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THE EXPECTATION OF X^{-1} AS A FUNCTION OF $\mathbb{E}(X)$ FOR AN EXPONENTIAL FAMILY ON THE POSITIVE LINE

BY GÉRARD LETAC AND VANAMAMALAI SESHADRI

Université Paul Sabatier and McGill University

If the distribution of X belongs to a natural exponential family on the positive real line, this note studies the expectation of the reciprocal of X as a function of the expectation m of X and characterizes the cases where this function is an affine function of m^{-1} as gamma, inverse-Gaussian, Ressel or Abel families.

1. Natural exponential families. Let us first recall a few features of the natural exponential families on the line $\mathbb{R}$. All proofs can be found in [4] (consult also [1] and [2]).

If μ is a positive Radon measure on $\mathbb{R}$, we consider its Laplace transform L defined on $\mathbb{R}$ by

$$L(\theta) = \int_{-\infty}^{+\infty} \exp(\theta x)\mu(dx) \leq \infty.$$

Hölder's inequality implies easily that the set $D = \{\theta | L(\theta) < \infty\}$ is an interval and that $k(\theta) = \log L(\theta)$ is convex on D.

We denote by $\mathscr{M}$ the set of measures μ such that the interior Θ of D is nonvoid and such that μ is not concentrated on one point. For μ in $\mathscr{M}$, $k(\theta)$ is real analytic on Θ, and for θ in Θ, one considers the probability distribution

$$P(\theta)(dx) = \exp\{\theta x - k(\theta)\}\mu(dx). \tag{1.1}$$

The set $F = F(\mu) = \{P(\theta);\ \theta \in (\Theta)\}$ is called the *natural exponential family* generated by μ. Now observe that

$$k'(\theta) = \int_{-\infty}^{\infty} xP(\theta)(dx). \tag{1.2}$$

Hence the image M_F of Θ by $\theta \mapsto k'(\theta)$ is called the *domain of the means* of F. Since μ is in $\mathscr{M}$, then k is strictly convex on Θ; the map $\theta \mapsto k'(\theta)$ is a bijection from Θ onto M_F, which is an open interval, and we denote by

$$\psi\colon M_F \to \Theta$$

its inverse map. Note that ψ is real analytic on M_F. We also denote

$$Q(m, F) = P(\psi(m));$$

therefore, the map $M_F \to F$, defined by $m \mapsto Q(m, F)$, provides another parametrization of F by its domain of the means. Finally, we introduce the

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variance function of F as a function V defined on M_F by

$$V(m) = \int_{-\infty}^{\infty} (x-m)^2 Q(m,F)(dx).$$

It is easily proved that

$$V(m) = 1/\psi'(m). \tag{1.3}$$

This function V occupies a central position in the classification of natural exponential families since V characterizes F. More precisely one has the following result.

PROPOSITION 1.1. *Let F_1 and F_2 be two natural exponential families on $\mathbb{R}$ with variance functions V_1 and V_2, respectively, and assume that there exists a nonvoid open interval I contained in $M_{F_1} \cap M_{F_2}$ such that V_1 and V_2 coincide on I. Then $F_1 = F_2$.*

For a proof see, for instance, [2]. [2], [3] and [4] give all natural exponential families on $\mathbb{R}$ such that V is the restriction to M_F of a polynomial of degree less than or equal to 3.

2. The expectation of X^{-1}. We restrict ourselves now to the natural exponential families concentrated on $(0, +\infty)$. Here is our main result.

THEOREM 2.1. *Let μ in $\mathscr{M}$ be such that $\mu((-\infty, 0]) = 0$ and $M_F = (a, b) \subset (0, +\infty)$. Then* (i)

$$\varphi(m) = \int_0^{\infty} x^{-1} Q(m,F)(dx)$$

is finite for all m in M_F if and only if

$$\int_0^1 x^{-1}\mu(dx) < \infty. \tag{2.1}$$

(ii) *If* (2.1) *is true, denoting $G(m) = k(\psi(m))$, one has for all m in M_F,*

$$\varphi(m) = \exp(-G(m)) \int_a^m \exp(G(x)) \frac{dx}{V(x)} \tag{2.2}$$

$$= \frac{1}{m} + \exp(-G(m)) \int_a^m \exp(G(x)) \frac{dx}{x^2}, \tag{2.3}$$

and (iii) *φ is a solution of the differential equation*

$$V(m)\varphi'(m) + m\varphi(m) = 1. \tag{2.4}$$

COMMENT. Recall that a positive random variable X always satisfies $1 \le \mathbb{E}(X)\mathbb{E}(X^{-1})$ by applying Schwarz's inequality to the product $\sqrt{X}\sqrt{X^{-1}}$. Hence (2.3) gives an explicit expression for the positive difference $\varphi(m) - 1/m$.

