \theta) \right| \rightarrow 0$$

in probability. Hence (2.2)–(2.4) give that the distribution of $n^{1/2}(F_n - sF_n)$ in $l^\infty(D_P)$ is asymptotically the same as the distribution of

$$\frac{1}{2}n^{1/2}(F_n - F)(t) + \frac{1}{2}n^{1/2}(F_n - F)(2\theta - t) + n^{1/2}f(t)(\theta_n - \theta),$$

which converges in law in $l^\infty(D_P)$ to $Z_P(t)$ by part (b) in Definition 2.2. Hence, part (ii) is proved.

The proof of part (i) is similar. In analogy with (2.2) we write

$$\begin{aligned} &n^{1/2}(F_{n, sP_n} - sF_{n, sP_n})(t) \\ &= \frac{1}{2}n^{1/2}[(P_{n, sP_n} - sP_n)(I(-\infty, t] + I(-\infty, 2\theta - t))] \\ (2.5) \quad &+ \frac{1}{2}n^{1/2}[(P_{n, sP_n} - sP_n)(I(-\infty, 2\hat{\theta}_n - t) - I(-\infty, 2\theta - t))] \\ &+ \frac{1}{2}n^{1/2}sP_n(I(-\infty, 2\hat{\theta}_n - t) - I(-\infty, 2\theta_n - t)) \\ &:= \text{I} + \text{II} + \text{III}, \end{aligned}$$

where $\hat{\theta}_n = \theta(P_{n, sP_n})$.

Obviously, $Y((-\infty, t]) = U(sF(t))$, $t \in D_{sP}$, has uniformly continuous paths a.s.; hence it follows from Lemma 2.1 that ω -a.s., for all $\varepsilon > 0$,

$$(2.6) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \widehat{\Pr} \left\{ \sup_{|u-v| \leq \delta} n^{1/2} |(P_{n, sP_n(\omega)} - sP_n(\omega))[u, v]| > \varepsilon \right\} = 0$$

and likewise for $(u, v]$, $[u, v)$ and (u, v) . By parts (a) and (c) in Definition 2.2, $\hat{\theta}_n - \theta \rightarrow 0$ in $\widehat{\Pr}$ probability ω -a.s. and therefore (2.6) gives

$$(2.7) \quad \sup_{t \in D_{sP}} (|II|) \rightarrow 0 \quad \text{in } \widehat{\Pr} \text{ probability } \omega\text{-a.s.}$$

As for (III) in (2.5), we write (for V_n as in (1.2))

$$\begin{aligned} & n^{1/2} (sF_n(2\hat{\theta}_n - t) - sF_n(2\theta_n - t)) \\ &= \frac{n^{1/2}}{2} [F_n(2\hat{\theta}_n - t) - F_n(2(\theta_n - \hat{\theta}_n) + t) - F_n(2\theta_n - t) + F_n(t)] \\ & \quad + O(n^{-1/2}) \\ (2.8) \quad &= \frac{1}{2} [V_n(F(2\hat{\theta}_n - t)) - V_n(F(2\theta_n - t))] \\ & \quad - \frac{1}{2} [V_n(F(2(\theta_n - \hat{\theta}_n) + t)) - V_n(F(t))] \\ & \quad + \frac{n^{1/2}}{2} [F(2\hat{\theta}_n - t) - F(2(\theta_n - \hat{\theta}_n) + t) - F(2\theta_n - t) + F(t)] \\ & \quad + O(n^{-1/2}). \end{aligned}$$

Stute (1982), Lemmas 2.4 and 2.6, proves inequalities that imply the following: For all $c > 0$,

$$(2.9) \quad \limsup_n \sup_{|u-v| \leq c/\log \log n} |V_n(u) - V_n(v)| \leq 8c^{1/2} \quad \text{a.s.}$$

[An easy proof of this fact follows from Inequalities 1 and 2 in Mason, Shorack and Wellner (1983), which are based on Stute, loc. cit. Alternatively, one can use strong approximation, i.e., KMT, as in their Section 4.] By our hypothesis on F there exist $c_n \rightarrow \infty$ such that

$$(2.10) \quad \sup_{|t-s| \leq c_n/n^{1/2}} |F(t) - F(s)| \leq c/(\log \log n)^{1+\delta},$$

for all $n \geq 27$ and some $\delta > 0$. Since $\{n^{1/2}(\hat{\theta}_n - \theta_n)\}_{n=1}^\infty$ is $\widehat{\Pr}$ -stochastically bounded ω -a.s. by Definition 2.2, it follows from (2.9) and (2.10) that the V_n terms in (2.8) converge uniformly in t to 0 in $\widehat{\Pr}$ probability ω -a.s. As for the last term of (2.8), using a.s. convergence of θ_n to θ , the ω -a.s. stochastic boundedness of $n^{1/2}(\hat{\theta}_n - \theta_n)$ and the uniform differentiability of F , we have

that ω -a.s.

$$\left\| \frac{n^{1/2}}{2} \left[F(2\hat{\theta}_n - t) - F(2\theta_n - t) - F(2(\theta_n - \hat{\theta}_n) + t) + F(t) \right] - n^{1/2}(\hat{\theta}_n - \theta)(f(t) + f(2\theta - t)) \right\|_{\infty} \rightarrow 0$$

in $\widehat{\Pr}$ probability.

