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ON KARLIN'S CONJECTURE FOR RANDOM REPLACEMENT SAMPLING PLANS

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In 1974 Karlin introduced the concept of random replacement schemes and conjectured that the componentwise monotonicity of the replacement probabilities (condition A) is equivalent to a corresponding ordering of expectations of all functions ϕ from a certain class $\mathscr{L}_K$ (condition B). In this paper it is shown that A implies B for sample sizes $n \leq 5$ and—provided the sample space is sufficiently large—also for $n \geq 6$. By a counterexample it is shown that $\mathscr{L}_K$ is not suitable for A being implied by B, i.e. one direction of Karlin's conjecture is disproved.

1. Introduction. Let $\mathscr{P} = \{P_\theta: \theta \in \Theta\}$ be a class of probability measures on a measurable space, where Θ is an arbitrary parameter space, and let $\mathscr{L}$ be a class of real functions ϕ, integrable w.r.t. $\mathscr{P}$. Then $\mathscr{L}$ induces a partial ordering on Θ, namely $\theta \leq_{\mathscr{L}} \theta'$ iff $E_\theta \phi \leq E_{\theta'} \phi$ for all $\phi \in \mathscr{L}$. If, conversely, a partial ordering $\leq_\Theta$ on Θ is given, it may be interesting to search for classes $\mathscr{L}$ which characterize this partial ordering in the sense above. If such a class exists, then of course the set $\mathscr{L}_0$ of all integrable ϕ such that $E_\theta\phi \leq E_{\theta'}\phi$ for all $\theta, \theta' \in \Theta$ with $\theta \leq_\Theta \theta'$ is the largest of these classes.

In the context of sampling from finite populations, a beautiful result illustrating this correspondence is Theorem 12.A.1 in Marshall and Olkin (1979), cf. also Snijders (1976). Here for a certain finite sample space Ω an ordering for $\mathscr{P}$ by dominance is characterized by the set of all Schur-convex functions on Ω. Karlin (1974) considered this problem for sampling plans with random replacement which will be described here in the simplified—but for the present purposes equivalent—form used by van Zwet (1983): Let $n, N \in \mathbb{N}$ with $2 \leq n \leq N$ be given and put $\Omega = \{1, \cdots, N\}^n$, $\Theta = [0, 1]^{n-1}$. For $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{n-1}) \in \Theta$ let $P_{\boldsymbol{\theta}}$ be the following distribution: Consider an urn containing N balls numbered by $1, \cdots, N$. Take n drawings according to the following scheme:

(a) The first ball is drawn with equal probabilities. For $1 \leq i \leq n - 1$

(b1) replace with probability θ_i the ith ball drawn and remove it with probability $1 - \theta_i$ and

(b2) draw the $(i + 1)$th ball with equal probabilities.

Then P_θ is the distribution of $\mathbf{X} = (X_1, \cdots, X_n)$ where the random variable X_i represents the number x_i of the ball resulting from the ith draw.

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Further, let $\mathscr{L}_K$ denote the set of real functions ϕ on Ω which are symmetric in their arguments and satisfy

$$2\phi(u, v, y_3, \cdots, y_n) \le \phi(u, u, y_3, \cdots, y_n) + \phi(v, v, y_3, \cdots, y_n) \quad \text{for all} \quad (u, v, y_3, \cdots, y_n) \in \Omega. \tag{1}$$

Karlin conjectured that $\mathscr{L}_K$ characterizes the componentwise real ordering on Θ, which will be denoted simply by $\boldsymbol{\theta} \le \boldsymbol{\theta}'$. As affirmation he proved a set of results comparing an arbitrary $\boldsymbol{\theta}$ with the vectors $\boldsymbol{\theta}_1 = (1, \cdots, 1)$ and $\boldsymbol{\theta}_0 = (0, \cdots, 0)$, i.e. with sampling with and without replacement, respectively. Van Zwet (1983) showed, using conditional expectations and a discrete version of Jensen's inequality, that for a certain class $\mathscr{L}_Z$ different from $\mathscr{L}_K$ one has

$$E_{\boldsymbol{\theta}}\psi \le E_{\boldsymbol{\theta}_1}\psi \quad \forall \boldsymbol{\theta} \in \Theta, \quad \forall \psi \in \mathscr{L}_Z.$$

