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ON NON-NULL DISTRIBUTIONS CONNECTED WITH TESTING REALITY OF A COMPLEX NORMAL DISTRIBUTION

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Certain polynomials of a skew-symmetric matrix are considered. These polynomials can be expressed in terms of the zonal polynomials on the Hermitian matrices, and they are used to obtain a series expansion for the density of the non-null distribution of the maximal invariant corresponding to the problem of testing for reality of the covariance matrix of a complex multivariate normal distribution.

1. Introduction. In an article by Andersson, Brøns and Jensen (1983) 10 fundamental tests concerning the structure of covariance matrices in multivariate analysis are treated. Each of the 10 problems is invariant under a group of linear transformations, and the distribution of the maximal invariant under the null hypothesis was found in terms of a density with respect to a Lebesgue measure. A series expansion for the density of the distribution of the maximal invariant under the alternative hypothesis has been obtained for some of the 10 problems by James (1964) and Constantine (1963) by use of zonal polynomials and hypergeometric functions; it concerns the tests for independence and the tests for identity of two sets of variates where the simultaneous covariance matrix has real or complex structure. In this article one of the remaining non-null distribution problems are treated by using methods similar to those of James and Constantine. It concerns the test that a $2m \times 2m$ covariance matrix with complex structure has real structure; this test was considered for the first time by Khatri (1965).

Andersson and Perlman (1984) study the non-null distribution of the maximal invariant and we use their results as a starting point. The problem is the evaluation of a certain integral over the group of nonsingular $m \times m$ matrices; this integral is a function having two skew-symmetric matrices as arguments. The theory of group representations is used to define polynomials of a skew-symmetric matrix. The polynomials can be expressed in a simple way in terms of the complex zonal polynomials on the Hermitian matrices. Finally the polynomials are used to obtain a series expansion for the integral.

2. Statement of the problem. Let $x_1, \ldots, x_N$ be independent observations from a $2m$-dimensional real normal distribution with mean 0 and unknown covariance matrix of the form

$$\Sigma = \begin{pmatrix} \Sigma_1 & -\Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix}, \tag{1}$$

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where Σ_1 is a real positive-definite $m \times m$ matrix and $\Sigma_2 \in A(m)$, the set of $m \times m$ skew-symmetric matrices. Σ is assumed positive-definite.

Let H_1 denote the hypothesis that Σ has the form (1); in this case we say that Σ has complex structure. For $m > 1$ Andersson and Perlman (1984) have considered the problem of testing H_0: that Σ has real structure, that is, that $\Sigma_2 = 0$ against H_1. They reduce the testing problem by invariance to the maximal invariant statistics, and the non-null distribution of this statistics is obtained in terms of an expression containing an integral, which is not evaluated. It concerns an integral of the form

$$I(B_1, B_2) = \int_{\mathrm{GL}(m,\mathbb{R})} \varphi(L)\exp\left(-\tfrac{1}{2}\operatorname{tr}(B_1 L B_2 L')\right) d\beta(L), \tag{2}$$

where B_1 and B_2 belong to $A(m)$, and $\mathrm{GL}(m,\mathbb{R})$ is the group of nonsingular $m \times m$ matrices,

$$\varphi(L) = |L|^{2N}\exp\left(-\tfrac{1}{2}\operatorname{tr}(LL')\right), \qquad L \in \mathrm{GL}(m,\mathbb{R}). \tag{3}$$

Finally β is a Haar measure on $\mathrm{GL}(m,\mathbb{R})$ normalized such that the integral of $\varphi(L)$ over $\mathrm{GL}(m,\mathbb{R})$ with respect to β is 1.

In the present article we obtain an explicit expression for $I(B_1, B_2)$.

To simplify (2), we consider skew-symmetric matrices of a special form. For the rest of the article we let $n = [m/2]$. For real numbers $\lambda_1, \dots, \lambda_n$ let Λ denote the matrix of the form

$$\begin{aligned} \Lambda &= \operatorname{diag}\left[\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}\right] && \text{if } m = 2n, \\ \Lambda &= \operatorname{diag}\left[\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}, 0\right] && \text{if } m = 2n+1. \end{aligned} \tag{4}$$

For every $B \in A(m)$ there exists a matrix $H \in O(m)$, the group of orthogonal matrices, such that $HBH' = \Lambda$, where Λ has the form (4) [see Andersson and Perlman (1984)]. Using that β is a Haar measure, it follows that we only have to consider $I(\Lambda, \Gamma)$, where Λ and Γ has the form (4).