Hence (III) in (2.5) is asymptotically equivalent (in $\widehat{\Pr}$ probability, ω -a.s.) to $n^{1/2}(\hat{\theta}_n - \theta)(sf)(t)$. This and (2.7) give the asymptotic equivalence between the $l^{\infty}(D_{sP})$ -valued random variables $n^{1/2}(F_{n, sP_n(\omega)} - sF_{n, sP_n(\omega)})$ and

$$\begin{aligned} & \frac{1}{2}n^{1/2}(P_{n, sP_n(\omega)} - sP_n(\omega))(I(-\infty, t] - I(-\infty, 2\theta - t]) \\ & + n^{1/2}(\hat{\theta}_n - \theta)(sf)(t). \end{aligned}$$

By Lemma 2.1 and the joint convergence implied by Definition 2.2, this last sequence of processes converges in $\widehat{\Pr}$ law, ω -a.s. to the process Z_{sP} . \square

REMARK 2.4. Schuster and Barker (1987) suggest considering a smoothed symmetric bootstrap. This requires proving that the limit in part (i) of Theorem 2.3 holds also for the processes

$$n^{1/2}(F_{n, (sP_n)^*\lambda_n} - s^{\hat{\theta}_n}F_{n, (sP_n)^*\lambda_n})(t), \quad t \in \mathbb{R},$$

where λ_n is uniform over $[-a_n, a_n]$, $a_n \rightarrow 0$, or any other approximate identity. It is easy to check that, in the notation of Lemma 2.1, $\sup_{J \in \mathcal{J}} |((sP_n)^*\lambda_n - sP)(J)| \rightarrow 0$ a.s. and therefore that $n^{1/2}(P_{n, (sP_n)^*\lambda_n} - (sP_n)^*\lambda_n) \rightarrow_{\mathcal{L}} Y$ in $l^{\infty}(\mathbb{R})$ ω -a.s. by the same lemma. Since $n^{1/2}\|F_{n, sP_n} - F_{n, (sP_n)^*\lambda_n}\|_{\infty} \rightarrow 0$ in $\widehat{\Pr}$ probability ω -a.s., $n^{1/2}(\hat{\theta}_n - \theta_n)$ converges in conditional law a.s. jointly with $n^{1/2}(F_{n, (sP_n)^*\lambda_n} - (sF_n)^*\lambda_n)$. These two observations allow us to proceed just as in the proof of Theorem 2.3 and conclude that the processes above have indeed the same limit as the original ones.

Theorem 2.2 and the fact that if F is not symmetric then $\|F - sF\|_{\infty} \neq 0$ immediately give the following corollary.

COROLLARY 2.5. Let $P \in \Pi$, let θ be a $n^{1/2}$ -strongly bootstrap consistent location parameter for Π and suppose the cdf of $\|Z_{sP}\|_{\infty}$ is continuous. Let Π_s be the set of symmetric probability measures in Π . Let $t_{n, \alpha}(\omega)$ be defined by

$$(2.11) \quad t_{n, \alpha}(\omega) := \inf \left[t : \widehat{\Pr} \left\{ \|n^{1/2}(F_{n, sP_n(\omega)} - sF_{n, sP_n(\omega)})\|_{\infty} \geq t \right\} \geq \alpha \right].$$

Consider the test

$$(2.12) \quad \mathcal{H}_0: P \in \Pi_s \quad \text{versus} \quad \mathcal{H}_1: P \in \Pi - \Pi_s$$

with rejection region $\|n^{1/2}(F_n - sF_n)\|_\infty \geq t_{n,\alpha}$. Then

$$(2.13) \quad \begin{aligned} \Pr\{\|n^{1/2}(F_n - sF_n)\|_\infty \geq t_{n,\alpha} | \mathcal{H}_0\} &\rightarrow \alpha, \\ \Pr\{\|n^{1/2}(F_n - sF_n)\|_\infty \geq t_{n,\alpha} | \mathcal{H}_1\} &\rightarrow 1. \end{aligned}$$

If t_α is defined as $P\{\|Z_{sP}\|_\infty > t_\alpha\} = \alpha$ and the cdf of $\|Z_{sP}\|_\infty$ is strictly increasing at t_α , then $t_{n,\alpha}(\omega) \rightarrow t_\alpha$ a.s.

Consider now the local alternatives

$$Q_n = P + \Delta/n^\lambda, \quad 0 < \lambda \leq 1/2,$$

where $Q_n, P \in \Pi$, P is symmetric but Q_n is not. Under some modified consistency hypotheses on $\theta(Q_n)$, we can obtain asymptotic consistency of the tests of symmetry described above. It is convenient to use the special construction described in Section 1 in order to give a meaning to the a.s. bootstrap. Letting ξ_{ni}, ξ_i be the uniform random variables from Section 1, suppose that we have

$$(2.14) \quad n^\lambda(\theta(Q_n) - \theta(P)) \rightarrow \eta(P, \Delta, \lambda), \quad 0 < \lambda \leq 1/2,$$

$$(2.15) \quad n^{\alpha(\lambda)}[\theta_n(F_{Q_n}^{-1}(\xi_{n1}), \dots, F_{Q_n}^{-1}(\xi_{nn})) - \theta(Q_n)] \rightarrow \tilde{\theta}(P, \Delta, \lambda) \quad \text{a.s.}$$

for some random variable $\tilde{\theta}(P, \Delta, \lambda)$ and $0 < \alpha(\lambda) \leq 1/2$, with $\alpha(1/2) = 1/2$, and

$$(2.16) \quad \begin{aligned} \omega\text{-a.s.}, n^{1/2}[\theta_n((sF_{n,Q_n})^{-1}(\omega)(\xi_{n1}), \dots, (sF_{n,Q_n})^{-1}(\omega)(\xi_{nn})) \\ - \theta_n(X_{n1}(\omega), \dots, X_{nn}(\omega))] \rightarrow \tilde{\theta}(sP) \\ \text{a.s. where } X_{ni} = F_{Q_n}^{-1}(\xi_i). \end{aligned}$$

Then we have the following proposition.