In this paper we discuss for the random replacement scheme described above the general case $\boldsymbol{\theta} \le \boldsymbol{\theta}'$ and show that provided $2 \le n \le 5$, $N \ge n$, and for $n \ge 6$, N sufficiently large, it holds that

$$\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta, \quad \boldsymbol{\theta} \le \boldsymbol{\theta}' \quad \text{implies} \quad E_{\boldsymbol{\theta}}\phi \le E_{\boldsymbol{\theta}'}\phi \quad \forall \phi \in \mathscr{L}_K. \tag{2}$$

Furthermore, we give an example showing that the set $\mathscr{L}_s$ of all ϕ on Ω which are symmetric in their arguments is not large enough to separate the elements of Θ; in particular the converse of (2) does not hold.

For that purpose we will show first that the class

$$\mathscr{L}_0 = \{\phi: \Omega \to \mathbb{R} \mid E_{\boldsymbol{\theta}}\phi \le E_{\boldsymbol{\theta}'}\phi \ \forall \boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta, \boldsymbol{\theta} \le \boldsymbol{\theta}'\} \tag{3}$$

characterizes the componentwise real ordering on Θ and derive sufficient conditions for $\phi \in \mathscr{L}_0$, if $\phi \in \mathscr{L}_s$. We then prove that these conditions are satisfied for $\phi \in \mathscr{L}_K$ under the assumptions on n and N specified above.

In the sequel we write A^c for the complement of a set A, $\sum A_m$ denotes the union of the pairwise disjoint sets A_m, $A \backslash B = A \cap B^c$ and $|A|$ is the cardinality of A. The usual conventions for sums and products over empty index sets are adopted. For $\mathbf{v} = (v_1, \cdots, v_p) \in \mathbb{R}^p$ and $\mathbf{m} = (m_1, \cdots, m_p) \in \mathbb{N}^p$ we will denote by $([v_1, m_1], \cdots, [v_p, m_p])$ the vector whose first m_1 components are v_1, the next m_2 components are v_2 etc.

2. Tne class $\mathscr{L}_0$ and partial results on Karlin's conjecture. Let Ω, $P_{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \Theta$ correspond to the random replacement model as described in Section 1. For $I \subset I_0 = \{1, \cdots, n-1\}$, $\boldsymbol{\theta} \in \Theta$ define

$$\begin{aligned} B(I) &= \{\mathbf{x} = (x_1, \cdots, x_n) \in \Omega: x_j \ne x_i \ \forall j > i, \forall i \in I\}, \\ b(I) &= |B(I)|, \ f_I(\boldsymbol{\theta}) = \textstyle\prod_{i \in I} (1 - \theta_i) \prod_{j \in I^c} \theta_j. \end{aligned} \tag{4}$$

With these notations one obtains

$$P_{\boldsymbol{\theta}}(\mathbf{X} = \mathbf{x}) = \textstyle\sum_{I \subset I_0} f_I(\boldsymbol{\theta}) (b(I))^{-1} 1_{B(I)}(\mathbf{x}). \tag{5}$$

(5) can easily be proved by introducing independent zero-one variables Y_i indicating whether the ith ball is removed or replaced and calculating the

probabilities $P_\theta(\mathbf{X} = \mathbf{x};\ Y_i = 0,\ i \in I;\ Y_j = 1, j \in I^c)$ as conditional probabilities. (Thus $i \in I$ iff the ith ball drawn is removed.) $b(I)$ can be calculated in the following way:

Let $I = \{i_1, \cdots, i_\ell\} \neq \varnothing$, $i_1 < i_2 < \cdots < i_\ell$. Then

$$b(I) = \begin{cases} |\Omega| = N^n, & \text{if } I = \varnothing, \\ N^{i_1}(N-1)^{n-i_1}, & \text{if } I = \{i_1\}, \\ N^{i_1}(N-\ell)^{n-i_\ell} \prod_{\nu=2}^{\ell} (N+1-\nu)^{i_\nu - i_{\nu-1}}, & \text{otherwise.} \end{cases} \tag{6}$$

