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CONSTRUCTION OF 2^m4^n DESIGNS VIA A GROUPING SCHEME[1]

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We develop a method for grouping the $2^k - 1$ factorial effects in a 2-level factorial design into mutually exclusive sets of the form (s, t, st), where st is the generalized interaction of effects s and t. As an application, we construct orthogonal arrays $OA(2^k, 2^m4^n, 2)$ of size 2^k, m constraints with 2 levels and n constraints with 4 levels satisfying $m + 3n = 2^k - 1$, and strength 2. The maximum number of constraints with 4 levels in the construction cannot be further improved. In this sense our grouping scheme is optimal. We discuss the advantages of the present approach over other construction methods.

1. Introduction. In this paper we use a "grouping" method to construct factorial designs with 2-level and 4-level factors from those with 2-level factors. Consider a saturated fractional factorial design with $N = 2^k$ runs (rows) and $2^k - 1$ variables (columns). Each variable has two levels denoted by 0 and 1. The $2^k - 1$ variables can formally be represented as the $2^k - 1$ factorial effects of k factors (see Section 2). Since for any two columns of the design, each possible level combination appears equally often, the design is also called an orthogonal array $OA(N, 2^{N-1}, 2)$ of size N, $N - 1$ constraints with 2 levels, and strength two [Rao (1947)]. Take three columns of the form $(\alpha, \beta, \alpha\beta)$, where the column $\alpha\beta$ is obtained as the sum (mod 2) of the column α and the column β. (In factorial design, $\alpha\beta$ is called the generalized interaction of effects α and β.) Replace these three 2-level columns by one 4-level column according to the rule

$$(1)\qquad \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} \;\rightarrow\; \begin{matrix} 0 \\ 1 \\ 2 \\ 3. \end{matrix}$$

This 4-level column is orthogonal to the remaining 2-level columns in $OA(N, 2^{N-1}, 2)$ in the sense that each of the eight pairs (i, j), $i = 0, 1, 2, 3$ and $j = 0, 1$ in the two columns appears equally often (Addelman, 1962).

More generally, if among the $2^k - 1$ columns there are n exclusive sets of the form $(\alpha_i, \beta_i, \alpha_i\beta_i)$, $i = 1, \dots, n$, by applying the previous method of replacement to each set, $3n$ 2-level columns in $OA(N, 2^{N-1}, 2)$ are replaced by n 4-level columns. The resulting design is still saturated and is an orthogonal array $OA(N, 2^m4^n, 2)$ of size N, m constraints with 2 levels and n constraints with

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4 levels satisfying $m + 3n = N - 1$, and strength two. The method was employed by Addelman (1962) to construct many 2^m4^n designs. By using a general result of Bose and Bush (1952), he also showed that the upper bound on n, the number of 4-level columns, is $(2^k - 1)/3$ for even k and $(2^k - 5)/3$ for odd k, where $N = 2^k$. However, he did not give any systematic method for grouping the $2^k - 1$ effects (columns) or its proper subset into exclusive sets of the form $(\alpha, \beta, \alpha\beta)$. Nor did he show that the previous upper bound on n is attainable with a proper grouping scheme. The main purpose of this paper is to develop such a grouping scheme in Section 2. In Section 3 we consider other methods for constructing OA($N, 2^m4^n$, 2) and discuss the advantages of the present approach over the others.

2. Grouping of factorial effects. The $2^k - 1$ constraints in OA($N, 2^{N-1}$, 2), $N = 2^k$, can be represented as $1^{x_1}2^{x_2} \cdots k^{x_k}$, where $x_i = 0$ or 1 and at least one x_i is 1. In statistical design of experiments, i denotes the main effect of factor i and ij the interaction effect of factors i and j, etc. [Raghavarao (1971)]. Therefore we call these $2^k - 1$ constraints the $2^k - 1$ factorial effects (i.e., main effects and interactions of different orders) of factors $1, 2, \dots, k$. We use the identity I to denote $1^{x_1} \cdots k^{x_k}$ with $x_i = 0$ for all i. These $2^k - 1$ elements and the identity I form an Abelian group, denoted by B_k, of size 2^k by the multiplication rules: for $s = 1^{x_1} \cdots k^{x_k}$ and $t = 1^{y_1} \cdots k^{y_k}$, $s \cdot t = 1^{z_1} \cdots k^{z_k}$, $z_i = x_i + y_i \pmod 2$.