PROPOSITION 2.6. *The following holds under the assumptions of the previous paragraph:*

- (i) *The processes $n^{1/2}(F_{n,sP_{n,Q_n}(\omega)} - sF_{n,sP_{n,Q_n}(\omega)})(t)$ converge in law in $l^\infty(\mathbb{R})$ a.s. to the limiting process of Theorem 2.2.*
- (ii) *For $\lambda = 1/2$,*

$$\begin{aligned} n^{1/2}(F_{n,Q_n} - sF_{n,Q_n})(\cdot) &\rightarrow \frac{1}{2}(U(F(\cdot)) + U(1 - F(\cdot))) + \frac{1}{2}d(\Delta, \cdot) \\ &\quad + (\eta(P, \Delta, 1/2) + \tilde{\theta}(P, \Delta, 1/2))f(\cdot), \end{aligned}$$

where $d(\Delta, t) = \Delta(-\infty, t] - \Delta[2\theta - t, \infty)$. For $0 < \lambda < 1/2$,

$$n^{1/2}\|F_{n,Q_n} - sF_{n,Q_n}\|_\infty \rightarrow \infty \quad \text{a.s.},$$

assuming $\|\frac{1}{2}d(\Delta, t) + \eta(P, \Delta, \lambda)f(t)\|_\infty \neq 0$ if $\alpha(\lambda) > \lambda$, $\|\frac{1}{2}d(\Delta, t) + (\eta(P, \Delta, \lambda) + \tilde{\theta}(P, \Delta, \lambda))f(t)\|_\infty \neq 0$ a.s. if $\alpha(\lambda) = \lambda$ and $|\tilde{\theta}(P, \Delta, \lambda)| > 0$ a.s. if $\alpha(\lambda) < \lambda$.

As a consequence, the tests described in Corollary 2.5 have asymptotic power 1 against the local alternatives $P + \Delta/n^\lambda$ if $\lambda < 1/2$ and asymptotic power $P(\|Z_\Delta\|_\infty > t_\alpha)$ if $\lambda = 1/2$, where $P(\|Z_P\|_\infty > t_\alpha) = \alpha$ and $Z_{\Delta, P}$ is the limit in part (ii), Proposition 2.6.

We omit the proof of Proposition 2.6 since it follows very closely that of Theorem 2.2.

3. Asymptotic theory for the test of Schuster and Barker. The location parameter of Schuster and Narvarte (1973) departs from typical smooth statistics in that $n^{1/2}(\theta(P_n) - \theta(P))$ is not even asymptotically normal. The limiting processes of Theorem 2.3 for this parameter are not Gaussian either. Given $P \in \mathcal{P}(\mathbb{R})$ and $a \in \mathbb{R}$, let

$$(3.1) \quad \begin{aligned} D^+(a) &= D^+(a, P) = \sup_{t \in \mathbb{R}} P[I(-\infty, t] - I[2a - t, \infty)] \\ D^-(a) &= D^-(a, P) = \sup_{t \in \mathbb{R}} P[I[2a - t, \infty) - I(-\infty, t]]. \end{aligned}$$

Note that $D^+(-\infty^+) = D^-(+\infty^-) = 0$, $D^+(+\infty^-) = D^-(-\infty^+) = 1$, D^+ is nondecreasing and D^- is nonincreasing. D^+ is left continuous and D^- is right continuous. If P has a bounded density, then both D^+ and D^- are continuous. We then let

$$(3.2) \quad \begin{aligned} \theta^* &= \theta^*(P) = \sup\{a: D^+(a) < D^-(a)\}, \\ \theta^{**} &= \theta^{**}(P) = \inf\{a: D^+(a) > D^-(a)\} \end{aligned}$$

and

$$(3.3) \quad \theta = \theta(P) = (\theta^*(P) + \theta^{**}(P))/2.$$

θ is the Schuster–Narvarte location parameter. Obviously, $a = \theta(P)$ minimizes the function

$$(3.4) \quad D(a, P) = D(a) = \|F - s^a F\|_\infty$$

and therefore

$$(3.5) \quad \|F - s^\theta F\|_\infty = \inf_{Q \text{ symmetric}} \|F - Q\|_\infty.$$

(It is easy to show that $\inf_{Q \text{ sym}} \|F - Q\|_\infty = \inf_a \|F - s^a F\|_\infty$.) In this minimum distance sense, the test for symmetry based on the statistic $n^{1/2}\|F_n - s^\theta F_n\|_\infty$, that is, the Schuster–Barker test, is most natural. We refer to Schuster and Barker (1987) for practical ways to compute this statistic as well as the location parameter $\theta_n = \theta(P_n)$.