The following theorem shows that we can restrict attention to the extremal points of the unit cube, i.e. $\boldsymbol{\theta} \in \Theta_0 = \{0, 1\}^{n-1}$. Therefore, when discussing $E_\theta \phi$ as a function of $\boldsymbol{\theta} \in \Theta_0$ we will write

$$E_\theta \phi = \alpha_\phi(I) = (b(I))^{-1} \sum_{\mathbf{x} \in B(I)} \phi(\mathbf{x}), \tag{7}$$

using the 1-1-correspondence between $\boldsymbol{\theta} \in \Theta_0$ and $I \subset I_0$ given by

$$I = I_\theta = \{i \in I_0 \colon \theta_i = 0\}.$$

Note that for $\boldsymbol{\theta} \in \Theta_0$

$$P_\theta(\mathbf{X} = \mathbf{x}) = \begin{cases} (b(I_\theta))^{-1}, & \text{if } \mathbf{x} \in B(I_\theta), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

THEOREM 1. *Let for $I \subset I_0$, $\phi\colon \Omega \to \mathbb{R}$, $\alpha_\phi(I)$ as in* (7), *$\mathscr{L}_0$ as in* (3),

$$\mathscr{L}_1 = \{\phi\colon \Omega \to \mathbb{R} \mid E_\theta \phi \le E_{\theta'} \phi\ \forall \boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta_0,\ \boldsymbol{\theta} \le \boldsymbol{\theta}'\}$$

and

$$\mathscr{L}_2 = \{\phi\colon \Omega \to \mathbb{R} \mid \alpha_\phi(I \cup \{i\}) \le \alpha_\phi(I)\ \forall I \subset I_0,\ \forall i \in I^c\}.$$

Then $\mathscr{L}_0 = \mathscr{L}_1 = \mathscr{L}_2$.

PROOF. It is easy to see that $\mathscr{L}_0 \subset \mathscr{L}_1 = \mathscr{L}_2$. In order to show that $\mathscr{L}_2 \subset \mathscr{L}_0$ let $i_0 \in I_0$ and $J = I_0 \backslash \{i_0\}$. Then, using (5), we get for all $\phi \in \mathscr{L}_2$

$$\begin{aligned} \frac{\partial}{\partial \theta_{i_0}} E_\theta \phi &= \sum_{I \subset I_0} \alpha_\phi(I) \frac{\partial}{\partial \theta_{i_0}} f_I(\boldsymbol{\theta}) \\ &= \sum_{I \subset J} \left(\prod_{i \in I} (1 - \theta_i) \prod_{j \in J \backslash I} \theta_j\right)(\alpha_\phi(I) - \alpha_\phi(I \cup \{i_0\})) \ge 0, \end{aligned}$$

hence $\phi \in \mathscr{L}_0$. □

EXAMPLE 1. As an example for a function ϕ which is in $\mathscr{L}_0$ but not in $\mathscr{L}_K$ take for $n = N = 3$

$$\phi(x_1, x_2, x_3) = \begin{cases} 1/3 \sum_{i=1}^3 x_i, & \text{if the } x_i \text{ are distinct,} \\ \text{median } (x_1, x_2, x_3), & \text{otherwise,} \end{cases}$$

cf. Marshall and Olkin (1979, page 339). Here we have $E_\theta \phi = 2$ for all $\boldsymbol{\theta} \in \Theta$, hence $\phi \in \mathscr{L}_0$; but

$$\phi(1, 1, 3) + \phi(2, 2, 3) = 3 < 2\phi(1, 2, 3) = 4.$$

THEOREM 2. *$E_\theta \phi \le E_{\theta'} \phi$ for all $\phi \in \mathscr{C}_0$ implies $\theta \le \theta'$.*

