The Annals of Statistics
1984, Vol. 12, No. 3, 1145–1150

TWO BOOKS ON MULTIVARIATE ANALYSIS

ROBB J. MUIRHEAD, *Aspects of Multivariate Statistical Theory*. John Wiley and Sons, New York, 1982, xix + 673 pages, $39.55.

MORRIS L. EATON, *Multivariate Statistics. A Vector Space Approach*. John Wiley and Sons, New York, 1983, xvi + 512 pages, $34.95.

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With the appearance of these two excellent books, the literature in the field of multivariate analysis has been greatly enriched. This is all the more remarkable since in recent years there has not exactly been a paucity of available books on this subject. On the contrary, since the classic by T. W. Anderson (1958), there has been a proliferation of high quality texts. A conservative count reveals more than sixteen books aimed at the general statistician (as opposed to, say, the psychometrician or other specialized applicators). But the field of multivariate analysis is extremely broad and offers something to everybody, from the highly theoretical to the very applied. The existing literature reflects this and by and large different books stress different aspects of the field. This makes it possible for a relatively large number of books to live peacefully together and complement each other rather than compete.

The above remarks apply in particular to the two books under review. The latter have a lot in common and yet there is almost no overlap, either with each other or with any of the other multivariate books. Neither book aims at a comprehensive treatment of the subject (but Muirhead's can claim somewhat greater completeness than Eaton's). The purpose of each of the two books is to emphasize certain aspects of multivariate analysis that have not been treated sufficiently in textbooks before. This aim is explicit in the title of Muirhead's book; it is somewhat less so in the title of Eaton's book, except for its subtitle. The aspects stressed by Muirhead are: (1) group methods and exterior differential forms in the treatment of distributional problems; (2) zonal polynomials and hypergeometric functions of matrix argument to express noncentral distributions; (3) asymptotic distributions. The aspects emphasized by Eaton are (1) vector space methods; (2) group methods for distributional problems, including invariant measures on locally compact groups and homogeneous spaces.

What the two books have in common is their emphasis on invariance and group methods. There is a good deal more on those topics here than one can find in other comparable books (with the exception, of course, of Farrell, 1976; and Giri, 1977, also has a bit on group methods). But there are great differences in

Received January 1984; revised February 1984.
AMS 1980 *subject classifications*. Primary G2H99; secondary G2E99.
Key words and phrases. Group methods, exterior differential forms, zonal polynomials, asymptotic distributions, invariant measures.

their treatments of these group methods. Although Muirhead considers invariance reductions under whatever group is appropriate, when it comes to the use of Haar measure he confines himself to the orthogonal group and the manifolds (Stiefel and Grassmann) on which this group acts transitively. Eaton, on the other hand, treats and uses Haar measure on locally compact groups in general, but does not think that the reader should have to understand how to cope with measures on manifolds that are not Euclidean (such as the orthogonal group and Stiefel manifold). Therefore, he does not introduce exterior differential forms, in contrast to Muirhead. This poses some limitations on the results that can be obtained. For instance, although Eaton shows that an orthogonally invariant measure on positive definite matrices S (such as the Wishart distribution) can be written as a product of Haar measure on the orthogonal group and some measure Q on the characteristic roots of S, the method does not permit obtaining Q explicitly.

Another good illustration of the difference in use of group methods by the two authors is their derivations of the Wishart distribution. Muirhead starts from the Gram-Schmidt decomposition $X = HT$ of the $n \times p$ (observation) matrix X, where H has orthonormal columns and T is upper triangular. Then by use of exterior differential forms, Lebesgue measure on X is shown to be the product of the invariant measure on the Stiefel mainfold of matrices H, and a measure on the matrices T (related to right Haar measure on the upper triangular group, but that is not used). For given distribution of X the distribution of T follows, and from this the distribution of $S \equiv X'X = T'T$. Eaton, on the other hand, considers the mapping $S = \phi(X) \equiv X'X$, introduces the measure $d\mu = | X'X |^{-(1/2)n}(dX)$ on the matrices X (where (dX) is Lebesgue measure), and considers the induced measure $\nu = \mu\phi^{-1}$ on the matrices S. Then if X has density $f(X'X)$ with respect to μ, S has density $f(S)$ with respect to ν. The induced measure ν is found indirectly as follows. The general linear group of $p \times p$ nonsingular matrices A acts both on the matrices X and on the matrices S by $X \to XA'$, $S \to ASA'$, the latter action being transitive. Under this action μ is invariant and ϕ is equivariant, and it follows that ν is invariant. Since invariant measure on a homogeneous space is unique up to a positive multiplicative constant, ν must be given by $d\nu = c \mid S \mid^{-(1/2)(p+1)}(dS)$, where c may be evaluated by integrating a Wishart density.