For the Schuster–Narvarte parameter, we let Π_{SN} be the set of probability measures P on \mathbb{R} satisfying the hypotheses of Theorem 2.3, and such that $D(a, P)$ attains its minimum at a single point [which is therefore $\theta(P)$]. This last condition holds even when the median is not unique, but it does not hold in general. For example, it can be seen that if an absolutely continuous cdf F has only one flat on its convex domain, or is symmetric with respect to some center, then $D(a)$ attains its minimum at a single point. We check this

property for symmetric distributions: If P is symmetric and θ is its center of symmetry, then $D^+(a) = \sup_{t \in \mathbb{R}} (F(2a - t) - F(2\theta - t))$ is strictly positive for $a > \theta$ and 0 for $a \leq \theta$; similarly $D^-(a)$ is strictly positive for $a < \theta$ and 0 for $a \geq \theta$, so that $D^+(a) = D^-(a)$ if and only if $a = \theta$, that is, D attains its minimum only at θ .

To justify the test for large n , in view of Theorem 2.3 and Corollary 2.5, it suffices to prove that the Schuster–Narvarte parameter θ is $n^{1/2}$ -bootstrap consistent and that the limit $\|Z_{sP}\|_\infty$ has a continuous distribution, strictly increasing on its convex domain. The proofs of these assertions follow.

Let $P \in \Pi$, $\theta = \theta(P)$ and $\theta_n = \theta(P_n)$. We show first that

$$(3.6) \quad \theta_n \rightarrow \theta \quad \text{a.s.}$$

By the Glivenko–Cantelli theorem, for all $a > 0$,

$$D^+(a, P_n) \rightarrow D^+(a, P), \quad D^-(a, P_n) \rightarrow D^-(a, P) \quad \text{a.s.}$$

For each $\varepsilon > 0$, taking $a = \theta + \varepsilon$ (note $\theta = \theta^* = \theta^{**}$), it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} D^+(\theta + \varepsilon, P_n) &= D^+(\theta + \varepsilon, P) > D^-(\theta + \varepsilon, P) \\ &= \lim_{n \rightarrow \infty} D^-(\theta + \varepsilon, P_n) \end{aligned}$$

so that from some $n(\omega)$ ($< \infty$ a.s.) on, $D^+(\theta + \varepsilon, P_n) > D^-(\theta + \varepsilon, P_n)$; hence $\theta + \varepsilon > \theta^{**}(P_n) \geq \theta(P_n)$. Similarly, taking $a = \theta - \varepsilon$ gives $\theta - \varepsilon \leq \theta^*(P_n) \leq \theta(P_n)$ from some n on, and (3.6) follows.

To prove part (b) in Definition 2.2, we let $\bar{P}_n = n^{-1} \sum_{i=1}^n \delta_{F^{-1}(\xi_{ni})}$ and $\bar{\theta}_n = \theta(\bar{P}_n)$. In view of (1.3) it will suffice to prove

$$n^{1/2}(\bar{\theta}_n - \theta) \rightarrow \tilde{\theta}(P) \quad \text{a.s.}$$

for some random variable $\tilde{\theta}(P)$. We define

$$\begin{aligned} E_n^+(a) &= \sup_{t \in \mathbb{R}} [U_n(F(t)) + U_n(1 - F(t)) + 2af(t)], \\ (3.7) \quad E_n^-(a) &= \sup_{t \in \mathbb{R}} [-U_n(F(t)) - U_n(1 - F(t)) - 2af(t)], \quad a \in \mathbb{R}. \end{aligned}$$

$E^+(a)$ and $E^-(a)$ are defined by the same expressions, with U_n replaced by U . We then have

$$\begin{aligned} n^{1/2} \bar{P}_n(I(-\infty, t] - I[2\theta + 2a/n^{1/2} - t, \infty)) \\ &= n^{1/2}(\bar{P}_n - P)(I(-\infty, t] - I[2\theta - t, \infty)) \\ &\quad + n^{1/2}(\bar{P}_n - P)(I[2\theta - t, \infty) - I[2\theta + 2a/n^{1/2} - t, \infty)) \\ &\quad + n^{1/2}(F(t) - F(t - 2a/n^{1/2})). \end{aligned}$$

Therefore

$$\begin{aligned} & |n^{1/2}D^+(\theta + a/n^{1/2}, \bar{P}_n) - E_n^+(a)| \\ & \leq 3n^{1/2} \left\| (\bar{P}_n - P)\{t\} \right\|_\infty + \left\| U_n(F(2\theta - t)) - U_n(F(2\theta + 2a/n^{1/2} - t)) \right\|_\infty \\ & \quad + \left\| n^{1/2}(F(t) - F(t - 2a/n^{1/2})) - 2af(t) \right\|_\infty. \end{aligned}$$

Now, the differentiability properties of F and (1.3) together with the uniform continuity of the sample paths of U , give

$$n^{1/2}D^+(\theta + a/n^{1/2}, \bar{P}_n) - E_n^+(a) \rightarrow 0 \quad \text{a.s. for all } a \in \mathbb{R}.$$