PROOF. Let $i_0 \in I_0$. It is sufficient to demonstrate the existence of a $\phi \in \mathscr{C}_0$ with $E_\theta \phi = \theta_{i_0}$ for all $\theta \in \Theta$. If

$$\alpha_\phi(I) = 1 - 1_I(i_0) \quad \text{for all} \quad I, \tag{9}$$

then

$$E_\theta \phi = \textstyle\sum_I f_I(\theta)\alpha_\phi(I) = \sum_{I: i_0 \notin I} f_I(\theta) = \theta_{i_0}$$

and $\phi \in \mathscr{C}_0$ by Theorem 1. Hence it is sufficient to demonstrate the existence of a ϕ satisfying (9). Define

$$I_{\mathbf{x}} = \{i \in I_0 \colon x_j \ne x_i \text{ for all } j > i\}.$$

Then $\Omega = \sum_I \{\mathbf{x} \colon I_{\mathbf{x}} = I\}$, so that for $\theta \in \Theta_0$

$$\alpha_\phi(I_\theta) = \textstyle\sum_I \sum_{\mathbf{x} \in \{I_{\mathbf{x}} = I\}} \phi(\mathbf{x}) P_\theta(\mathbf{X} = \mathbf{x}).$$

Choose $\phi(x) = \psi(I)$ on $\{\mathbf{x} \colon I_{\mathbf{x}} = I\}$. Then

$$\alpha_\phi(I_\theta) = \textstyle\sum_I \psi(I) P_\theta(I_{\mathbf{X}} = I).$$

The constants $\psi(I)$ are then uniquely determined by

$$\textstyle\sum_I \psi(I) P_\theta(I_{\mathbf{X}} = I) = 1 - 1_{I_\theta}(i_0) \quad \text{for all} \quad \theta \in \Theta_0. \tag{10}$$

In fact, if the I are arranged in an order stronger than the inclusion order, from $P_\theta(I_{\mathbf{X}} \supset I_\theta) = 1$ and $P_\theta(I_{\mathbf{X}} = I_\theta) > 0$, $\theta \in \Theta_0$, it follows that (10) defines a system of linear equations in $\psi(I)$ with a coefficient matrix in triangular form and positive diagonal elements. □

For discussing whether $\mathscr{C}_K \subset \mathscr{C}_0$, it is appropriate to find first a representation of $\alpha_\phi(I)$ tailored to the symmetry of ϕ. To this end the following definitions are useful: For $1 \le p \le n$ let

$$K_p = \{\mathbf{k} = (k_1, \cdots, k_p) \in \mathbb{N}^p \colon \textstyle\sum_{j=1}^p k_j = n,\ k_1 \ge \cdots \ge k_p\}, \quad K = \sum_{p=1}^n K_p,$$

$$\pi \colon \begin{cases} K \to \{1, \cdots, n\} \\ \mathbf{k} \to p, \quad \text{if} \quad \mathbf{k} \in K_p, \end{cases}$$

and

$$G(p, N) = \{\mathbf{y} = (y_1, \cdots, y_p) \in \{1, \cdots, N\}^p \colon y_i \ne y_j \text{ for } i \ne j\}.$$

Define $\kappa \colon \Omega \to K$ as the *symmetric* function for which $\kappa([y_1, k_1], \cdots, [y_p, k_p]) = (k_1, \cdots, k_p)$ for all $\mathbf{k} \in K_p$, $\mathbf{y} \in G(p, N)$. Let further, for $\mathbf{k} \in K$ and $I \subset I_0$, with $p = \pi(\mathbf{k})$,

$$\Phi(\mathbf{k}) = \textstyle\sum_{\mathbf{y} \in G(p,N)} \phi([y_1, k_1], \cdots, [y_p, k_p]),$$

$$A(\mathbf{k}, I) = \{\mathbf{x} \in \kappa^{-1}(\{\mathbf{k}\}) \cap \{1, \cdots, p\}^n \colon x_i \ne x_j\ \forall j > i,\ \forall i \in I\},$$

$$a(\mathbf{k}, I) = (p!)^{-1} |A(\mathbf{k}, I)|.$$

Note that the numbers $a(\mathbf{k}, I)$ do not depend on N and that

$$a(\mathbf{k}, I) = 0 \quad \text{for all } \mathbf{k} \text{ with } \pi(\mathbf{k}) \le |I|. \tag{11}$$

LEMMA 1. *For* $\phi \in \mathscr{L}_s$ *one has*

(a) $$b(I)\alpha_\phi(I) = \sum_{\mathbf{k}\in K} a(\mathbf{k}, I)\Phi(\mathbf{k}),$$

(b) $$b(I) = \sum_{\mathbf{k}\in K} (\pi(\mathbf{k}))! \binom{N}{\pi(\mathbf{k})} a(\mathbf{k}, I).$$

