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ESTIMATION IN SOME COUNTING PROCESS MODELS WITH MULTIPLICATIVE STRUCTURE[1]

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It is assumed that we observe one realization of an r-dimensional counting process with intensities that are products of a predictable weight process, a common function of time and parameters β_i, $i = 1, \dots, r$, which distinguish the components. Provided the realization observed brings increasing information on β as the observed time grows, strong consistency of a partial ML estimator is proved. For such realizations it is also proved that the estimate, after applying a random normalization, is asymptotically standard normal.

1. Introduction. We will consider an r-dimensional counting process ($r \geq 2$),

$$N(t) = (N_1(t), \dots, N_r(t)),$$

defined on a probability space $(\Omega, \mathscr{A}, P)$ and adapted to the filtration $\mathscr{A}_t$, $t \in [0, \infty[$. The components, N_i, are integer-valued, right-continuous functions with jumps of size 1 only. Two components can not jump at the same time, and $N_i(0) = 0$, $i = 1, \dots, r$. The basic assumption is that the process has an intensity

$$\lambda(t) = (\lambda_1(t), \dots, \lambda_r(t))$$

of the form

$$\lambda_i(t) = \exp\{\beta_i\}\alpha(t)Y_i(t), \tag{1.1}$$

$i = 1, \dots, r$. The weight processes Y_i are predictable (relative to the filtration $\mathscr{A}_t$) and nonnegative. The function α is an unknown nuisance parameter. The parameters $\beta_1, \dots, \beta_r$ describe the proportionality between the intensities of the component processes. We will use the arbitrary normalization $\beta_r = 0$ and write $\beta = (\beta_1, \dots, \beta_{r-1})$.

Our aim is to estimate β when one realization of N is observed during a long period of time. If new information is added as time goes on it should be possible to obtain reasonable estimates. In this paper we will verify this belief and give asymptotic results for one particular estimator.

The model has the structure of a Cox regression model [cf. Cox (1972)]. Andersen and Gill (1982) studied a situation where many independent realizations of a process were observed during a bounded time interval and Pons and Turckheim (1987) studied a Cox periodic regression model where the function α was periodic with period 1 and one realization was observed during a long period of time. In these two cases it is reasonable to try to estimate both the parameter β and the underlying cumulative intensity $\int_0^u \alpha(t)\,dt$. Here we will only consider

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estimates of β. It turns out that the asymptotic behaviour of the estimate is different and that other kind of limit theorems are needed in the proofs. Both Andersen and Gill (1982) and Pons and de Turckheim (1987) used a more general form for the intensities than the one defined by (1.1).

2. Estimation of β. Assume that we have observed the process N and the weight process Y up till time $t = u$. Following Cox (1972) and Andersen and Gill (1982), we will use a partial ML estimate, i.e., we will estimate β with the value of γ that maximizes the function

$$H(\gamma, u) = \sum_1^r \gamma_i N_i(u) - \int_0^u \ln\left(\sum_1^r \exp\{\gamma_j\} Y_j(t)\right) d\bar{N}(t), \tag{2.1}$$

where $\gamma_r = 0$ and $\bar{N}(t) = \Sigma N_i(t)$. This estimate is denoted by $\hat{\beta}(u)$. [Observe that $d\bar{N}(t) \neq 0$ implies that $\Sigma \exp\{\gamma_j\} Y_j(t) \neq 0$ with probability 1 and that (2.1) is well defined.]

The estimate will solve the equations

$$G_i(\gamma, u) = \partial H(\gamma, u)/\partial \gamma_i = N_i(u) - \int_0^u p_i(\gamma, t)\, d\bar{N}(t) = 0, \tag{2.2}$$

$i = 1, \dots, r - 1$, where

$$p_i(\gamma, t) = \exp\{\gamma_i\} Y_i(t) \Big/ \left(\sum_1^r \exp\{\gamma_j\} Y_j(t)\right).$$