The group B_k has another interpretation. Its 2^k elements are the 2^k subsets of a set of size k with the empty set being the identity I. For any two subsets s and t, $s \cdot t$ is their symmetric difference. Our grouping scheme may find other applications in this context.

The groups B_k can be constructed iteratively as follows. Define $B_1 = \begin{pmatrix} I \\ 1 \end{pmatrix}$ and, for $k \geq 2$,

$$B_k = \begin{pmatrix} B_{k-1} \\ B_{k-1} \cdot k \end{pmatrix}, \tag{2}$$

where $B_{k-1} \cdot k$ consists of the elements of the form $w \cdot k$, $w \in B_{k-1}$. For example,

$$B_2 = \begin{pmatrix} I \\ 1 \\ 2 \\ 12 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} I \\ 1 \\ 2 \\ 12 \\ 3 \\ 13 \\ 23 \\ 123 \end{pmatrix}.$$

The order in which the factorial effects appear in B_k is called the Yates order.

Our grouping scheme works as follows. For $k = 2$, the three effects (1, 2, 12) are already of the form $(\alpha, \beta, \alpha \cdot \beta)$. For $k = 3$, only three out of seven effects can be grouped into a set of the form $(\alpha, \beta, \alpha \cdot \beta)$, say, (1, 2, 12). We then use mathematical induction. Given a grouping scheme for B_k, $k \geq 2$, we will find a grouping scheme for B_{k+2}. From (2), we can write

$$B_{k+2} = \begin{pmatrix} B_k \\ B_k \cdot (k+1) \\ B_k \cdot (k+2) \\ B_k \cdot (k+1)(k+2) \end{pmatrix}. \tag{3}$$

It is shown in Theorem 2 that, for $k \geq 2$, all the elements $w_1, \ldots, w_N$, $N = 2^k$, in B_k can be expressed as $w_i = w_{\pi(i)} \cdot w_{\tau(i)}$, where π and τ are two permutations of $\{1, \ldots, N\}$. Therefore the elements in $B_k \cdot (k+1)$, $B_k \cdot (k+2)$ and $B_k \cdot (k+1)(k+2)$ can be grouped into the exclusive sets $(w_{\pi(i)} \cdot (k+1), w_{\tau(i)} \cdot (k+2), w_i \cdot (k+1)(k+2))$, $i = 1, \ldots, N$, each of which is of the form $(\alpha, \beta, \alpha \cdot \beta)$. An algorithm for finding the two permutations π and τ is given in the proof of Theorem 2.

Theorem 1 summarizes the scheme.

THEOREM 1. (i) *For even $k \geq 2$, all the $2^k - 1$ effects can be grouped into $(2^k - 1)/3$ mutually exclusive sets of the form $(\alpha, \beta, \alpha \cdot \beta)$.*

(ii) *For odd $k \geq 3$, only $(2^k - 5)$ effects can be grouped into $(2^k - 5)/3$ mutually exclusive sets of the form $(\alpha, \beta, \alpha \cdot \beta)$.*

PROOF. For even k, we can apply the induction step (3) for grouping until reaching $k = 2$, thus exhausting all the effects. This proves (i). For odd k, it can be repeated until reaching $k = 3$. Since in B_3, only three out of seven effects can be of the form $(\alpha, \beta, \alpha \cdot \beta)$, there are four effects that cannot be included in the grouping scheme. This proves (ii). □

Results from applying this grouping scheme to B_k, $k = 2$ to 5, are given in Table 1.

TABLE 1

Sets of the form $(\alpha, \beta, \alpha \cdot \beta)$ in $B_k \setminus \{I\}$

k	
2	(1, 2, 12)
3	(1, 2, 12)
4	(1, 2, 12), (3, 4, 34), (13, 24, 1234), (23, 124, 134), (123, 14, 234)
5	(1, 2, 12), (4, 5, 45), (234, 1235, 145), (1234, 135, 245), (134, 15, 345), (14, 25, 1245), (124, 235, 1345), (24, 35, 2345), (34, 125, 12345)
6	(1, 2, 12), (3, 4, 34), (13, 24, 1234), (23, 124, 134), (123, 14, 234), (5, 6, 56), (15, 26, 1256), (125, 16, 256), (25, 126, 156), (35, 46, 3456), (345, 36, 456), (45, 346, 356), (135, 246, 123456), (12345, 136, 2456), (245, 12346, 1356), (235, 1246, 13456), (1345, 236, 12456), (1245, 1346, 2356), (1235, 146, 23456), (2345, 1236, 1456), (145, 2346, 12356)