PROOF. Since ϕ is symmetric, one has with $D(\mathbf{k}, I) = B(I) \cap \kappa^{-1}(\{\mathbf{k}\})$

$$\sum_{\mathbf{x}\in D(\mathbf{k},I)} \phi(\mathbf{x}) = a(\mathbf{k}, I) \sum_{\mathbf{y}\in G(\pi(\mathbf{k}),N)} \phi([y_1, k_1], \cdots, [y_p, k_p])$$
$$= a(\mathbf{k}, I)\Phi(\mathbf{k}).$$

From (8) one therefore obtains for $\theta \in \Theta_0$ and $I = I_\theta$ that

$$\alpha_\phi(I) = E_\theta\phi = \sum_{\mathbf{k}\in K} \sum_{\mathbf{x}\in D(\mathbf{k},I)} \phi(\mathbf{x}) P_\theta(\mathbf{X} = \mathbf{x})$$
$$= (b(I))^{-1} \sum_{\mathbf{k}\in K} \sum_{\mathbf{x}\in D(\mathbf{k},I)} \phi(\mathbf{x})$$
$$= (b(I))^{-1} \sum_{\mathbf{k}\in K} a(\mathbf{k}, I)\Phi(\mathbf{k}).$$

This is (a), and implies (b) taking $\phi = 1$. □

EXAMPLE 2. We show by a counterexample that the set $\mathscr{L}_s$ is not large enough to separate the elements of Θ; in particular, one direction of Karlin's conjecture is disproved: Let for $3 \le n \le N$

$$\boldsymbol{\theta}' = (\theta', 0, \cdots, 0), \quad \boldsymbol{\theta}'' = (0, 1, 0, \cdots, 0).$$

Obviously, the distributions of $\kappa(\mathbf{X})$ under $P_{\theta'}$ and $P_{\theta''}$ are both concentrated on

$$K_{n-1} \cup K_n = \{\mathbf{k}^{(n-1)}, \mathbf{k}^{(n)}\}$$

where

$$\mathbf{k}^{(n-1)} = ([2, 1], [1, n-2]), \quad \mathbf{k}^{(n)} = ([1, n]). \tag{12}$$

Straightforward calculations show that

$$P_{\theta'}(\kappa(\mathbf{X}) = \mathbf{k}^{(n-1)}) = \theta'(n-1)/N$$

and

$$P_{\theta''}(\kappa(\mathbf{X}) = \mathbf{k}^{(n-1)}) = (n-2)/(N-1).$$

With $\theta' = N(n-2)(N-1)^{-1}(n-1)^{-1}$ both distributions are identical; hence

$$E_{\theta'}\phi = E_{\theta''}\phi \quad \text{for all} \quad \phi \in \mathscr{L}_s,$$

but $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}''$ are not comparable.

Under the assumption that a set of combinatorial inequalities—only depending on n and N—holds, Lemma 1 enables us to give a sufficient condition for $\phi \in \mathscr{L}_s$ to be in $\mathscr{L}_0$ which depends on ϕ only through $\Phi(\mathbf{k})$.

THEOREM 3. *Assume that*

$$\gamma(\mathbf{k}, I, i) = a(\mathbf{k}, I)b(I \cup \{i\}) - a(\mathbf{k}, I \cup \{i\})b(I) \geq 0 \tag{13}$$
$$\textit{for all} \quad I \subset I_0, \quad i \in I^c, \quad \mathbf{k} \in K_p, \quad 1 \leq p \leq n-2.$$

If $\phi \in \mathscr{L}_s$ *and—with* $\mathbf{k}^{(n-1)}$, $\mathbf{k}^{(n)}$ *as in* (12)—

$$\min_{1 \leq p \leq n-2} \min_{\mathbf{k} \in K_p} (N-p)!\Phi(\mathbf{k}) \geq (N-n+1)!\Phi(\mathbf{k}^{(n-1)}) \tag{14}$$
$$\geq (N-n)!\Phi(\mathbf{k}^{(n)}),$$

then $\phi \in \mathscr{L}_0$.