The $r' = (r - 1)$-dimensional vector

$$G(\beta, u) = (G_1(\beta, u), \dots, G_{r'}(\beta, u))$$

is a martingale with the predictable quadratic variation

$$W(\beta, u) = \int_0^u \Pi(\beta, t) \bar{\lambda}(t)\, dt,$$

where $\bar{\lambda}(t) = \Sigma \lambda_i(t)$ and $\Pi(\beta, t)$ is the $r' \times r'$ matrix with elements

$$\Pi_{ij}(\beta, t) = \partial p_i(\beta, t)/\partial \beta_j = \delta_i^j p_i(\beta, t) - p_i(\beta, t) p_j(\beta, t).$$

In the asymptotic expressions it will be convenient to replace $W(\beta, u)$ with an $r' \times r'$ matrix $S(\beta, u)$, which does not involve the nuisance parameter α,

$$S(\beta, u) = \int_0^u \Pi(\beta, t)\, d\bar{N}(t).$$

We will also need the optional quadratic variation $U(\beta, u)$, which is a $r' \times r'$ matrix with elements

$$U_{ij}(\beta, u) = \sum_{k=1}^r \int_0^u \left(\delta_i^k - p_i(\beta, t)\right)\left(\delta_j^k - p_j(\beta, t)\right) dN_k(t).$$

2.1. *Consistency.* In order to find consistent estimates of β based on one realization of the process it is necessary that the realization adds new information as u increases. This will, of course, not always be the case. If, e.g., the weight process equals 0 from a finite time and onward we will only observe a finite number of jumps. Since information on β can only be obtained when the process jumps we cannot, in this case, expect to obtain consistent estimates. We will consider a subset F of Ω on which the realizations carry sufficiently much new information as u increases. We will make conditions that guarantee that the number of jumps grows large asymptotically and that we can use the information to distinguish between the components of the β parameter. Let $F \in \mathscr{A}$ be a set on which

(i) $\int_0^u \bar{\lambda}(t)\,dt \to \infty$ as $u \to \infty$, and
(ii) the smallest eigenvalue ν of $W(\beta, u)/\int_0^u \bar{\lambda}(t)\,dt$ is bounded away from 0 for large u-values.

LEMMA 2.1. *$S(\beta, u)W^{-1}(\beta, u) \to I$, $S(\beta, u)U^{-1}(\beta, u) \to I$ and $W(\beta, u)U^{-1}(\beta, u) \to I$ a.s. on F.*

PROOF. If the norm of a matrix A is defined on $\|A\| = \sup_{|x|=1}|Ax'|$, then

$$\left\|S(\beta, u)W^{-1}(\beta, u) - I\right\| \le \nu^{-1}\left\|S(\beta, u) - W(\beta, u)\right\| \Big/ \int_0^u \bar{\lambda}(t)\,dt.$$

The (i, j)th element of $S(\beta, u) - W(\beta, u)$ equals

$$R_{ij}(u) = \int_0^u \Pi_{ij}(\beta, u)\big(d\bar{N}(t) - \bar{\lambda}(t)\,dt\big)$$

and is a martingale with predictable quadratic variation

$$\int_0^u \Pi_{ij}^2(\beta, t)\bar{\lambda}(t)\,dt \le \int_0^u \bar{\lambda}(t)\,dt.$$

Using Theorem 4.2, we see that $R_{ij}(u)/\int_0^u \bar{\lambda}(t)\,dt \to 0$ a.s. on F. Thus

$$\left\|S(\beta, u) - W(\beta, u)\right\| \Big/ \int_0^u \bar{\lambda}(t)\,dt \to 0$$

and $S(\beta, u)W^{-1}(\beta, u) \to I$ a.s. on F.

It is easy to verify that $U_{ij}(\beta, u) - S_{ij}(\beta, u)$, $i, j = 1, \dots, r'$, are martingales with predictable quadratic variations which are majorized by $\int_0^u \bar{\lambda}(t)\,dt$. Thus $\|U(\beta, u) - S(\beta, u)\|/\int_0^u \bar{\lambda}(t)\,dt \to 0$ a.s. on F. Using the same argument as above, it follows that also $S(\beta, u)U^{-1}(\beta, u)$ and consequently $W(\beta, u)U^{-1}(\beta, u)$ tend to I a.s. on F. □