PROOF OF THEOREM 2.1. (i) Suppose that $\varphi(m)$ is finite for all m in M_F. Clearly, since μ is concentrated on $(0, +\infty)$, Θ contains a half-line $(-\infty, \theta_0)$. Hence for $\theta < 0$ and $\theta < \theta_0$,

$$\int_0^1 x^{-1}\mu(dx) \le \exp\{k(\theta) - \theta\} \int_0^1 x^{-1} \exp\{\theta x - k(\theta)\}\mu(dx)$$
$$\le \exp\{k(\theta) - \theta\} \int_0^\infty x^{-1} P(\theta)(dx) < \infty.$$

Now suppose that $\int_0^1 x^{-1}\mu(dx) < \infty$. If θ is in Θ,

$$\int_0^\infty x^{-1}P(\theta)(dx) \le \exp\{|\theta| - k(\theta)\} \int_0^1 x^{-1}\mu(dx)$$
$$+ \int_1^\infty \exp\{\theta x - k(\theta)\}\mu(dx) < \infty.$$

To prove (ii), observe that for t in Θ and m in M_F an application of Fubini's theorem on

$$\exp(k(t)) = \int_a^\infty \exp(tx)\mu(dx) \tag{2.5}$$

gives

$$\int_{-\infty}^{\psi(m)} \exp(k(t))\, dt = \int_a^\infty x^{-1} \exp\{x\psi(m)\}\mu(dx). \tag{2.6}$$

Note that since $G(m) = k(\psi(m))$, using (1.3) and (1.2) one easily obtains

$$G'(m) = \frac{k'(\psi(m))}{V(m)} = \frac{m}{V(m)}. \tag{2.7}$$

The change of variable $t = \psi(x)$ in the left-hand side of (2.6), and the fact that $V(x) = 1/\psi'(x)$ gives

$$\int_{-\infty}^{\psi(m)} \exp(k(t))\, dt = \int_a^m \exp\{k(\psi(x))\} \frac{dx}{V(x)}$$
$$= \int_a^m \exp(G(x)) \frac{dx}{V(x)}.$$

Hence,

$$\varphi(m) = \exp(-G(m)) \int_a^m \exp(G(x)) \frac{dx}{V(x)}.$$

Now to obtain (2.3) we observe that if

$$v(x) = \exp(G(x)), \qquad v'(x) = \frac{x \exp(G(x))}{V(x)}$$

[using (1.3) and (2.7)]. Therefore, integrating the second member of (2.2) by parts with this $v'(x)$ and $u(x) = x^{-1}$ will give (2.3), provided that we show that

$$\lim_{x \downarrow a} x^{-1} \exp G(x) = 0,$$

which is equivalent to

$$\lim_{\theta \mapsto -\infty} \frac{1}{k'(\theta)} \exp k(\theta) = 0. \tag{2.8}$$

To prove (2.8) we introduce $\nu(dx) = x^{-1}\mu(dx)$; condition (2.1) implies that ν is in $\mathcal{M}$. If

$$A_j = \int_0^{\infty} x^j \exp(\theta x)\nu(dx) \quad \text{for } j = 0, 1, 2,$$

Schwarz's inequality implies $A_1^2 \le A_0 A_2$, and we have $[1/k'(\theta)]\exp k(\theta) = A_1^2/A_2$. Since $A_0 \to 0$ as $\theta \to -\infty$, (2.8) is proved.

(iii) Clearly φ is real analytic in M_F; differentiating with respect to m in (2.2) gives (2.4). □

The next corollary shows that φ characterizes F.

COROLLARY 2.2. *Let μ_1 and μ_2 in $\mathcal{M}$ be such that $\mu_j(]-\infty, 0]) = 0$ and $\int_0^1 x^{-1}\mu_j(dx) < \infty$, $j = 1, 2$; denote $F_j = F(\mu_j)$. Assume that there exists a nonvoid open interval I contained in $M_{F_1} \cap M_{F_2}$ such that φ_{F_1} and φ_{F_2} coincide on I. Then $F_1 = F_2$.*

PROOF. (2.4) and (2.3) show that $1 - m\varphi_{F_1} = V_{F_1}(m)\varphi'_{F_1}(m) < 0$. Hence $\varphi_{F_1} = \varphi_{F_2}$ on I implies $V_{F_1} = V_{F_2}$ on I, since $\varphi'_{F_1} \neq 0$ on M_{F_1} and this implies $F_1 = F_2$ by Proposition 1.1. □

We now make the following remark.