THEOREM 2. *For any $k \geq 2$, there exist two permutations π and τ of $\{1,\dots,N\}$, $N = 2^k$, such that $w_i = w_{\pi(i)} \cdot w_{\tau(i)}$ for $i = 1$ to N, where w_i are the N elements of B_k.*

PROOF. We prove this by induction. For $k = 2$, it follows from

$$(4)\qquad \begin{array}{c} I\cdot I=I\\ 1\cdot 12=2\\ 2\cdot 1=12\\ 12\cdot 2=1\end{array} \quad\text{or}\quad \begin{array}{c} I\cdot I=I\\ 1\cdot 2=12\\ 2\cdot 12=1\\ 12\cdot 1=2\end{array}.$$

For $k = 3$, it follows from

$$(5)\qquad \begin{array}{r} I\cdot I=I\\ 1\cdot 2=12\\ 2\cdot 3=23\\ 12\cdot 23=13\\ 3\cdot 12=123\\ 13\cdot 1=3\\ 23\cdot 123=1\\ 123\cdot 13=2.\end{array}$$

For $k \geq 4$,

$$B_k = \begin{pmatrix} B_{k-2} \\ B_{k-2}\cdot(k-1) \\ B_{k-2}\cdot k \\ B_{k-2}\cdot k(k-1)\end{pmatrix}.$$

By induction, there exist two permutations π and τ of $\{1,\dots,2^{k-2}\}$ such that

$$w_i = w_{\pi(i)}\cdot w_{\tau(i)} \quad \text{for } i = 1,\dots,2^{k-2},\ w_i \in B_{k-2}.$$

The rest of the elements in $B_k \setminus B_{k-2}$ can be represented by

$$w_i\cdot(k-1) = \{w_{\pi(i)}\cdot k\}\cdot\{w_{\tau(i)}\cdot k(k-1)\},$$
$$w_i\cdot k = \{w_{\pi(i)}\cdot k(k-1)\}\cdot\{w_{\tau(i)}\cdot(k-1)\},$$
$$w_i\cdot k(k-1) = \{w_{\pi(i)}\cdot(k-1)\}\cdot\{w_{\tau(i)}\cdot k\}.$$

This completes the proof. □

REMARK. 1. Theorem 2 does not hold for $k = 1$ since $\left\{\begin{smallmatrix} I\cdot I=I\\ 1\cdot 1=I\end{smallmatrix}\right\}$ and $\left\{\begin{smallmatrix} I\cdot 1=1\\ 1\cdot I=1\end{smallmatrix}\right\}$. This explains why the induction in the proof of Theorem 1 has to stop at $k = 3$.

REMARK. 2. The permutations π and τ in Theorem 2 are not unique. Therefore the proposed grouping scheme does not give a unique decomposition of the factorial effects into sets of the form $(\alpha, \beta, \alpha \cdot \beta)$. For example, the grouping for $k = 4$ in Table 1 is based on the π and τ (for $k = 2$) given by the right system of (4). If the left system of (4) is used for π and τ, the grouping for $k = 4$ will be (1, 2, 12), (3, 4, 34), (13, 124, 234), (23, 14, 1234), (123, 24, 134).