LEMMA 2.2. *For any $\varepsilon > 0$ and almost all realizations in F there exist a $\rho > 0$ and a $u_0 < \infty$ such that $\|S(\gamma, u)S^{-1}(\beta, u) - I\| \le \varepsilon$ if $|\gamma - \beta| \le \rho$ and $u > u_0$.*

PROOF. If $Y_i(t)Y_j(t) \neq 0$ and $\max|\gamma_j - \beta_j| = \delta$ it is easy to verify that

$$\begin{aligned}\exp\{-4\delta\} &\le p_i(\gamma, t)p_j(\gamma, t)/\big(p_i(\beta, t)p_j(\beta, t)\big)\\ &= \exp\{\gamma_i - \beta_i + \gamma_j - \beta_j\}\left(\sum_1^r \exp\{\beta_k\}Y_k(t)\right)^2 \Bigg/ \left(\sum_1^r \exp\{\gamma_k\}Y_k(t)\right)^2\\ &\le \exp\{4\delta\}.\end{aligned}$$

If $Y_i(t)Y_j(t) = 0$, then either $p_i(\gamma, t) = 0$ or $p_j(\gamma, t) = 0$ for all values of γ. This implies that for any $\eta > 0$ there exists a $\rho > 0$ such that

$$|\Pi_{ij}(\gamma, t) - \Pi_{ij}(\beta, t)| \le \eta|\Pi_{ij}(\beta, t)| \tag{2.3}$$

for $i, j = 1, \dots, r'$ if $|\gamma - \beta| \le \rho$ and $i \neq j$. That (2.3) holds also when $i = j$ follows from the fact that $\Pi_{ii}(\gamma, t) = p_i(\gamma, t)\Sigma_{j\neq i}p_j(\gamma, t)$.

Using (2.3), we conclude that

$$\begin{aligned}|S_{ij}(\gamma, u) - S_{ij}(\beta, u)| &\le \int_0^u |\Pi_{ij}(\gamma, t) - \Pi_{ij}(\beta, t)|\, d\overline{N}(t)\\ &\le \eta\int_0^u |\Pi_{ij}(\beta, t)|\, d\overline{N}(t) = \eta|S_{ij}(\beta, t)|,\end{aligned}$$

since $\Pi_{ij}(\beta, t)$ has the same sign for all t. Then

$$\begin{aligned}\|S(\gamma, t) - S(\beta, t)\|^2 &\le \sum_{ij}\big(S_{ij}(\gamma, t) - S_{ij}(\beta, t)\big)^2 \le \eta^2\sum_{ij} S_{ij}^2(\beta, t)\\ &\le r\eta^2\max_i \sum_j S_{ij}^2(\beta, t) \le r\eta^2\|S(\beta, t)\|^2.\end{aligned}$$

From Lemma 2.1 it follows that for almost all realizations in F there exists a u_0 such that

$$\|S(\beta, u)\| \le (1 + \eta)\|W(\beta, u)\| \le (1 + \eta)\int_0^u \overline{\lambda}(t)\, dt$$

and

$$\|S^{-1}(\beta, u)\| \le (1 - \eta)^{-1}\|W^{-1}(\beta, u)\| \le (1 - \eta)^{-1}\left(\nu\int_0^u \overline{\lambda}(t)dt\right)^{-1}$$

if $u \ge u_0$. For these realizations

$$\begin{aligned}\|S(\gamma, u)S^{-1}(\beta, u) - I\| &\le \|S(\gamma, u) - S(\beta, u)\|\, \|S^{-1}(\beta, u)\|\\ &\le \sqrt{r}\eta\|S(\beta, u)\|\, \|S^{-1}(\beta, u)\|\\ &\le \sqrt{r}\eta(1 + \eta)/(\nu(1 - \eta))\end{aligned}$$

if $|\gamma - \beta| \le \rho$ and $u \ge u_0$. We can choose η such that $\varepsilon = \sqrt{r}\eta(1 + \eta)/(\nu(1 - \eta))$. This proves the lemma. □

THEOREM 2.3. *$\hat{\beta}(\mu) \to$ a.s. on F.*

PROOF. The martingale $G_i(\beta, u)$ has the predictable quadratic variation

$$W_{ii}(\beta, u) = \int_0^u \Pi_{ii}(\beta, u)\bar{\lambda}(t)\,dt \le \int_0^u \bar{\lambda}(t)\,dt.$$

