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BOUNDS ON AREs FOR RESTRICTED CLASSES OF DISTRIBUTIONS DEFINED VIA TAIL-ORDERINGS[1]

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It is shown that large classes of asymptotic relative efficiencies (AREs) are isotonic with respect to various partial orderings on the heaviness of tails of symmetric distributions. The orderings include those of van Zwet (1964), Lawrence (1975), Barlow and Proschan (1975), and a new one that generalizes all three. Characterizations in terms of these orderings are given for many familiar families of distributions with restricted tail and central behavior. By restricting attention to such distributions, finite bounds are obtained for AREs such as that of some robust estimates to the sample mean, which could be unbounded otherwise. Similar results are shown to hold for the approximate Bahadur efficiencies of Kolmogorov-type tests.

1. Introduction. It is well known that in estimating the location parameter of a symmetric distribution F, the sample mean $\bar{X}$ is not efficient compared to some of its nonparametric competitors if the data contains gross error. Specifically if δ is some robust estimator such as the α-trimmed mean $\bar{X}_\alpha$ or the Wilcoxon estimator W, the *asymptotic relative efficiency* (ARE) $e_{\delta,\bar{X}}(F)$ of δ to $\bar{X}$ is unbounded above if F is allowed to have arbitrarily heavy tails. The reason, of course, is that unlike these competitors, the asymptotic variance of $\bar{X}$ is highly sensitive to heavy tails since it is directly proportional to the variance of F. An interesting question that has not received much attention in the literature is how $\sup_F e_{\delta,\bar{X}}(F)$ is affected by limiting the tail behavior of F. We examine this question here for a number of pairs of estimators.

Since the ARE of any δ to $\bar{X}$ is proportional to the variance of F, it is clear that if finite upper bounds are to be obtained, we need to restrict at least to the class of distributions with finite second moments. It is not difficult to see, however, that this class is still too large because the supremum of $e_{\delta,\bar{X}}(F)$ over the family of truncated Cauchy distributions is infinite. This family clearly has moments of all orders.

In fact, since AREs are scale invariant, this example indicates that even restricting attention to distributions with supports on the interval $(-1, 1)$ is not enough. Rescaling the truncated Cauchy distributions so that they have supports in $(-1, 1)$ forces the densities to take large values at their centers. It therefore appears that finite bounds on $e_{\delta,\bar{X}}(F)$ can be obtained only by simultaneously controlling the behavior of the distributions at both the centers as well as the

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tails. The strongly unimodal distributions turn out to be satisfactory. Other, more general classes are introduced in Section 3. They are the symmetric analogues of the failure-rate distributions.

Restricting attention to these classes also makes it possible to obtain finite bounds for AREs of the form $e_{\tilde{X},\delta}(F)$ for large classes of δ, where $\tilde{X}$ is the sample median. It is well known that, in this case, the AREs are unbounded whenever the density of F is unbounded at the center.

It will be seen that the bounds obtained are not only numerically small, but also sharp in most cases. The key to the proofs lies in the twin properties that (a) many AREs are isotonic (or order-preserving) relative to certain partial orderings, and (b) in terms of these orderings, the distribution classes are not heavier-tailed than the double exponential.

In Section 2 we describe the four tail-orderings needed and their basic properties. Three of these are known from earlier authors, while the fourth one is new and generalizes the others.

Section 4 contains the results for location estimates. In Section 5 we apply the same techniques to the corresponding problem of obtaining finite bounds for the approximate Bahadur efficiencies of pairs of Kolmogorov-type two-sample tests for location shifts when the distributions are restricted to be strongly unimodal (but not necesarily symmetric). Sinha and Wieand (1977) studied this problem for completely unrestricted F.