Thus the difference between the two authors in their use of group methods may be summarized by stating that Muirhead uses exterior differential forms on differentiable manifolds, whereas Eaton uses properties of invariant measures on locally compact groups and on homogenous spaces on which these groups act. This also brings up a matter of notation. It used to be so that in the computation of an exterior differential form involving a matrix X the symbol dX could stand either for a matrix of elements dx_{ij} or for a product of such elements, depending on the context. In Muirhead's book, however, the product of the dx_{ij} is denoted (dX) to distinguish it from the matrix dX (according to Muirhead, credit for the introduction of this symbol goes to Alan T. James). To the best of my knowledge this is the first time such a careful and helpful distinction in notation has been made in print.

Muirhead's choice of multivariate problems treated in his book includes most of the classical ones: inference on mean vectors, including MANOVA, on covariance matrices, correlations (ordinary, multiple, partial, canonical), and others. More unusual is the inclusion of James-Stein estimators and a discussion of admissibility. But what sets the book especially apart from other books at this level is its extensive treatment of zonal polynomials and their application to infinite series expressions for noncentral distributions. This reflects of course the strong influence of A. T. James and A. G. Constantine. In addition, Muirhead works out asymptotic versions of many of the distributions. This represents much of his own research. Altogether, Muirhead has collected in this book an impressive list of results. Zonal polynomials may be defined in various ways and Muirhead follows James (1968) by defining them as eigenfunctions of a differential operator. At first contact the motivation for this definition must seem rather mysterious to most students. They may be interested in a Ph.D. dissertation of Takemura (1982) in which zonal polynomials are defined and treated in a new way that is quite attractive for statisticians.

Eaton's book (an outgrowth of Eaton, 1972) is even more unconventional than Muirhead's and more difficult to summarize. It emphasizes methods (more than specific results), the two main ones being vector space methods and group methods. A substantial part of the book is devoted to the algebraic and analytic prerequisites. Groups are used throughout, both to derive invariant procedures and, through the use of Haar measure, to deal with distributional problems. Vector space approach, also called coordinate-free approach, was introduced by Kruskal (1961) for univariate linear problems. The idea is to define a random vector X as a random element of a finite dimensional vector space V with inner product $(\cdot, \cdot)$, without having to commit oneself to a basis for V. Then the mean of X is a linear functional on V and can therefore be written $(\mu, \cdot)$ for some $\mu \in V$. The covariance Σ of X is a nonnegative definite linear transformation $V \to V$ such that $(x, \Sigma y)$ equals the covariance of (x, X) and (y, X) for all $x, y \in V$. This approach takes more preparation but probably pays off, especially when it comes to covariances of random matrices. Therefore, the vector space approach may be motivated even more in multivariate than in univariate linear models.

Some of Eaton's applications of invariance and group methods have been mentioned earlier in this review. Another one that deserves special mention links invariance with sufficiency and independence (Section 7.6). The bulk of applications of vector space and group methods takes place in Chapter 9, entitled "inferences for means in multivariate linear models." This is not a conventional treatment of the MANOVA problem. The various models are special cases of a general linear model in which the distribution of a random vector X in V is one of a family whose members are characterized by a density f (according to which X may but need not be normal), a mean vector μ known to lie in a given subspace M of V, and a covariance Σ of which it is assumed that $\Sigma(M) = M$. The latter condition guarantees that maximum likelihood estimators coincide with Gauss-Markov and least squares estimators. Different models arise from assuming various special structures for Σ, such as the usual MANOVA model, block diagonal covariance structure, intraclass covariance structure, symmetry models, and complex covariance structure (the latter is given a very nice exposition by Eaton!). A few linear models that are not special cases of the general model described above are also discussed. In addition to inference on means, there is a small amount of material on other topics such as Wishart distribution and correlations, which is of a more conventional form.