PROOF. It follows from

$$\gamma(\mathbf{k}^{(n)}, I, i) = b(I \cup \{i\}) - b(I) < 0$$

and Lemma 1(b) that

$$0 = b(I)b(I \cup \{i\}) - b(I \cup \{i\})b(I)$$
$$= \sum_{p=1}^{n-1} \sum_{\mathbf{k} \in K_p} p!\binom{N}{p}\gamma(\mathbf{k}, I, i) + n!\binom{N}{n}\gamma(\mathbf{k}^{(n)}, I, i)$$
$$< \sum_{p=1}^{n-1} \sum_{\mathbf{k} \in K_p} p!\binom{N}{p}\gamma(\mathbf{k}, I, i).$$

With Lemma 1(a) and assumption (14) this yields

$$b(I)b(I \cup \{i\})[\alpha_\phi(I) - \alpha_\phi(I \cup \{i\})]$$
$$= \sum_{p=1}^{n} \sum_{\mathbf{k} \in K_p} \gamma(\mathbf{k}, I, i)\Phi(\mathbf{k})$$
$$\geq \Phi(\mathbf{k}^{(n-1)}) \sum_{p=1}^{n-1} \sum_{\mathbf{k} \in K_p} \frac{(N-n+1)!}{(N-p)!} \gamma(\mathbf{k}, I, i)$$
$$+ \gamma(\mathbf{k}^{(n)}, I, i)\Phi(\mathbf{k}^{(n)})$$
$$\geq \Phi(\mathbf{k}^{(n)}) \sum_{p=1}^{n} \sum_{\mathbf{k} \in K_p} \frac{(N-n)!}{(N-p)!} \gamma(\mathbf{k}, I, i) = 0.$$

This shows that $\phi \in \mathscr{L}_2$ so that, by Theorem 1, $\phi \in \mathscr{L}_0$. □

The next two lemmas will be useful for asymptotic considerations.

LEMMA 2. *Let* $I \subset I_0$, $\mathbf{k} \in K_p$, $1 \leq p \leq n-1$, *be such that* $a(\mathbf{k}, I) > 0$. *Then* $a(\mathbf{k}, I) > a(\mathbf{k}, I \cup \{i\})$ *for all* $i \in I^c$.

PROOF. Since $A(\mathbf{k}, I \cup \{i\}) \subset A(\mathbf{k}, I)$, one has to demonstrate the existence of an $\mathbf{x} \in A(\mathbf{k}, I) \cap (A(\mathbf{k}, I \cup \{i\}))^c$. Let $\mathbf{z} \in A(\mathbf{k}, I)$ be arbitrary and assume

$\mathbf{z} \in A(\mathbf{k}, I \cup \{i\})$. Because of $\pi(\mathbf{k}) \leq n - 1$ and $i \in I \cup \{i\}$ we have

$$I_{\mathbf{z}}^c = \{i' \in I_0\colon \exists j > i' \text{ with } z_j = z_{i'}\} \neq \varnothing$$

and $i \notin I_{\mathbf{z}}^c$. Let $i_1 = \max I_{\mathbf{z}}^c$. Then $i \neq i_1$ and there is an $i_2 > i_1$ with $z_{i_1} = z_{i_2}$. Putting $\mathbf{x} = \sigma(\mathbf{z})$ where σ is the permutation on $\{1, \cdots, n\}$ with

$$\sigma(i) = i_1, \quad \sigma(i_1) = i, \quad \sigma(j) = j \quad \text{for all other } j, \quad \text{if} \quad i_2 > i,$$

$$\sigma(i+1) = i_1, \quad \sigma(i_1) = i + 1, \quad \sigma(j) = j \quad \text{for all other } j, \quad \text{if} \quad i_2 = i,$$

$$\sigma(i) = i_1, \quad \sigma(i+1) = i_2, \quad \sigma(i_1) = i, \quad \sigma(i_2) = i+1, \quad \sigma(j) = j \quad \text{for all other } j, \quad \text{if} \quad i_2 < i,$$

by a cumbersome, but easy calculation one verifies that in all three cases $\mathbf{x} \in A(\mathbf{k}, I) \cap (A(\mathbf{k}, I \cup \{i\}))^c$. □

LEMMA 3. *Let $n \geq 3$. Then there exists $N_0 = N_0(n) \geq n$ such that $\gamma(\mathbf{k}, I, i) \geq 0$ for all $I \subset I_0$, $i \in I^c$, $\mathbf{k} \in \sum_{p=1}^{n-2} K_p$, $N \geq N_0$.*