It follows from Theorem 4.2 that $G_i(\beta, u)/\int_0^u\bar{\lambda}(t)\,dt \to 0$ a.s. on F. Thus $G(\beta, u)W^{-1}(\beta, u) \to 0$ a.s. on F. By Lemma 2.1 also $G(\beta, u)S^{-1}(\beta, u) \to 0$ a.s. on F. A Taylor expansion yields

$$G(\gamma, u)S^{-1}(\beta, u)(\gamma - \beta)' = G(\beta, u)S^{-1}(\beta, u)(\gamma - \beta)' - (\gamma - \beta)S(\gamma^*, u)S^{-1}(\beta, u)(\gamma - \beta)'$$

for some γ^* such that $|\gamma^* - \beta| \le |\gamma - \beta|$. Combining this with Lemma 2.2, we find that on F if ρ is small and u is large $G(\gamma, u)S^{-1}(\beta, u)(\gamma - \beta)' < 0$ for all $|\gamma - \beta| = \rho$. According to Theorem 4.1 the equation $G(\gamma, u)S^{-1}(\beta, u) = 0$ and thus $G(\gamma, u) = 0$ then has a solution in the set $|\gamma - \beta| < \rho$. □

After combining Lemma 2.2 and Theorem 2.3 we obtain the following lemma.

LEMMA 2.4. *$S(\hat{\beta}(u), u)S^{-1}(\beta, u) \to I$ a.s. on F.*

2.2. *Asymptotic distribution of $\hat{\beta}(u)$.* We have the following theorem.

THEOREM 2.5. *Assume that there exists an $\mathscr{A}_0$-measurable scalar function $b(u)$, a positive definite matrix D and an $\mathscr{A}$-measurable random matrix Ψ such that*

(i) $b(u) \to \infty$,
(ii) $U(\beta, u)/b(u) \to \Psi$ *in probability and*
(iii) $EU(\beta, u)/b(u) \to D$

as $u \to \infty$, then $(\hat{\beta}(u) - \beta)S^{1/2}(\hat{\beta}(u), u)$ is asymptotically $N(0, I)$-distributed conditionally on $F \cap \{\Psi > 0\}$.

PROOF. We start by applying Theorem 4.3 to the r'-dimensional martingale $G(\beta, \cdot\,)$. Since $|\Delta G(\beta, u)| \le 1$ for all u all conditions of the theorem are satisfied. Thus $G(\beta, u)U^{-1/2}(\beta, u)$ is asymptotically $N(0, I)$-distributed conditionally on the event $F \cap \{\Psi > 0\}$. Using Lemma 2.1, we see that the same is true for $G(\beta, u)W^{-1/2}(\beta, u)$ and $G(\beta, u)S^{-1/2}(\beta, u)$.

The next step of the proof is to show that $(\hat{\beta}(u) - \beta)S^{1/2}(\beta, u)$ and $G(\beta, u)S^{-1/2}(\beta, u)$ are symptotically equivalent on F. A Taylor expansion yields for any $a \in R^{r'}$:

$$\begin{aligned} 0 &= G(\hat{\beta}(u), u)S^{-1/2}(\beta, u)a' \\ &= G(\beta, u)S^{-1/2}(\beta, u)a' \\ &\quad - (\hat{\beta}(u) - \beta)S^{1/2}(\beta, u)\big(S(\gamma_a^*, u)S^{-1}(\beta, u)\big)a', \end{aligned}$$

where $|\gamma_a^* - \beta| \le |\hat{\beta}(u) - \beta|$. Since $\hat{\beta}(u) \to \beta$ on F it follows from Lemma

2.2 that $S(\gamma_a^*, u)S^{-1}(\beta, u) \to I$ as $u \to \infty$. Thus $G(\beta, u)S^{-1/2}(\beta, u)a'$ and $(\hat{\beta}(u) - \beta)S^{1/2}(\beta, u)a'$ are asymptotically equivalent on F for all $a \in R^{r'}$. This implies the desired result. By Lemma 2.4 $(\hat{\beta}(u) - \beta)S^{1/2}(\hat{\beta}(u), u)$ is asymptotically equivalent to these two variables on F. The theorem is thus proved. □