We adopt the following notation in this paper. Distributions and their cumulative distribution functions (cdfs) are denoted by the same upper case letters F, G, etc. and their densities by the respective lower case letters. All cdfs are assumed to be absolutely continuous. The center of symmetry of F is denoted by μ_F and the inverse of F is defined by

$$F^{-1}(x) = \tfrac{1}{2}[\inf\{t\colon F(t) \geq x\} + \sup\{t\colon F(t) \leq x\}].$$

All the orderings appearing in the literature are defined on the class $\mathscr{F}$ of symmetric absolutely continuous F such that F is strictly increasing on $S_F = \{x\colon 0 < F(x) < 1\}$. For the t-ordering to be defined below, we will enlarge its domain of definition to the class $\mathscr{F}^*$ of symmetric absolutely continuous F such that F is strictly increasing in a neighborhood of μ_F. If X and Y are random variables (r.v.'s) with cdfs F and G, and F precedes G according to the ordering "$\lesssim$", we sometimes write $X \lesssim Y$ in place of $F \lesssim G$.

2. Tail-orderings on symmetric distributions.

Three partial orderings on the location-scale families in $\mathscr{F}$ have appeared in the literature. We recall the definitions.

DEFINITION 2.1. Let $G \in \mathscr{F}$. Then

$F <_s G \Leftrightarrow G^{-1}F(x)$ is convex for $x > \mu_F$;

$F <_r G \Leftrightarrow (G^{-1}F(x) - \mu_G)/(x - \mu_F)$ is non-decreasing for $x > \mu_F$;

$F <_{su} G \Leftrightarrow G^{-1}F(x + y) + \mu_G \geq G^{-1}F(x) + G^{-1}F(y)$ for $x, y > \mu_F$.

The s-ordering was introduced by van Zwet (1964). If a r.v. X has a continuous cdf F, $Y = G^{-1}F(x)$ has cdf G and so will have heavier tails if $G^{-1}F(x)$ is convex for $x > \mu_F$. The r-ordering was defined by Lawrence (1975). When $\mu_F = \mu_G = 0$, this reduces to requiring $G^{-1}F(x)$ to be star-shaped on $[0, \infty)$. (Recall that a function $f(x)$ is *star-shaped* on $[0, \infty)$ if $f(ax) \le af(x)$ for all $0 \le a \le 1$ and all $x \ge 0$.) The su-ordering can be found in Barlow and Proschan (1975) as an ordering for positive r.v.'s. We modified it here into an ordering on symmetric cdfs. It says that $G^{-1}F(x) - \mu_G$ is superadditive on (μ_F, ∞). Since star-shapedness and superadditivity are successive generalizations of convexity for functions passing through the origin (cf. Bruckner and Ostrow, 1962), it follows that

$$F <_s G \Rightarrow F <_r G \Rightarrow F <_{su} G. \tag{2.1}$$

We note that in Definition 2.1, F need not belong to $\mathscr{F}$ for the definitions to make sense. However the s-ordering will not be a true partial ordering on the location-scale families unless the distributions are restricted to be in $\mathscr{F}$.

Since every cdf with a unimodal density is convex-concave (i.e. convex for $x < \mu_F$ and concave for $x > \mu_F$), it is obvious that the uniform density s-precedes any unimodal density. In fact, van Zwet (1964) proved that

$$\begin{aligned} &U\text{-shaped density} <_s \text{Uniform} <_s \text{Normal} \\ &\qquad\qquad <_s \text{Logistic} <_s \text{Double exponential}. \end{aligned} \tag{2.2}$$

Except for the first of this ordering, all the rest correspond to our intuitive feeling of heavy-tailedness. It is easily verified that the reason for the first part of (2.2) is due to the fact that the U-shaped density is convex at its median (and hence is less peaked than the uniform density). This provides one example of the property of the s-ordering that it orders peakedness as well as tailedness. Another illustration is the result (Latta, 1979): Double exponential $\not<_s$ Cauchy. Here, although the Cauchy is quite obviously heavier-tailed, the double exponential is more peaked since it has a cusp at the median. These examples are partly explained by the fact that the s-ordering preserves kurtosis, a measure which is known to be influenced by both peakedness and tailedness (c.f. Kaplansky, 1945, Chissom, 1970, and Darlington, 1970).