Let F be a natural exponential family on $(0, +\infty)$ fulfilling condition (2.1). Let $c > 0$, $h(x) = cx$ and $h(F)$ be the image of F by h. Then if $M_F = (a, b) \subset (0, +\infty)$,

$$M_{h(F)} = (ca, cb) \quad \text{and} \quad \varphi_{h(F)}(m) = \frac{1}{c}\varphi_F\left(\frac{m}{c}\right).$$

3. Examples and applications.

EXAMPLE 3.1 (The gamma families). Let $p > 0$,

$$\mu_p(dx) = x^{p-1}\mathbb{1}_{(0,+\infty)}(x)\frac{dx}{\Gamma(p)}.$$

It is easily seen that $\Theta(\mu) = (-\infty, 0)$, $k(\theta) = -p\log(-\theta)$, $\psi(m) = -p/m$, $M_F = (0, +\infty)$ and $V(m) = m^2/p$. Clearly μ_p fulfills (2.1) if and only if $p > 1$. In this case, a direct computation, or (2.2) or (2.3), gives, for $F = F(\mu_p)$,

$$\varphi(m) = \frac{p}{p-1}\frac{1}{m} \quad \text{for } m > 0. \tag{3.1}$$

EXAMPLE 3.2 (The inverse-Gaussian families). Let $p > 0$, $\mu_p(dx) = p(2\pi)^{-1/2}x^{-3/2}\exp(-p^2/2x)\mathbb{1}_{(0,+\infty)}(x)\,dx$ (a stable law with parameter $\frac{1}{2}$). Clearly $\Theta = (-\infty, 0)$. A not entirely trivial computation gives the known result

$$k(\theta) = -p\sqrt{-2\theta}\,;$$

one deduces from this $\psi(m) = -p^2/(2m^2)$, $M_F = (0, +\infty)$ and $V(m) = m^3/p^2$. A direct computation of φ from the definition is rather tedious, but one can use one form of (2.5) to get

$$\varphi(m) = \exp\left(\frac{p^2}{m}\right)\int_0^{\infty}\exp\left(-p\left(2s + \frac{p^2}{m^2}\right)^{1/2}\right)ds,$$

or use (2.2) or (2.3). We obtain

$$\varphi(m) = \frac{1}{m} + \frac{1}{p^2} \quad \text{for } m > 0. \tag{3.2}$$

EXAMPLE 3.3 (The Ressel families). Let $p > 0$,

$$\mu_p(dx) = \frac{px^{p+x-1}}{\Gamma(p+x+1)}\mathbb{1}_{(0,+\infty)}(x)\,dx.$$

(See [2] for further details about this distribution.) Again we have $\Theta = (-\infty, 0)$ and $M_F = (0, +\infty)$, but the explicit computation of k is rather intractable, since it is obtained as the solution of an implicit equation. However, the main interest of this μ_p is the simplicity of the variance function of $F = F(\mu_p)$, which is

$$V(m) = \frac{m^2}{p}\left(1 + \frac{m}{p}\right), \qquad m > 0.$$

Even if we don't know k, *we can compute* $G(m)$ *up to a constant*, since $G'(m) = m/V_F(m)$, and we can apply (2.2) or (2.3). Therefore, we get

$$G(m) = C + p\log\frac{m}{m+p}.$$

Now from the definition of μ_p, one has $\int_0^1 x^{-1}\mu_p(dx) < \infty$ if and only if $p > 1$. Therefore, using (2.3) we get, for $p > 1$,

$$\varphi(m) = \frac{1}{m} + \left(\frac{m+p}{m}\right)^p\int_0^m \frac{x^{p-2}}{(x+p)^p}\,dx.$$

We make the change of variable $y = x/(x+p)$ and we get

$$\varphi(m) = \frac{p}{p-1}\frac{1}{m} + \frac{1}{p(p-1)} \quad \text{for } m > 0. \tag{3.3}$$

EXAMPLE 3.4 (The Abel families). Let $p > 0$,

$$\mu_p(dx) = \sum_{n=0}^{\infty} \frac{p(n+p)^{n-1}}{n!}\delta_n,$$

where δ_n is the Dirac unit mass on n. Here we have a point mass on 0 and if the distribution of X belongs to $F(\mu_p)$, the Abel family with parameter p (see [2] and [3]), clearly $\mathbb{E}(1/X) = +\infty$. Note also that from Lagrange's formula, one has the equality

$$\sum_{n=0}^{\infty} p\frac{(n+p)^{n-1}}{n!}h^n e^{-nh} = e^{ph},$$

for $|h|$ small enough, k is therefore not an elementary function. However, it can be proved that $M_{F(\mu_p)} = (0, +\infty)$ and that $V(m) = m(1 + m/p)^2$.