To conclude this section, we give a method of grouping for the case of even k which does not require the construction of the two permutations π and τ. Take $k = 4$. Let $a_1 = (1, 2, 12)$ and $a_2 = (3, 4, 34)$ be the two *generating* sets. Denote the two cyclic permutations of a_2 by $a_2^2 = (4, 34, 3)$ and $a_2^3 = (34, 3, 4)$. For two sets $a = (\alpha, \beta, \gamma)$ and $b = (\alpha', \beta', \gamma')$, define their product $a * b = (\alpha\alpha', \beta\beta', \gamma\gamma')$. Then the 15 effects in $B_4 \setminus \{I\}$ can be grouped into the 5 sets $a_1, a_2, a_1 * a_2, a_1 * a_2^2, a_1 * a_2^3$. By induction, for $k = 2m$, the $2^k - 1$ effects in B_k can be grouped into $l = (2^k - 1)/3$ mutually exclusive sets of the form $(\alpha, \beta, \alpha \cdot \beta)$, denoted by $c_1, \dots, c_l$. Let $a_{m+1} = (k+1, k+2, (k+1)(k+2))$. Then the effects in $B_{k+2} \setminus B_k = \{B_k \cdot (k+1), B_k \cdot (k+2), B_k \cdot (k+1)(k+2)\}$ can be grouped into $c_i * a_{m+1}, c_i * a_{m+1}^2, c_i * a_{m+1}^3, i = 1, \dots, l$, each of which is of the form $(\alpha, \beta, \alpha \cdot \beta)$. The grouping for $k = 6$ in Table 1 is constructed from the grouping for $k = 4$ in this fashion. Note that the order in which the sets in B_k appear in this grouping scheme resembles the Yates order for 4^m factorial design, where $4^m = 2^k$. For $k = 6$, it is $(a_1, a_2, a_1a_2, a_1a_2^2, a_1a_2^3, a_3, a_1a_3, a_1a_3^2, a_1a_3^3, a_2a_3, a_2a_3^2, a_2a_3^3, a_1a_2a_3, a_1a_2a_3^2, a_1a_2a_3^3, a_1a_2^2a_3, a_1a_2^2a_3^2, a_1a_2^2a_3^3, a_1a_2^3a_3, a_1a_2^3a_3^2, a_1a_2^3a_3^3)$.

3. Comparison with other construction methods. The grouping scheme in Section 2 provides a general way of replacing sets of three 2-level columns of the form $(\alpha, \beta, \alpha\beta)$ by 4-level columns according to (1). The following result is a direct consequence of Theorem 1.

COROLLARY. *The following designs can be constructed by using the grouping scheme in Section* 2:

(i) *for even* $k \geq 2$, $\mathrm{OA}(2^k, 2^m4^n, 2)$, $m + 3n = 2^k - 1$, $n = 1, \dots, (2^k - 1)/3$;
(ii) *for odd* $k \geq 3$, $\mathrm{OA}(2^k, 2^m4^n, 2)$, $m + 3n = 2^k - 1$, $n = 1, \dots, (2^k - 5)/3$.

Another approach is to construct 2^m4^n designs from 4-level factorial designs. For example, the orthogonal arrays in Corollary (i) can be obtained by first constructing $\mathrm{OA}(2^k, 4^{n_0}, 2)$, $n_0 = (2^k - 1)/3$, and then replacing a 4-level factor by three 2-level factors as in (1). The orthogonal arrays in Corollary (ii) can be obtained by suitable replacement from $\mathrm{OA}(2^k, 8^14^{n_0}, 2)$, $n_0 = (2^k - 5)/3 - 1$, k odd, which Chacko and Dey (1981) constructed by extending a result of Addelman and Kempthorne (1961).

Although the arrays in the Corollary can be constructed by other methods, the present approach to construction has some advantages. First, it is elementary and simple. (On the other hand, the Chacko–Dey construction is based on the complicated group-theoretic method of Addelman and Kempthorne). Second, the replacement method relates each 4-level factor to three factorial effects in a 2-level design. This makes it easier to study the aliasing patterns of main effects and interactions in the $2^m 4^n$ design since they can be easily derived from those of the 2^k design. Knowing the aliasing patterns is crucial to the statistical analysis of experimental design data. This method of introducing a 4-level factor is extensively used by Taguchi [see, for example, Taguchi (1986)] in teaching design techniques to nonstatisticians. Use of the grouping scheme has the additional advantage that the construction of designs with 2-, 4- and 8-level factors is quite straightforward. For example, $\mathrm{OA}(16, 2^8 8, 2)$ is obtained by replacing the seven 2-level factors denoted by $(1, 2, 3, 12, 13, 23, 123)$ in $\mathrm{OA}(16, 2^{15}, 2)$ by an 8-level factor. Similarly, we can construct $\mathrm{OA}(32, 4^8 8, 2)$ from $\mathrm{OA}(32, 2^7 4^8, 2)$ and $\mathrm{OA}(32, 2^a 4^n 8, 2)$, $a + 3n = 24$, from $\mathrm{OA}(32, 2^m 4^n, 2)$ with $n \leq 8$.

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