Since

$$(3.8) \quad |E_n^+(a) - E^+(a)| \leq \left\| (U_n - U)(F(t)) - (U_n - U)(F(2\theta - t)) \right\|_\infty \rightarrow 0 \quad \text{a.s.},$$

we conclude, using monotonicity and continuity of $E^+(a)$, that

$$(3.9) \quad n^{1/2}D^+(\theta + a/n^{1/2}, \bar{P}_n) \rightarrow E^+(a) \quad \text{for all } a \in \mathbb{R}, \text{ a.s.}$$

Similarly,

$$(3.10) \quad n^{1/2}D^-(\theta + a/n^{1/2}, \bar{P}_n) \rightarrow E^-(a) \quad \text{for all } a \in \mathbb{R}, \text{ a.s.}$$

It is shown in Rao, Schuster and Littell (1975), proof of Lemma 1, that for each a , $P\{E^+(a) = E^-(a)\} = 0$. Since E^+ and E^- are continuous, $E^+(a) \nearrow$, $E^-(a) \searrow$, $E^+(\infty^-) = E^-(\infty^+) = +\infty$ a.s. and $E^+(-\infty^+) = E^-(-\infty^-) = -\infty$, it follows that $E^+(r) \neq E^-(r)$ for all r rational, ω -a.s. Therefore

$$E^+(a) = E^-(a)$$

has a.s. a *unique* solution $\tilde{\theta}$ (which is clearly a random variable). Given $\varepsilon > 0$, $E^-(\tilde{\theta} + \varepsilon) < E^+(\tilde{\theta} + \varepsilon)$ a.s. so that by (3.9) and (3.10), eventually a.s.

$$D^-(\theta + (\tilde{\theta} + \varepsilon)/n^{1/2}, \bar{P}_n) < D^+(\theta + (\tilde{\theta} + \varepsilon)/n^{1/2}, \bar{P}_n),$$

implying $\limsup_{n \rightarrow \infty} n^{1/2}(\theta^{**}(\bar{P}_n) - \theta) \leq \tilde{\theta}$ a.s. Similarly, using $a = \tilde{\theta} - \varepsilon$, we obtain $\liminf_{n \rightarrow \infty} n^{1/2}(\theta^*(\bar{P}_n) - \theta) \geq \tilde{\theta}$ a.s. Therefore

$$\lim_{n \rightarrow \infty} n^{1/2}(\theta(\bar{P}_n) - \theta) = \tilde{\theta} \quad \text{a.s.}$$

and part (b) in Definition 2.2 is proved.

Rao, Schuster and Littell (1975), Theorem 4, proved weak convergence of $n^{1/2}(\theta(P_n) - \theta)$ and obtained the law of $\tilde{\theta}$, which is not normal:

$$(3.11) \quad P\{\tilde{\theta} \leq t\} = P\left[\sup_{0 \leq u \leq 1/2} (W(u) + 2^{1/2}tf(F^{-1}(u))) \right. \\ \left. + \inf_{0 \leq u \leq 1/2} (W(u) + 2^{1/2}tf(F^{-1}(u))) \geq 0 \right],$$

where W is Brownian motion. The above proof is somewhat similar to theirs. The nice thing about it is that it allows bootstrapping, that is, part (c) in

Definition 2.2, which we see next. Let $\bar{P}_{n, sP_n} = n^{-1} \sum_{i=1}^n \delta_{(sF_n)^{-1}(\xi_{ni})}$, so that we can apply the "special construction" [leading to (1.3)] to $\bar{P}_{n, sP_n(\omega)}$ for each ω . Define

$$\bar{E}_n^+(a) = \sup_{t \in \mathbb{R}} (U_n(sF_n(t)) + U_n(1 - sF_n(t)) + 2a(sf(t)))$$

and likewise $\bar{E}_n^-(a)$. As in the proof of part (b) above, we have, with $\theta_n = \theta(P_n)$,

$$\begin{aligned} & \left| n^{1/2} D^+(\theta_n + a/n^{1/2}, \bar{P}_{n, sP_n}) - \bar{E}_n^+(a) \right| \\ & \leq 3n^{1/2} \left\| (\bar{P}_{n, sP_n} - sP_n)(t) \right\|_{\infty} \\ & \quad + \left\| U_n(sF_n(2\theta_n + 2a/n^{1/2} - t)) - U_n(sF_n(2\theta_n - t)) \right\|_{\infty} \\ & \quad + \left\| n^{1/2}(sF_n(t) - sF_n(t - 2a/n^{1/2})) - 2a(sf)(t) \right\|_{\infty} + O(n^{-1/2}) \end{aligned}$$

ω -a.s.

(Note that sF_n has jumps of size $1/2n$ a.s.) Call the terms at the right of the inequality respectively (I), (II) and (III). Stute's exponential bound by (2.9) and (2.10) makes (III) ω -a.s. $(\widehat{\text{Pr}})$ -weak convergence equivalent to

$$\begin{aligned} & \left\| \frac{n^{1/2}}{2} (F(t) - F(t - 2a/n^{1/2})) \right. \\ & \quad \left. + \frac{n^{1/2}}{2} (F(2\theta_n + 2a/n^{1/2} - t) - F(2\theta_n - t)) - 2a(sf)(t) \right\|_{\infty}, \end{aligned}$$