The next step is to express $I(\Lambda, \Gamma)$ by means of an integral over $T(m)$, the group of upper-triangular matrices with positive diagonal elements. Let μ be the right Haar measure on $T(m)$ given by

$$\frac{1}{c}\prod_{i=1}^{m} t_{ii}^{-i}\, dT, \tag{5}$$

where t_{ii}, $i = 1, \dots, m$, are the diagonal elements of T and

$$c = 2^{m(N-1)}\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left(N - \tfrac{1}{2}(i-1)\right). \tag{6}$$

Then the integral of $\varphi(T)$ over $T(m)$ w.r.t. μ is 1.

Consider now the Iwasawa decomposition, that is, the one-to-one and onto mapping given by

$$(H, T) \to HT, \qquad O(m) \times T(m) \to \mathrm{GL}(m,\mathbb{R}). \tag{7}$$

By this mapping β is the transformed measure of $\mu \otimes \alpha$, where α is the normed Haar measure on $O(m)$ [see Bourbaki (1963), Chapters 7 and 8]. Using this, we can write $I(\Lambda, \Gamma)$ as

$$\int_{T(m)} \varphi(T) \int_{O(m)} \exp\left(-\tfrac{1}{2}\mathrm{tr}(\Gamma H T \Lambda T' H')\right) d\alpha(H)\, d\mu(T). \tag{8}$$

An integral over $O(m)$ of the form occurring in (8) has been studied by Harish and Chandra [see Helgason (1984), page 328], but we have not been able to use their result.

Expanding as a power series, the integral above can be expressed as an infinite sum of terms of the form

$$(k!)^{-1} \int_{T(m)} \varphi(T) \int_{O(m)} \left(-\tfrac{1}{2}\mathrm{tr}(\Gamma H T \Lambda T' H')\right)^k d\alpha(H)\, d\mu(T). \tag{9}$$

3. The polynomials $C_{\bar{k}}$. As a function of $\lambda_1, \dots, \lambda_n$, the integral (9) is a homogeneous symmetric polynomial of degree k. We shall show that it is possible to obtain an explicit expression for the integral by selecting a suitable basis for the vector space of these polynomials. Since the integral over $O(m)$ contains the term $T\Lambda T'$ it will be convenient to consider polynomials as functions of a skew-symmetric matrix, instead of just skew-symmetric matrices of the form (4).

First, let $P_n(k)$ be the set of ordered sequences $\bar{k} = (k_1, \dots, k_n)$, where $k_i \in \mathbb{N} \cup \{0\}$, $k_1 \geq k_2 \geq \cdots \geq k_n$ and $k_1 + \cdots + k_n = k$. An element in $P_n(k)$ is also called a partition of k in at most n parts.

In the Appendix it is shown how it is possible to define for each $\bar{k} \in P_n(k)$ a polynomial $C_{\bar{k}}$, which is a homogeneous polynomial in the different element of $B \in A(m)$, and that these polynomials have the following essential properties:

$$C_{\bar{k}}(HBH') = C_{\bar{k}}(B) \quad \text{for } H \in O(m), \tag{10}$$

$$\int_{T(m)} \varphi(T) C_{\bar{k}}(TBT')\, d\mu(T) = d(N, \bar{k}) C_{\bar{k}}(B), \tag{11}$$

where

$$d(N, \bar{k}) = 2^{2k} \frac{\prod_{i=1}^m \Gamma\left(N + l_i - \frac{1}{2}(i-1)\right)}{\prod_{i=1}^m \Gamma\left(N - \frac{1}{2}(i-1)\right)}, \tag{12}$$

with $\bar{k} = (l_1, l_1, \dots, l_n, l_n)$ for m even and $\bar{k} = (l_1, l_1, \dots, l_n, l_n, 0)$ for m odd;

$$\int_{O(m)} \mathrm{tr}(B_1 H B_2 H')^{2k} d\alpha(H) = \sum_{\bar{k} \in P_n(k)} c(\bar{k}) C_{\bar{k}}(B_1) C_{\bar{k}}(B_2), \tag{13}$$

$$\int_{O(m)} \mathrm{tr}(B_1 H B_2 H')^{2k+1} d\alpha(H) = 0, \tag{14}$$

where $B_1, B_2 \in A(m)$ and α is the normed Haar measure on $O(m)$. The coefficients $c(\bar{k})$ are found in the next section.