One property, related to truncation, might be desirable of any tail-ordering. Given a cdf G with symmetric unimodal density g, suppose we truncate g symmetrically about its center and rescale the result to yield a density f with cdf F. Since F is essentially G with the latter's tails removed, it seems desirable that a tail-ordering should judge F to be lighter-tailed than G. The following example shows that this is not necessarily true of the su-ordering and hence also of the s- and r-orderings.

EXAMPLE 2.1. Let $c, \varepsilon > 0$, $c < d < \frac{1}{2}$ and G be a cdf with symmetric unimodal density g defined on $(0, \infty)$ by

$$g(x) = \begin{cases} 1 + \varepsilon - \varepsilon x/c, & 0 \le x \le c \\ 1, & c < x \le d \\ \text{continuous, decreasing, for } x \ge d. & \end{cases}$$

Let F be obtained from G by truncating g at $\pm c$ and rescaling to have support in $(-\frac{1}{2}, \frac{1}{2})$. Then

$$F(x) = \tfrac{1}{2} + 2x(1 + \varepsilon - \varepsilon x)(2 + \varepsilon)^{-1}, \quad 0 \leq x \leq \tfrac{1}{2}.$$

If d is sufficiently close to ½, there is $0 < \gamma < \frac{1}{2}$ such that

$$G^{-1}F(\tfrac{1}{2} - \gamma) = (1 - 2\gamma)(1 + \tfrac{1}{2}\,\varepsilon + \varepsilon\gamma)(2 + \varepsilon)^{-1} - \tfrac{1}{2}\,\varepsilon c$$

and

$$G^{-1}F(\tfrac{1}{4} - \tfrac{1}{2}\gamma) = (\tfrac{1}{2} - \gamma)(1 + \tfrac{3}{4}\,\varepsilon + \tfrac{1}{2}\,\varepsilon\gamma)(2 + \varepsilon)^{-1} - \tfrac{1}{2}\,\varepsilon c.$$

It is easy to see that for c sufficiently small, we have

$$G^{-1}F(\tfrac{1}{2} - \gamma) < 2G^{-1}F(\tfrac{1}{4} - \tfrac{1}{2}\gamma)$$

thus proving that $G^{-1}F$ is not superadditive on $(0, \frac{1}{2})$. Hence $F \nless_{\text{su}} G$.

Another operation which does not preserve these orderings is scale-mixing. The interested reader is referred to Loh (1982) for details.

The reason that the s, r, and su-orderings are not preserved by truncation is that their definitions are too stringent. A weaker ordering will hopefully take care of this. Our choice for such an ordering is motivated by the following. Suppose we have two symmetric cdfs F, G centered at the origin and such that $F(x) \geq G(x)$ for $x > 0$. Because G puts more mass away from 0 than F, one would like to say that G have heavier tails than F. (Birnbaum, 1948, called G "less peaked" than F, although here we are using "peakedness" to mean "sharpness" of the density at the center.) This ordering is clearly not scale-free since, starting with any symmetric F, we can always produce such a G by making G a rescaled version of F. Some matching of the scale parameters is thus necessary. This may be achieved in many ways, but as we are interested in an ordering for tail behavior, the most natural course seems to be to "match" the centers of F and G. Since the symmetry condition already makes the values of the cdfs agree at 0, we choose to match their first derivatives, i.e., given any two location-scale families of symmetric cdfs, we compare their relative tail behavior by selecting a member from each family such that $\mu_F = \mu_G = 0$, and $F'(0) = G'(0)$. Then if $F(x) \geq G(x)$ for $x > 0$, we will say that F has lighter tails than G. This is the essence of the following definition. Recall the definition of $\mathscr{F}^*$ is Section 1.