With so much to praise in Muirhead's and Eaton's books, it is only fair to mention some minor flaws and omissions. In Muirhead's Chapter 10 on the multivariate linear model, I failed to find any mention, let alone treatment, of simultaneous confidence sets for parametric functions along the lines of Roy and Bose (1953), Mudholkar (1966), and others. Not mentioned by either Muirhead or Eaton is the method of finding the distribution of certain statistics by random transformation of random variables (as distinct from transformation of densities), as in, for instance, Kshirsagar (1959) for the Wishart matrix, Wijsman (1959) for the multiple correlation coefficient, and Katti (1961) for Wilks' lambda.

Probably some of the detail in Eaton's book could safely have been omitted; for instance, much of the standard material in Chapter 1 on vector spaces, details on the noncentral χ^2 distribution in Chapter 3, and the proof of Proposition 5.1. After all, the student to whom this text is directed must have a good mathematical background as evidenced by the amount of topology assumed in Chapter 6 (for instance, on page 208 "quotient topology" is mentioned but not defined). On the other hand, propositions are not always completely stated; i.e., one sometimes has to scan preceding paragraphs for hypotheses assumed but not stated. For instance, in Proposition 6.9 it is not stated that μ is invariant and $\nu = \mu\phi^{-1}$, and in Proposition 7.5 it is not stated that G is compact (this could prove disastrous for a researcher applying this proposition to a noncompact group). Also, the list of notation on pages XV–XVI and the index seem somewhat incomplete. Finally, I expected but did not find something on the GMANOVA problem in Chapter 9 (in spite of the fact that Kariya, 1978, is listed in the bibliography).

Direct verification that a space is a left homogeneous space (called "topological homogeneous" by Nachbin, 1976, Section III 3) is usually difficult since one has to verify that the mapping $g \to gx$, for fixed x, is open. Thus, the statement in Example 6.15 on page 209 of Eaton's book "That $\mathscr{X}$ is a left homogeneous space is easily verified" may be disputed. Fortunately, there is a nice theorem, stated for instance in Bourbaki (1963), page 97, Lemme 2, in which the only additional condition from which the desired result follows is that G be second countable. The generalized inverse A^- of A that is defined on page 87 is in fact the Moore-Penrose generalized inverse, usually denoted A^+. On the other hand, A^- usually denotes the generalized inverse introduced by C. R. Rao (see Rao and Mitra, 1971) which only has to satisfy the equation $AA^-A = A$ (and is not necessarily unique). At closer inspection, it turns out that Propositions 2.16 and 2.17 are valid even for any Σ_{22}^-, and not just for Σ_{22}^+.

In Section 9.6 of Eaton's book there are two examples, in each of which a maximal invariant is claimed and then verified. But this is not the way that maximal invariants are found in practice. It would be a help for most students to use these examples for the illustration of techniques for finding maximal invariants. One such technique is to choose on the orbit of x a point, say $t(x)$, in a unique "canonical" way. Then $t(x)$, or any $1 - 1$ function of it, is a maximal invariant. This is illustrated by Eaton in Example 7.4. Another technique is invariance reduction in steps if the group is composed of several subgroups (Lehmann, 1959, Section 6.2). This is also extremely useful but never stated by Eaton as a general method. He does use it in the proof of Proposition 10.6, but that comes a bit late. Both techniques could be nicely illustrated in Section 9.6.

Muirhead and Eaton have done a very nice and careful job writing their books, and the fruits of their labor will be received warmly by researchers in multivariate analysis. Students too will reap the benefits. Muirhead's book is perhaps best used as a reference work; it contains a large amount of information, not easily digested by reading the book from beginning till end. Eaton's book, on the other hand, can and should be read from cover to cover. It is eminently suited for self-study. Whether either of the books will be highly successful as a text in a graduate course depends on the level of the audience. It should be borne in mind that neither book has any numerical examples or exercises, only exercises of a theoretical nature. For those of us who face mixed audiences of M.S. and Ph.D. students, it will be necessary to supplement these highly theoretical texts with one that concentrates on applications. Fortunately, there are now several very good applied multivariate books to choose from.

Publisher John Wiley and Sons has done an excellent job in making these books physically attractive. I was especially grateful for the listing of proposition numbers in the top margin of pages in Eaton's book. This makes it considerably easier to hunt for a previous proposition when it is referenced later. Why not go a step further and extend the same format to the numbered examples and/or definitions? Both books obviously have been proofread very carefully: I found very few typos.

In summary, Muirhead and Eaton have done the multivariate community a great service by publishing their excellent accounts of the aspects of multivariate analysis they care for most strongly.

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