From (11) and (12) it follows (see the Appendix) that when T is distributed on $T(m)$ such that TT' has the Wishart(I, m) distribution, then

$$E_T C_{\bar{k}}(TBT') = d\left(\frac{m}{2}, \bar{k}\right) C_{\bar{k}}(B). \tag{15}$$

From (10) it follows that $C_{\bar{k}}(\Lambda)$ is a homogeneous symmetric polynomial in $\lambda_1, \dots, \lambda_n$. In the Appendix it is shown that the term with the highest weight is $\lambda_1^{2k_1} \dots \lambda_n^{2k_n}$. In the next section we shall see, using (10) and (11), that $C_{\bar{k}}(\Lambda)$ can be expressed by means of some well-known polynomials. It is then clear that (10)–(14) make it possible to evaluate the integral (9).

4. Evaluation of the integral. Consider a $m \times m$ matrix $X = (x_{i,j})$ of independent standard normal variables $x_{i,j}$, and define the generating function g by

$$g(z, \Gamma, \Lambda) = E_X \exp(z \operatorname{tr}(\Gamma X \Lambda X')) \tag{16}$$

for z sufficiently small, and Γ, Λ have the form (4).

Let

$$\det_{ij} X = \det\begin{pmatrix} x_{2i-1,2j-1} & x_{2i-1,2j} \\ x_{2i,2j-1} & x_{2i,2j} \end{pmatrix}$$

for $i = 1, \dots, n$; $j = 1, \dots, n$.

It is then seen that

$$\operatorname{tr}(\Gamma X \Lambda X') = -2 \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \gamma_j \det_{ij} X$$

and using this, we get that

$$g(z, \Gamma, \Lambda) = \prod_{i=1}^{n} \prod_{j=1}^{n} \left(1 - 4z^2 \lambda_i^2 \gamma_j^2\right)^{-1} \tag{17}$$

From Takemura (1984), page 93, we find that $g(z, \Gamma, \Lambda)$ is a sum of terms of the form

$$2^{2k} z^{2k} \sum_{\bar{k} \in P_n(k)} S_{\bar{k}}(\lambda_1^2, \dots, \lambda_n^2) S_{\bar{k}}(\gamma_1^2, \dots, \gamma_n^2), \tag{18}$$

where $S_{\bar{k}}$ is the Schur function corresponding to the partition $\bar{k} \in P_n(k)$. These functions are the same as the zonal polynomials on the Hermitian matrices with the matrices restricted to be diagonal.

We get another expression for g by expanding (16) in a sum of terms of the form

$$(k!)^{-1} z^k E_X\left(\operatorname{tr}\left((\Gamma X \Lambda X')^k\right)\right). \tag{19}$$

Now $E_X = E_T E_H$, where H has the uniform distribution on $O(m)$, and T is distributed on $T(m)$ such that TT' has the Wishart(I, m) distribution. Using

(13)–(15), we get that $g(z, \Gamma, \Lambda)$ can be written as

$$(2k)!^{-1} z^{2k} \sum_{\bar{k} \in P_n(k)} d\left(\frac{m}{2}, \bar{k}\right) c(\bar{k}) C_{\bar{k}}(\Lambda) C_{\bar{k}}(\Gamma). \tag{20}$$

Since the term of highest weight in both $C_{\bar{k}}(\Lambda)$ and $S_{\bar{k}}(\lambda_1^2, \ldots, \lambda_n^2)$ is of the form $\lambda_1^{2k_1} \ldots \lambda_n^{2k_n}$ it follows by comparing (18) and (20) that

$$C_{\bar{k}}(\Lambda) = S_{\bar{k}}(\lambda_1^2, \ldots, \lambda_n^2). \tag{21}$$

Again by comparing (18) and (20)

$$c(\bar{k}) = (2k)! 2^{2k} \Big/ d\left(\frac{m}{2}, \bar{k}\right). \tag{22}$$

Using (11)–(14) on (9), we get the following theorem.