DEFINITION 2.2. Let $G \in \mathscr{F}^*$. We say that $F <_t G$ if and only if the limit

$$L = \lim_{y \to \mu_F}(G^{-1}F(y) - \mu_G)/(y - \mu_F)$$

exists and

$$(G^{-1}F(x) - \mu_G)/(x - \mu_F) \geq L, \quad x \neq \mu_F. \tag{2.3}$$

The limit L has been called a subgradient (see e.g. Marshall and Olkin, 1979). In the case that $F(x)$ and $G(x)$ are both continuously differentiable, $\mu_F = \mu_G = 0$, and $f(0) = g(0)$, then $L = 1$ and (2.3) reduces to $F(x) \geq G(x)$, $x > 0$. The relationship between the t-ordering and the other orderings is given in the next theorem.

THEOREM 2.1.

$$F <_s G \Rightarrow F <_r G \Rightarrow F <_{su} G \Rightarrow F <_t G.$$

PROOF. Only the last implication needs to be proved. However, it is a well-known fact (see e.g. Hille and Phillips, 1957) that if $G^{-1}F(x) - \mu_G$ is superadditive for $x > \mu_F$, then L exists and (2.3) holds. Hence $F <_{su} G \Rightarrow F <_t G$.

We show next that the t-ordering is indeed a partial order on the location-scale families in $\mathscr{F}^*$. The proof of the following lemma is easy and is omitted.

LEMMA 2.1. *The condition* (2.3) *is location and scale-free.*

THEOREM 2.2. *The relation "$<_t$" is a partial order on the equivalence classes of location-scale families in $\mathscr{F}^*$.*

PROOF. In view of Lemma 2.1, we may assume that all cdfs are centerd at 0. To show antisymmetry, suppose that $F <_t G$ and $G <_t F$. We need to show that there is τ such that $F(x) = G(\tau x)$. Define $\tau = \lim_{y\to 0} y^{-1}G^{-1}F(y) = \lim_{z\to 0}(z^{-1}F^{-1}G(z))^{-1}$. The hypotheses imply that $0 < \tau < \infty$. Since $F <_t G$ implies that $F(x) \geq G(\tau x)$ for $x > 0$, while $G <_t F$ implies the reverse inequality, we obtain $F(x) = G(\tau x)$ for all x. The proof of transitivity is straightforward.

It was noted earlier that the double exponential does not s-precede the Cauchy. The same turns out to be true with the t-ordering. To see this, choose a member from each family so that they are both centered at 0 and their densities are equal at 0. Then L in Definition 2.2 is 1 but, because the double exponential density has a cusp at 0 while the Cauchy has more mass far away from 0, the two cdfs cross each other at some point in $(0, \infty)$, thereby violating (2.3). It is clear that this phenomenon is not restricted to the Cauchy and double exponential, but holds more generally with any pair F and G such that $f(x)$ has a cusp at μ_F while $g'(\mu_G) = 0 < g(\mu_G)$, and $g(x)/f(x) \to \infty$ as $|x| \to \infty$.

Like the s-ordering, the t-ordering therefore also takes into consideration the peakedness as well as the tailedness of a distribution. Sometimes when one distribution is highly peaked, the t-ordering can even forget about tail behavior in its pursuit of peakedness. Such instances occur when F and G are such that $f(\mu_F) < \infty$ and $g(\mu_G) = \infty$. Then L in Definition 2.2 is 0 and (2.3) is immediate. This yields $F <_t G$ regardless of the tails of either distribution. However, if we restrict attention to densities which are bounded from 0 and ∞ at their centers, these difficulties largely disappear, as (2.3) then genuinely compares tail probabilities. The following theorem gives conditions for a truncated distribution to t-precede its parent.