Now, instead of considering the Abel family $F(\mu_p)$, let us consider the shifted family F of $F(\mu_p)$ which is the image of $F(\mu_p)$ by the map $x \mapsto x + p$. [Thus, with the previous X, we are lead to compute $\mathbb{E}(1/(p + X))$.] It happens that the result is simple. Actually, one has

$$M_F = (p, +\infty), \qquad V(m) = (m-p)\frac{m^2}{p^2} \quad \text{for } m > p.$$

Hence $G'(m) = p(1/(m-p) - 1/m)$, and there exists C in $\mathbb{R}$ such that

$$G(m) = C + p\log\frac{m-p}{m}.$$

A computation, similar to Example 3.3, gives

$$\varphi(m) = \frac{p}{p+1}\frac{1}{m} + \frac{1}{p(p+1)} \quad \text{for } m > p. \tag{3.4}$$

We now remark that in the above examples, described by (3.1), (3.2), (3.3) and (3.4), φ is always an affine function of $1/m$. This is actually a characterization of the above four examples, up to a scale change. More precisely:

THEOREM 3.1. *Let F be a natural exponential family concentrated on* $(0, +\infty)$, *such that* (2.1) *is fulfilled and such that there exist α and β in $\mathbb{R}$ with*

$$\varphi(m) = \frac{\alpha}{m} + \beta \quad \textit{for all } m \textit{ in } M_F.$$

Then $M_F = (a, +\infty)$ with $a \geq 0$. Furthermore:

(a) *Either $a = 0$. In this case $\alpha \geq 1$, $\beta \geq 0$ and $\alpha + \beta > 1$*: (i) *If $\beta = 0$, F is a gamma family with parameter $p = \alpha/(\alpha - 1)$.* (ii) *If $\beta > 0$ and $\alpha = 1$, F is an inverse-Gaussian family with parameter $p = \beta^{-1/2}$.* (iii) *If $\beta > 0$ and $\alpha > 1$, then, denoting $p = \alpha/(\alpha - 1)$ and $c = \beta(\alpha - 1)^2/\alpha$, F is the image of the Ressel family with parameter p by $x \mapsto cx$.*

(b) *Or* $a > 0$. *In this case* $0 < \alpha < 1$, $\beta > 0$ *and* $a = (1 + \alpha)/\beta$. *Denoting* $p = \alpha/(1 - \alpha)$ *and* $c = a/p$, *F is the image of the Abel family with parameter p by* $x \mapsto cx + a$.

PROOF. Carry the explicit expression of φ into the differential equation (2.4). If α would be 0, one would get $1 = \beta m$ for all m in the nonvoid open interval M_F; an impossibility. Since $\alpha \neq 0$, one gets

$$V(m) = m^2\left(1 - \frac{1}{\alpha} + \frac{\beta}{\alpha}m\right) \quad \text{on } M_F \subset (0, +\infty).$$

The classification of the natural exponential families F on $\mathbb{R}$ such that V is the restriction to M_F of a polynomial with degree less than or equal to 3 (see [2], [3] and [4]) shows that M_F must be a half-line $(a, +\infty)$. The above classification gives also the results stated in the remainder of the theorem. But a direct discussion of this remaining part can also be done as follows.

(a) If $a = 0$, using (2.3) one gets

$$\frac{\alpha - 1}{m} + \beta > 0 \quad \text{for all } m > 0.$$

This implies $\beta \geq 0$ (let $m \to +\infty$), $\alpha \geq 1$ (let $m \to 0$) and $\alpha + \beta > 1$. The rest of the discussion comes from Examples 3.1, 3.2 and 3.3.

(b) If $a > 0$, from the classification one has $\lim_{m \downarrow a} V(m) = 0$ and $a = (1 - \alpha)/\beta$. Since $\beta = \lim_{m \to \infty} \varphi(m)$, one deduces $\beta > 0$ and $\alpha < 1$; $\lim_{m \to \infty} V(m) \geq 0$ implies $0 < \alpha$.

The rest of the discussion comes from Example 3.4. □

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LABORATOIRE DE STATISTIQUE
ET PROBABILITES
UNIVERSITE PAUL SABATIER
31062 TOULOUSE CEDEX
FRANCE

DEPARTMENT OF MATHEMATICS
AND STATISTICS
MCGILL UNIVERSITY
805 SHERBROOKE STREET WEST
MONTREAL, P.Q.
CANADA H3A 2K6