Theorem. *Let $I(\Lambda, \Gamma)$ be given by* (2). *Then*

$$I(\Lambda, \Gamma) = \sum_{k=0}^{\infty} \sum_{\bar{k} \in P_n(k)} \alpha(\bar{k}) S_{\bar{k}}(\lambda_1^2, \ldots, \lambda_n^2) S_{\bar{k}}(\gamma_1^2, \ldots, \gamma_n^2), \tag{23}$$

where $S_{\bar{k}}$ is the Schur function corresponding to the partition $\bar{k} \in P_n(k)$,

$$\alpha(\bar{k}) = \frac{\prod_{i=1}^{m} \Gamma\left(N + l_i - \frac{1}{2}(i-1)\right) \prod_{i=1}^{m} \Gamma\left(\frac{1}{2}m - \frac{1}{2}(i-1)\right)}{\prod_{i=1}^{m} \Gamma\left(N - \frac{1}{2}(i-1)\right) \prod_{i=1}^{m} \Gamma\left(\frac{1}{2}m + l_i - \frac{1}{2}(i-1)\right)} \tag{24}$$

and $\bar{k} = (l_1, l_1, \ldots, l_n, l_n)$ for m even and $\bar{k} = (l_1, l_1, \ldots, l_n, l_n, 0)$ for m odd.

Some manipulation (see the Appendix) with the coefficients $\alpha(\bar{k})$ gives that

$$I(\Lambda, \Gamma) = {}_2\tilde{F}_1\left(N, N - \tfrac{1}{2}, \left[\frac{m}{2} + \frac{1}{2}\right] - \tfrac{1}{2}; \lambda_1^2, \ldots, \lambda_n^2, \gamma_1^2, \ldots, \gamma_n^2\right), \tag{25}$$

where ${}_2\tilde{F}_1$ is the complex hypergeometric function as defined in James (1964), page 488.

APPENDIX

Definition of $C_{\bar{k}}$. Let $V(k)$ be the real vector space of homogeneous polynomials $f(B)$ of degree k in the $m(m-1)/2$ different elements of $B \in A(m, \mathbb{R})$.

For each $L \in \mathrm{GL}(m, \mathbb{R})$ a transformation $D(L)$ of $A(m, \mathbb{R})$ (as a vector space) is defined by

$$D(L)(B) = LBL'. \tag{26}$$

The transformations $D(L)$ define a representation, D, of $\mathrm{GL}(m, \mathbb{R})$ on $A(m, \mathbb{R})$. By considering the trace of $D(L)$ it is easy to see that D is equivalent to the irreducible, integral representation of $\mathrm{GL}(m, \mathbb{R})$ corresponding to the partition $(1, 1)$ [see Littlewood (1940), Chapter 10].

For each $L \in \mathrm{GL}(m, \mathbb{R})$ a transformation $T(L)$ of $V(k)$ is defined by

$$f \to T(L)f, \qquad (T(L)f)(B) = f(L^{-1}BL^{-1'}). \tag{27}$$

The transformations $T(L)$ define a representation, T, of $\mathrm{GL}(m, \mathbb{R})$ on $V(k)$.

The elements of $P_n(k)$ are ordered lexicographically [see Constantine (1963), page 1272]. For $\bar{k} \in P_n(k)$ we let $2\bar{k} = (2k_1, \dots, 2k_n) \in P_n(2k)$ and $\bar{k}2 = (k_1, k_1, \dots, k_n, k_n) \in P_{2n}(2k)$.

It now follows from Thrall (1942), page 380, that the representation T decomposes into the irreducible representations of $\mathrm{GL}(m, \mathbb{R})$ corresponding to the partitions $\bar{k}2$, each of which is contained exactly once, and $\bar{k}$ runs through $P_n(k)$. Let $V(\bar{k})$ be the invariant irreducible subspace of $V(k)$ in which the irreducible representation of $\mathrm{GL}(m, \mathbb{R})$ corresponding to the partition $\bar{k}2$ acts.

For each $\bar{k} \in P_n(k)$ it is seen that (27) with L restricted to be orthogonal defines a representation of $O(m)$ on $V(\bar{k})$; by this representation $V(\bar{k})$ decomposes into a direct sum of irreducible invariant subspaces $V(\bar{k}, i)$, $i = 1, \dots, n(\bar{k})$. It follows from Littlewood (1940), page 240, that if $\bar{k}$ is a partition in even parts, that is, each k_i is even, then exactly one of the subspaces, say $V(\bar{k}, 1)$ has the following property: It is one-dimensional and the corresponding representation of $O(m)$ is the identity representation; if $\bar{k}$ is not a partition in even parts none of the subspaces has this property. We will now only consider the space $V(2k)$. Using a method similar to that of Constantine (1963), pages 1272–1273, it can be shown that a polynomial f, which generates $V(2\bar{k}, 1)$, has the form $f(\Lambda) = d(\bar{k})\lambda_1^{2k_1} \dots \lambda_n^{2k_n}$ + terms of lower weight. Here is Λ of the form (4), and terms $\lambda_1^{l_1} \dots \lambda_n^{l_n}$ are ordered corresponding to the ordering of the partitions $\bar{l} \in P_n(2k)$.