3. Examples. As a first example we will consider a system of n particles that moves between two states (A and B) independently of each other. The intensity for a jump from state A to state B (B to A) at time t is assumed to be $\mu_1\theta(t)$ $[\mu_2\theta(t)]$. Here the unknown nuisance parameter θ describes a common jump proneness that depends on time. Let $N_1(t)$ $[N_2(t)]$ denote the number of jumps from A to B (B to A) up till time t. If the system starts at $t = 0$ with n_A particles in state A and n_B particles in state B, then just before time t there are $F(t) = n_A + N_2(t-) - N_1(t-)$ particles in state A. Now $N(t) = (N_1(t), N_2(t))$ is a two-dimensional counting process with intensities $\lambda_1(t) = \exp\{\beta\}\alpha(t)F(t)$ and $\lambda_2(t) = \alpha(t)(n - F(t))$, where $\beta = \ln(\mu_1/\mu_2)$ and $\alpha(t) = \mu_2\theta(t)$.

We can now use the results obtained to study the behaviour of the estimate $\hat{\beta}(u)$ that solves the equation

$$N_1(u) = \int_0^u \exp\{\beta\}F(t)/(\exp\{\beta\}F(t) + n - F(t))\, d\overline{N}(t).$$

Let $\tau_1, \tau_2, \ldots$ be the successive jump times of the process. Then $(F(0), F(\tau_1), F(\tau_2), \ldots)$ is a time-homogeneous Markov chain with state space $(0, 1, \ldots, n)$. The estimate of β and its asymptotic properties will depend on this embedded chain. If $\int_0^u \alpha(t)\, dt \to \infty$ as $u \to \infty$, then

$$\int_0^u \overline{\lambda}(t)\, dt \geq (\exp\{\beta\} \wedge 1)n\int_0^u \alpha(t)\, dt \to \infty$$

and $\overline{N}(u)/\int_0^u \overline{\lambda}(t)\, dt \to 1$ a.s. as $u \to \infty$ (cf. Theorem 4.2). The chain will thus asymptotically have infinitely many jumps. The embedded chain will have a stationary distribution given by the positive probabilities g_f, $f = 0, \ldots, n$. It follows from standard ergodic theory that

$$S(\beta, u)/\overline{N}(u) \to \sum_{f=0}^{n} \exp\{\beta\}f(n-f)(\exp\{\beta\}f + n - f)^{-2}g_f = D_1 > 0 \tag{3.1}$$

a.s. as $u \to \infty$. This implies (by an argument analogous to the one used in the proof of Lemma 2.1) that $W(\beta, u)/\int_0^u \overline{\lambda}(t)\, dt$ is bounded away from 0 for large u-values with probability 1. By Theorem 2.3, $\hat{\beta}(u) \to \beta$ a.s. as $u \to \infty$. Next consider the martingale $T(u) = \int_0^u d\overline{N}(t)/P(t) - \int_0^u \alpha(t)\, dt$, where $P(t) = \exp\{\beta\}F(t) + n - F(t)$. Since the predictable quadratic variation of T is majorized by $(\exp\{\beta\} \wedge 1)^{-1}\int_0^u \alpha(t)\, dt/n$ it follows from Theorem 4.2 that $T(u)/\int_0^u \alpha(t)\, dt \to 0$ and

$$\left(\int_0^u d\overline{N}(t)/P(t)\right)\Big/\int_0^u \alpha(t)\, dt \to 1 \tag{3.2}$$

a.s. as $u \to \infty$. Using the same ergodic theory, we also find

$$\left(\int_0^u d\overline{N}(t)/P(t)\right)\bigg/\overline{N}(u) \to \sum_{f=0}^{n} (\exp\{\beta\}f + n - f)^{-1} g_f = D_2 > 0$$

a.s. as $u \to \infty$. Together with (3.1) and (3.2) this implies that

$$S(\beta, u)\bigg/\int_0^u \alpha(t)\,dt \to D = D_1/D_2 > 0$$

a.s. as $u \to \infty$. According to Lemma 2.1 also $U(\beta, u)/\int_0^u \alpha(t)\,dt$ and $W(\beta, u)/\int_0^u \alpha(t)\,dt$ tend to D a.s. By dominated convergence

$$EW(\beta, u)\bigg/\int_0^u \alpha(t)\,dt = EU(\beta, u)\bigg/\int_0^u \alpha(t)\,dt \to D.$$