which, by differentiability of F and by a.s. convergence of θ_n to θ , converges to 0 ω -a.s. Also, by (1.3), in (I) $\leq 3 \sup_{|t-s| \leq 1/n} |U_n(sF_n(t)) - U_n(sF_n(s))|$, U_n can be replaced by U . Hence (I) $\rightarrow 0$ $\widehat{\text{Pr}}$ a.s. for almost every ω by the uniform continuity of U , the Glivenko-Cantelli theorem for sF_n and the uniform continuity of F . The same applies to (II) with n^{-1} replaced by $n^{-1/2}$. The ω -set of probability 1, where $\widehat{\text{Pr}}$ -a.s. convergence to 0 takes place, can be made to work for all a simultaneously. So we have: For almost every ω , for all $a \in \mathbb{R}$,

$$n^{1/2} D^+(\theta_n + a/n^{1/2}, \bar{P}_{n, sP_n(\omega)}) - \bar{E}_n^+(a) \rightarrow 0 \quad \widehat{\text{Pr}}\text{-a.s.}$$

Using $\|sF_n - sF\|_{\infty} \rightarrow 0$ and (1.3) we obtain, as in (3.9), that ω -a.s., for all $a \in \mathbb{R}$,

$$n^{1/2} D^+(\theta_n + a/n^{1/2}, \bar{P}_{n, sP_n(\omega)}) \rightarrow E_{sP}^+(a) \quad \widehat{\text{Pr}}\text{-a.s.}$$

The same holds for D^- , and therefore proceeding as in the proof of part (b) it follows that ω -a.s.

$$\lim_{n \rightarrow \infty} n^{1/2} (\theta(\bar{P}_{n, sP_n}) - \theta(P_n)) = \tilde{\theta}(sP) \quad \widehat{\text{Pr}}\text{-a.s.},$$

which implies, in view of (1.3), part (c) in Definition 2.2. This finishes the proof of bootstrap consistency of θ_n .

Regarding (2.14)–(2.16), a similar proof (which is omitted) gives that (2.16) holds, and moreover

$$(3.12) \quad \eta(P, \Delta, \lambda) = \eta(P, \Delta), \quad 0 < \lambda \leq 1/2,$$

is the unique solution, if it exists, of the equation (in α)

$$(3.13) \quad \sup_{t \in \mathbb{R}} (2af(t) + d(\Delta, t)) = \sup_{t \in \mathbb{R}} (-2af(t) - d(\Delta, t)),$$

$$(3.14) \quad \alpha(\lambda) = \lambda,$$

$\tilde{\theta}(P, \Delta, 1/2)$ is the unique solution of the equation

$$(3.15) \quad \sup_{t \in \mathbb{R}} ((U(F(t)) + U(1 - F(t))) + 2af(t) + 2\eta(P, \Delta)f(t) + d(\Delta, t)) \\ = \sup_{t \in \mathbb{R}} (-(U(F(t)) + U(1 - F(t))) - 2af(t) \\ - 2\eta(P, \Delta)f(t) - d(\Delta, t)),$$

and for $0 < \lambda < 1/2$, $\tilde{\theta}(P, \Delta, \lambda)$ is the unique solution, if it exists, of the equation

$$(3.16) \quad \sup_{t \in \mathbb{R}} (2af(t) + 2\eta(P, \Delta)f(t) + d(\Delta, t)) \\ = \sup_{t \in \mathbb{R}} (-2af(t) - 2\eta(P, \Delta)f(t) - d(\Delta, t)).$$

Finally, we check that the cdf of $\|Z_{sP}\|_\infty$ is continuous and strictly increasing on the convex domain of F . By the discussion following (3.10) above, the random variable $\tilde{\theta}(sP)$ is the value of α that minimizes the expression

$$\left\| \frac{1}{2}U(sF(t)) + \frac{1}{2}U(1 - sF(t)) + \alpha(sf)(t) \right\|_\infty.$$

In other words,

$$(3.17) \quad \|Z_{sP}\|_\infty = \left\| \frac{1}{2}U(t) + \frac{1}{2}U(1 - t) \right\|,$$

where $\|\cdot\|$ denotes the seminorm on $C[0, 1]$ defined as the sup distance to the one-dimensional subspace $E = \{\lambda \cdot (sf)((sF)^{-1}(t)): \lambda \in \mathbb{R}\}$. Let H be the cdf of $\|Z_{sP}\|_\infty$. Then H is log concave and $H(x) < 1$, for all $x < \infty$ [see, e.g., Theorem 1.1 and (1.13) in Hoffmann-Jørgensen, Shepp and Dudley (1979)]. Therefore H is strictly increasing on $[0, \infty)$. $C[0, 1]$ is separable for $\|\cdot\|$ so that by Corollary 2.2, loc. cit., $\inf\{x \geq 0: H(x) > 0\} = 0$. This, by Theorem 1.2, loc. cit. [see also Cirel'son (1975)] implies that H is continuous except perhaps at 0. But $H(0)$ is 0 or 1 (Theorem 1.1., loc. cit.), therefore it is 0, and H is continuous on $(-\infty, \infty)$.

In conclusion, we have shown the following theorem.

THEOREM 3.1. *The Schuster–Narvarte location parameter is $n^{1/2}$ -bootstrap consistent for Π_{SN} , with $\tilde{\theta}(P)$ given as the a.s. unique solution of $E^+(a) = E^-(a)$, and it also satisfies (2.14)–(2.16) with variables as defined in (3.12)–(3.16). Moreover, $\|Z_{sP}\|_\infty$ has a continuous cdf strictly increasing on*

$[0, \infty)$. Therefore the conclusions of Theorem 2.3, Corollary 2.5 and Proposition 2.6 hold for $n^{1/2}(F_n - sF_n)$ and for its symmetric bootstrap.