PROOF. It suffices to consider those $\mathbf{k} \in \sum_{p=1}^{n-2} K_p$ for which $a(\mathbf{k}, I) > 0$. By Lemma 2 we then have $1 > a(\mathbf{k}, I \cup \{i\})/a(\mathbf{k}, I)$. Now $b(I \cup \{i\})$ and $b(I)$ are polynomials in N of degree n, cf. (6), the coefficient of N^n being one in both cases. Therefore, there exists $N_1 = N_1(I, i, \mathbf{k}) \geq n$ such that

$$a(\mathbf{k}, I)b(I \cup \{i\}) - a(\mathbf{k}, I \cup \{i\})b(I) = \gamma(\mathbf{k}, I, i) \geq 0 \quad \text{for all} \quad N \geq N_1.$$

For $N_0 = \max_{I,i,\mathbf{k}} N_1(I, i, \mathbf{k})$ we then get the assertion. □

THEOREM 4.

(a) *If $2 \leq n \leq 5$ and $N \geq n$, then $\mathscr{S}_K \subsetneq \mathscr{S}_0$.*

(b) *To each $n \geq 6$ there exists $N_0 = N_0(n)$ such that $\mathscr{S}_K \subset \mathscr{S}_0$ for all $N \geq N_0$.*

PROOF. By Theorem 2.1 in Karlin (1974), cf. also Marshall and Olkin (1979), Chapter 12. A, the function

$$E_{\mathscr{S}(\mathbf{k})}\, \phi = |G(\pi(\mathbf{k}), N)|^{-1} \Phi(\mathbf{k}) = \frac{(N - \pi(\mathbf{k}))!}{N!} \Phi(\mathbf{k})$$

is Schur-convex on K for all $\phi \in \mathscr{S}_K$. For $\mathbf{k}^{(n-1)}$, $\mathbf{k}^{(n)}$ as in (12) we have

$$\mathbf{k}^{(n)} \prec \mathbf{k}^{(n-1)} \prec \mathbf{k} \quad \text{for all } \mathbf{k} \in \textstyle\sum_{p=1}^{n-2} K_p,$$

where "$\prec$" is the majorization order. This shows that for $\phi \in \mathscr{S}_K$ the inequalities (14) in Theorem 3 are satisfied.

For $2 \leq n \leq 5$ one can directly verify (13): Because of (11) nothing remains to prove, if $n = 2$ or $n = 3$.

For $n = 4 \leq N$ we only have to show that

$$a((4 - \rho, \rho), \varnothing)b(\{i\}) \geq a((4 - \rho, \rho), \{i\})b(\varnothing), \quad \rho = 1, 2; \quad i = 1, 2, 3;$$

and for $n = 5 \leq N$ three types of inequalities have to be checked, namely

$$a((5-\rho, \rho), \varnothing)b(\{i\}) \geq a((5-\rho, \rho), \{i\})b(\varnothing),$$
$$\rho = 1, 2; \quad 1 \leq i \leq 4,$$
$$a((4-\rho, \rho, 1), \varnothing)b(\{i\}) \geq a((4-\rho, \rho, 1), \{i\})b(\varnothing),$$
$$\rho = 1, 2; \quad 1 \leq i \leq 4,$$
$$a((4-\rho, \rho, 1), \{i_1\})b(\{i_1, i_2\}) \geq a((4-\rho, \rho, 1), \{i_1, i_2\})b(\{i_1\}),$$
$$\rho = 1, 2; \quad 1 \leq i_1 \neq i_2 \leq 4.$$

The calculation of $a(\mathbf{k}, I)$, $b(I)$ and the verification of the inequalities can be left to the reader.

Part (b) follows with the help of Lemma 3. □

REMARK. For $n = 6$ some of the inequalities (13) do not hold. By more refined arguments we could however show that the assertion of Theorem 4(a) holds true also for $n = 6$ and $n = 7$. These arguments are too messy to be reproduced here; nevertheless, they gave us the feeling that there exist pairs (n, N) for which $\mathscr{L}_K \not\subset \mathscr{L}_0$.

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