THEOREM 2.3. *Suppose the density $g(x)$ of G is symmetric, unimodal, and satisfies*

(2.4) $$\lim_{\varepsilon\to 0} gG^{-1}(\tfrac{1}{2} + \alpha\varepsilon)/gG^{-1}(\tfrac{1}{2} + \varepsilon) = 1 \quad \textit{for all} \quad 0 < \alpha < 1.$$

Let F be obtained from G by symmetric truncation. Then $F <_t G$.

PROOF. We assume without loss of generality that G is centered at 0. Let G be truncated at $\pm y$ and define $F(x) = \alpha^{-1}(G(x) - G(-y))$, $-y \le x \le y$, where $\alpha = 2G(y) - 1$. The unimodality of G implies that $F(\alpha x) \ge G(x)$, $x > 0$. Therefore we have $x^{-1}G^{-1}F(x) \ge \alpha^{-1}$, $x > 0$. Using L'Hospital's rule and (2.4), it can be seen that

$$\alpha^{-1} = \lim_{x \to 0} x^{-1}G^{-1}F(x).$$

The theorem follows.

We note that (2.4) holds if $g(x)$ is bounded. Therefore the t-ordering applies in Example 2.1.

The following results are shared by all the orderings. Here we will use "$\lesssim$" to denote any of the s, r, su, and t-orderings if a result holds for all of them.

THEOREM 2.4. *Let F be a symmetric* cdf *and define F_α, $0 < \alpha < 1/2$, to be the* cdf *obtained from F by truncation:*

$$F_\alpha(x) = (1 - 2\alpha)^{-1}(F(x) - 1/2) + 1/2.$$

Then $F \lesssim G \Rightarrow F_\alpha \lesssim G_\alpha$.

PROOF. Since the orderings are scale-free, we may assume that F and G possess the same α-quantiles. Then it is easy to check that $G_\alpha^{-1}F_\alpha = G^{-1}F$ from which the theorem follows.

The following example shows that convolutions are not preserved.

EXAMPLE 2.2. Let F, G be the uniform and normal distributions respectively. Then $F <_s G$. Let $F * F$ denote the two-fold convolution of F. Since $F * F$ has a triangular density and $G * G$ is normal, it may be seen that $F * F \not<_t G * G$. The reason is the same as for $DE \not<_t$ Cauchy, which is that $F * F$ is more peaked than $G * G$. We conclude that for all four orderings, $F \lesssim G$ does not necessarily imply $F * F \lesssim G * G$.

Given two samples $\{X_1, \cdots, X_n\}$, $\{Y_1, \cdots, Y_n\}$ of size n from F and G respectively such that $F \lesssim G$, this example shows that we cannot expect $\bar{X}_n \lesssim \bar{Y}_n$, where $\bar{X}_n$ is the mean of the X-sample. A natural question to ask then is whether an ordering exists for the cdfs of the two sample medians. It is clear from the above example too that the answer is no for $n = 2$. It turns out that for odd n, the answer is yes.

THEOREM 2.5. *Let F_n, G_n be the* cdfs *of the medians from samples of size n from F, G respectively. If n is odd and $F \lesssim G$, we have $F_n \lesssim G_n$.*

PROOF. The result is due to the fact that $G_n^{-1}F_n = G^{-1}F$ (cf. Barlow and Proschan, 1975, page 108).

We mentioned earlier that van Zwet's s-ordering preserves kurtosis. Lawrence

(1975) showed that the same holds for the r-ordering. It can be verified however that this is not true of the t-ordering (Loh, 1982).

3. Families with restricted tails. Ibragimov (1956) defined a strongly unimodal distribution as one whose convolution with any unimodal distribution is unimodal, and proved that such a distribution is either degenerate or possesses a density that is absolutely continuous and log-concave on its support. Strongly unimodal distributions are unimodal. In this and the next section we will be interested only in symmetric distributions, especially the family $\mathscr{F}$(SU) of symmetric strongly unimodal distributions. The uniform, normal, logistic and double exponential are prominent members of this family. Larger families which contain $\mathscr{F}$(SU) may be defined via the failure rate distributions used in reliability theory. We recall here some of the definitions.