4. Other location parameters. As long as a location parameter is $n^{1/2}$ -bootstrap consistent, it can be used in the bootstrap symmetry test of Corollary 2.5. Even if the Schuster–Narvarte parameter seems to be the most natural one to use because of its optimality for the Kolmogorov distance, the asymptotic properties (power included) of the test based on other parameters are quite similar to those of the Schuster–Barker test (Proposition 2.6 and Corollary 2.5). Here is a description, without proofs, of these characteristics for three more parameters of interest. The class Π of probability measures to which these tests apply is different for each parameter, and this should play an important part in deciding which test to use. For two of the parameters the limiting process Z of Theorem 2.3 is Gaussian, which indicates that some differentiability of $\theta(P)$ is present—not as strong, however, as the differentiability hypothesis in Romano (1988), (2.1).

4.1. The median. If $\theta(P)$ is defined to be $\theta(P) = m(P)$ the median of P , for $P \in \Pi_m$, the set of absolutely continuous measures on \mathbb{R} whose cdf's satisfy the Hölder continuity hypothesis of Theorem 2.3 and whose densities f are uniformly continuous and positive at their median (thus making the median unique), then (1) θ is $n^{1/2}$ -bootstrap consistent for Π_m , (2) the limiting process of Theorem 2.3 is

$$Z_{sP}(t) = \frac{1}{2} [U(sF(t)) + U(1 - sF(t))] - [U(1/2)/f(m)](sf)(t),$$

$$t \in D_{sP},$$

a Gaussian process such that the cdf of $\|Z_{sP}\|_\infty$ is continuous and strictly increasing on $[0, \infty)$, and (3) the limiting process $Z_{\Delta, P}$ of Proposition 2.6 is

$$Z_{\Delta, P}(t) = Z(t) + \frac{1}{2}d(\Delta, t) - F_\Delta(m)f(t)/f(m), \quad t \in D_{s(P+\Delta/n^{1/2})},$$

that is, a shift of $Z_P(t)$.

4.2. The Hodges–Lehmann location parameter. Given P , let X and X' be iid (P) . The Hodges–Lehmann (1963) location parameter θ is defined as the center of medians of $(X + X')/2$. In particular, $\theta(P_n)$ is the center of medians of the set of points $\{(X_i + X_j)/2\}_{i,j=1}^n$. It is essentially proved in Fine (1966) that the p.m. $s^\theta P$, obtained by centering P at its Hodges–Lehmann location, minimizes the expression

$$\rho^2(P, Q) = \int_{-\infty}^{\infty} (F_P - F_Q)^2(x) dx$$

over all Q symmetric, assuming $\int |x| dP < \infty$. A minimum distance test could thus be based on the statistic $n^{1/2}\rho(P_n, sP_n)$ but it would only be applicable if P had a finite first moment—and in this case Theorem 3.3 applies. On the contrary, the Kolmogorov distance test of Corollary 3.5 can be used without

moment requirements. If Π_{HL} is the set of absolutely continuous probability measures whose cdf's satisfy the Hölder condition of Theorem 3.3 and whose densities f are uniformly continuous and satisfy $\int_{-\infty}^{\infty} f(2\theta(P) - x) dF(x) > 0$ [implying that $(X + X')/2$ has a unique median], then the Hodges–Lehmann parameter is $n^{1/2}$ -bootstrap consistent for Π_{HL} , and

$$Z_{sP}(t) = \frac{1}{2}[U(sF(t)) + U(1 - sF(t))] - \left[\int_0^1 U(v) dv \middle/ \int_{-\infty}^{\infty} (sf)^2(v) dv \right] (sf)(t),$$

a Gaussian process such that $\|Z_{sP}\|_{\infty}$ has a continuous distribution strictly increasing on $[0, \infty)$. Also

$$Z_{\Delta, P}(t) = Z_P(t) + \frac{1}{2}d(\Delta, t) + \eta(P, \Delta)f(t)$$

with

$$\eta(P, \Delta) = - \int_{-\infty}^{\infty} F_{\Delta}(2\theta - x) dF(x) \middle/ \int_{-\infty}^{\infty} f^2(x) dx.$$

4.3. The Hodges location parameter. The Hodges (1965) location parameter $\theta(P)$ is defined as the center of medians of $(F^{-1}(x) + F^{-1}(1 - x))/2$ considered as a random variable on $([0, 1], \mathcal{B}, \lambda)$. In particular, $\theta(P_n)$ is the center of medians of the set of points $\{(X_{n,i} + X_{n,n+1-i})/2\}_{i=1}^n$, where $X_{n,1} \leq \dots \leq X_{n,n}$ is the order statistic of X_1, \dots, X_n . It is not difficult to prove that $s^{\theta}P$ minimizes, if it exists, the Wasserstein's distance

$$w(P, Q) = \int_{-\infty}^{\infty} |F_P(x) - F_Q(x)| dx$$

over all Q symmetric. If Π_{H} is the set of probability measures on \mathbb{R} with densities f bounded, uniformly continuous and strictly positive on D_P , then $\theta(P)$ is $n^{1/2}$ -bootstrap consistent for Π_{H} . Then, for P symmetric,