Definition.

(28) $C_{\bar{k}}$ is the polynomial which generates $V(2\bar{k}, 1)$ normed such that the coefficient to the term with highest weight is 1.

It follows that

$$C_{\bar{k}}(LBL') \in V(2\bar{k}) \quad \text{for each } L \in \mathrm{GL}(m, \mathbb{R}), \tag{29}$$

$$C_{\bar{k}}(HBH') = C_{\bar{k}}(B) \quad \text{for each } H \in O(m). \tag{30}$$

Proof of (11)–(15). Consider the transformation, E, from $V(2k)$ to $V(2k)$ given by

$$Ef(B) = \int_{\mathrm{GL}(m, \mathbb{R})} \varphi(L) f(LBL')\, d\beta(L). \tag{31}$$

Since $V(2\bar{k})$ is invariant under $T(L)$ for each L it follows that $Ef \in V(2\bar{k})$ when $f \in V(2\bar{k})$. Using the invariance of β, it is seen that Ef is $O(m)$-invariant. In particular we get that $EC_{\bar{k}}$ is proportional to $C_{\bar{k}}$, and note that since $C_{\bar{k}}$ is $O(m)$-invariant we have by the Iwasawa decomposition that $(EC_{\bar{k}})(B)$ is given by the left-hand side of (11).

To evaluate $d(N, \bar{k})$ we first note that [proceed as in Constantine (1963), page 1273] for $T \in T(m)$ the term of highest weight in $C_{\bar{k}}(T\Lambda T')$ becomes

$$\lambda_1^{2k_l} \dots \lambda_n^{2k_n} g(T),$$

where

$$g(T) = \left(t_{11}^2 t_{22}^2\right)^{k_1 - k_2} \left(t_{11}^2 \dots t_{44}^2\right)^{k_2 - k_3} \dots \left(t_{11}^2 \dots t_{2n,2n}^2\right)^{k_n}. \tag{32}$$

By comparing the coefficients of the terms of highest weight on both sides of (11) (with $B = \Lambda$) we get that $d(N, \bar{k})$ is the integral of $\varphi(T)g(T)$ over $T(m)$ w.r.t. μ and a direct calculation gives (12).

Schur's lemma [see Naimark and Stern (1982), pages 26 and 58] and the fact that an integral of $T(H)$ over $O(m)$ w.r.t. α is 0 when T is an irreducible representation different from the identity representation of $O(m)$ give (13) and (14).

The Wishart(I, m) distribution has the density

$$a(m)^{-1} |W|^{-1/2} \exp\left(-\tfrac{1}{2} \operatorname{tr} W\right) dW,$$

where

$$a(m) = 2^{m^2/2} \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma(m - (i-1)/2)).$$

The mapping $T \to TT'$ has the Jacobian

$$2^m \prod_{i=1}^{m} t_{ii}^{m-i+1}.$$

From this and (11) and (12) follows (15). □

Proof of (25). In James (1964) the function ${}_2\tilde{F}_1$ is expressed by the complex zonal polynomials $\tilde{C}_{\bar{k}}$ given by

$$\tilde{C}_{\bar{k}}(\lambda_1, \dots, \lambda_n) = \chi_{[\bar{k}]}(1) S_{\bar{k}}(\lambda_1, \dots, \lambda_n), \tag{33}$$

where

$$\chi_{[\bar{k}]}(1) = k! \frac{\prod_{i<j}(k_i - k_j - i + j)}{\prod_{i=1}^{n}(k_i + n - i)!}$$

and

$$S_{\bar{k}}(1, \dots, 1) = \frac{\prod_{i<j}(k_i - k_j - i + j)}{\prod_{i=1}^{n}(n - i)!}.$$

From the relation

$$(a/2)_{\bar{l}} = [[a/2]]_{\bar{k}} [[(a+1)/2] - 1/2]_{\bar{k}} \tag{34}$$

[see James (1964) for the definition of $[a]_{\bar{k}}$ and $(a)_{\bar{k}}$] and (23) one then obtains (25). □

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