Using $b(u) = \int_0^u \alpha(t)\,dt$ in Theorem 2.5, we conclude that $(\hat{\beta}(u) - \beta)S^{1/2}(\hat{\beta}(u), u)$ is asymptotically $N(0,1)$-distributed.

In the second example we will omit a detailed proof. Consider a two-dimensional counting process $N(t) = (N_1(t), N_2(t))$ with intensities $\lambda_1(t) = \exp\{\beta\}\alpha(t)F(t)$ and $\lambda_2(t) = \alpha(t)F(t)$, where $F(t) = 1 + N_1(t-) - N_2(t-)$. This is a birth-and-death process starting with one individual at time $t = 0$ and developing with proportional birth-and-death intensities. The number of individuals living just before time t equals $F(t)$. The estimate $\hat{\beta}(u) = \ln(N_1(u)/N_2(u))$. The set F can be chosen as the set of nonextinction, i.e., the set of realizations for which $F(u) \to \infty$ as $u \to \infty$. F will not be the complete sample space Ω, since there is always a positive probability of extinction. If $\alpha(t) \geq \alpha > 0$ for all t and $\beta > 0$, then the set F will have positive probability. Using known theory for the asymptotic behaviour of birth-and-death processes, the conditions of Theorem 2.5 can be verified. The conclusion is that $(N_1(u)N_2(u))/(N_1(u) + N_2(u))^{1/2}(\hat{\beta}(u) - \beta)$ is asymptotically $N(0,1)$-distributed conditionally on the set of nonextinction.

4. Some auxiliary theorems. In this section some of the results used above are stated.

THEOREM 4.1. *Let $f(\gamma)$ be a continuous function from R^r to R^r such that $f(\gamma)(\gamma - \beta)' < 0$ for all γ such that $|\gamma - \beta| = \rho$. Then there exists a $\hat{\gamma}$ such that $|\hat{\gamma} - \beta| < \rho$ and $f(\hat{\gamma}) = 0$.*

PROOF. A proof of this result can be found in Aitchison and Silvey (1958). □

THEOREM 4.2. *Let $M(u)$ be a scalar local square integrable martingale and let $A(u)$ be any process such that $A(u) \geq \langle M(u)\rangle(u)$ for all u. Then $M(u)/A(u) \to 0$ a.s. on the set* $\lim A(u) = \infty$.

PROOF. From Lepingle (1978) it follows that $M(u)$ has a finite limit if $\langle M\rangle(\infty) < \infty$ and that $M(u)/\langle M\rangle(u) \to 0$ a.s. on the set $\langle M\rangle(u) = \infty$. These two results combined prove the theorem. □

THEOREM 4.3. *Let $M(u)$ be a k-dimensional square integrable martingale on $(\Omega, \mathscr{A}, P)$ and let $b(u)$ be a scalar $\mathscr{A}_0$-measurable function such that*

(i) $b(u) \to \infty$,
(ii) $\sup_{t \leq u}|\Delta M(t)|/b^{1/2}(u) \to 0$,
(iii) $[M](u)/b(u) \to \Psi$ *in probability and*
(iv) $E[M](u)/b(u) \to D$

as $u \to \infty$, where D is a positive definite matrix and Ψ is an $\mathscr{A}$-measurable random matrix, then $M(u)/b^{1/2}(u)$ converges stably to Z^*. The characteristic function of Z^* equals $E\exp\{-s\Psi s'/2\}$. $M(u)[M]^{-1/2}(u)$ is asymptotically $N(0, I)$-distributed conditionally on $F \cap \{\Psi > 0\}$ if $F \in \mathscr{A}$.

PROOF. For a proof of this theorem we refer to Hutton and Nelson (1984, 1986) [cf. also Feigin (1985)]. □

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