$$\tilde{\theta}(P) = \inf \{ \alpha: \lambda \{ t \in [0, 1/2]: U(t) + U(1 - t)/f(F^{-1}(t)) \leq \alpha \} \geq 1/4 \}$$

and the process

$$Z_{sP}(t) = \frac{1}{2}[U(sF(t)) + U(1 - sF(t))] + \tilde{\theta}(sP)(sf)(t)$$

is not Gaussian. We do not know if $\|Z\|_{\infty}$ has a continuous cdf. The process $Z_{\Delta}(t)$ is

$$Z_{\Delta, P}(t) = Z(t) + \frac{1}{2}d(\Delta, t) + \eta(P, \Delta)(sf)(t),$$

where $\eta(P, \Delta)$ is the center of medians with respect to Lebesgue measure on $[0, 1]$ of

$$-\Delta(-\infty, F^{-1}(t))/2f(F^{-1}(t)) - \Delta(-\infty, F^{-1}(1 - t))/2f(F^{-1}(1 - t)).$$

Acknowledgments. We are grateful to D. Alemayehu and to V. de la Peña for bringing the paper of Schuster and Barker (1987) to our attention, and to R. Dudley for an interesting conversation on Gaussian processes.

REFERENCES

- ANTILLE, A., KERSTING, G. and ZUCCHINI, W. (1982). Testing symmetry. *J. Amer. Statist. Assoc.* **77** 639–646.
- BICKEL, P. J. and HODGES, J. L., JR. (1967). The asymptotic theory of Galton's test and a related sample estimate of location. *Ann. Math. Statist.* **38** 73–89.
- BOOS, D. D. (1982). A test for asymmetry associated with the Hodges–Lehmann estimator. *J. Amer. Statist. Assoc.* **77** 647–649.
- CARROLL, R. J. (1979). On estimating variances of robust estimators when the errors are asymmetric. *J. Amer. Statist. Assoc.* **74** 674–679.
- CIREL'SON, B. S. (1975). The density of the distribution of the maximum of a Gaussian process. *Theory Probab. Appl.* **20** 847–855.
- CSÖRGŐ, S. and HEATHCOTE, C. R. (1987). Testing for symmetry. *Biometrika* **74** 177–186.
- FINE, T. (1966). On the Hodges and Lehmann shift estimator in the two sample problem. *Ann. Math. Statist.* **37** 1814–1818.
- GINÉ, E. and ZINN, J. (1991). Gaussian characterization of uniform Donsker classes of functions. *Ann. Probab.* **19** 758–782.
- HÁJEK, J. and SIDAČ, A. (1967). *Theory of Rank Tests*. Academic, New York.
- HODGES, J. L., JR. (1967). Efficiency in normal samples and tolerance of extreme values for some estimates of location. In *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **1** 163–186. Univ. California Press, Berkeley.
- HODGES, J. L., JR. and LEHMANN, E. (1963). Estimates of location based on rank tests. *Ann. Math. Statist.* **34** 598–611.
- HOFFMANN-JØRGENSEN, J., SHEPP, L. A. and DUDLEY, R. M. (1979). On the lower tail of Gaussian seminorms. *Ann. Probab.* **7** 319–342.
- KOZIOL, J. A. (1983). Tests of symmetry about an unknown value based on the empirical distribution function. *Comm. Statist. Theory Methods* **12** 2823–2844.
- MASON, D. M., SHORACK, G. R. and WELLNER, J. A. (1983). Strong limit theorems for oscillation moduli of the uniform empirical process. *Z. Wahrsch. Verw. Gebiete* **65** 83–97.
- RAO, P. V., SCHUSTER, E. F. and LITTELL, R. C. (1975). Estimation of shift and center of symmetry based on Kolmogorov–Smirnov statistics. *Ann. Statist.* **3** 862–873.
- ROMANO, J. H. (1988). A bootstrap revival of some non-parametric distance tests. *J. Amer. Statist. Assoc.* **83** 698–708.
- ROMANO, J. H. (1989). Bootstrap and randomization tests of some nonparametric hypothesis. *Ann. Statist.* **17** 141–159.
- ROTHMAN, E. D. and WOODROOFE, M. (1972). A Cramér–von Mises type statistic for testing symmetry. *Ann. Math. Statist.* **43** 2035–2038.
- SCHUSTER, E. F. and BARKER, R. C. (1987). Using the bootstrap in testing symmetry versus asymmetry. *Comm. Statist. Simulation Comput.* **16** 69–84.
- SCHUSTER, E. F. and NARVARTE, J. A. (1973). A new nonparametric estimator of the center of a symmetric distribution. *Ann. Statist.* **1** 1096–1104.
- SHORACK, G. R. (1972). Functions of order statistics. *Ann. Math. Statist.* **43** 412–427.
- SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.
- SMIRNOV, N. V. (1947). On criteria for the symmetry of distribution laws of random variables. *Dokl. Akad. Nauk. SSSR* **56** 13–16.
- STUTE, W. (1982). The oscillation behavior of empirical processes. *Ann. Probab.* **10** 86–107.

UNIVERSITY OF CONNECTICUT
DEPARTMENT OF MATHEMATICS, U-9
STORRS, CONNECTICUT 06269