DEFINITION 3.1. *Let $H(t)$ be a* cdf *with $H(0) = 0$ and write $\bar{H}(t) = 1 - H(t)$. Then*
(i) *H is* IFR *if $\bar{H}(t+s)/\bar{H}(t)$ is nonincreasing in $t > 0$ for each $s \ge 0$.*
(ii) *H is* IFRA *if $t^{-1}\log \bar{H}(t)$ is nonincreasing in $t > 0$.*
(iii) *H is* NBU *if $\bar{H}(t+s) \le \bar{H}(t)\bar{H}(s)$ for $s, t \ge 0$.*
(iv) *H is* NBUE *if $\int_t^\infty \bar{H}(s)\,ds \le \mu\bar{H}(t)$, $t \ge 0$, where μ is the mean of H.*

In this definition we have restricted attention to cdfs on $[0, \infty)$ which are continuous at 0. It can be seen however that the only H which is discontinuous at 0 and satisfies any of the conditions (i) through (iv) above, is the point mass at 0.

The connection between the strongly unimodal distributions and the reliability distributions lies in the fact that for any cdf F, $1 - F(x)$ is log-concave whenever its density $f(x)$ is log-concave (see e.g. Marshall and Olkin, 1979). This implies that the *failure rate* $f(x)/(1 - F(x))$ of every strongly unimodal distribution is non-decreasing, and hence every $F \in \mathscr{F}$(SU) is IFR.

For any symmetric cdf F with center of symmetry μ_F, we write $\tilde{F}(x) = 2F(x + \mu_F) - 1$, $x \ge 0$. Then the following classes of symmetric distributions may be defined:

$$\begin{aligned} \mathscr{F}(\text{IFR}) &= \{F\colon \tilde{F} \text{ is IFR}\} \\ \mathscr{F}(\text{IFRA}) &= \{F\colon \tilde{F} \text{ is IFRA}\} \\ \mathscr{F}(\text{NBU}) &= \{F\colon \tilde{F} \text{ is NBU}\} \\ \mathscr{F}(\text{UIFR}) &= \{F\colon F \text{ is unimodal and } \tilde{F} \text{ is IFR}\}. \end{aligned} \tag{3.1}$$

The well-known properties of life distributions imply that

$$\mathscr{F}(\text{SU}) \subset \mathscr{F}(\text{UIFR}) \subset \mathscr{F}(\text{IFR}) \subset \mathscr{F}(\text{IFRA}) \subset \mathscr{F}(\text{NBU}). \tag{3.2}$$

Each of these classes properly contains the one preceding it.

The following theorem shows that in terms of the orderings, the distributions in (3.2) are not heavier-tailed than the double exponential (DE) distribution.

THEOREM 3.1. *Let F be symmetric. Then*

(i) $$F \in \mathscr{F}(\text{IFRA}) \Leftrightarrow F <_r \text{DE},$$

(ii) $$F \in \mathscr{F}(\text{NBU}) \Leftrightarrow F <_{\text{su}} \text{DE}.$$

If F is in addition continuous,

(iii) $$F \in \mathscr{F}(\text{IFR}) \Leftrightarrow F <_s \text{DE},$$

and

(iv) $$F \in \mathscr{F}(\text{UIFR}) \Leftrightarrow \text{Uniform} <_s F <_s \text{DE}.$$

Hence

(v) $$F \in \mathscr{F}(\text{SU}) \Leftrightarrow \text{Uniform} <_s F <_s \text{DE}.$$

PROOF. The proof is similar to the one given in Barlow and Proschan (1975) for orderings on cdfs of positive r.v.'s.

This result associates each of the s, r, and su-orderings with a class of failure rate distributions. It turns out that this can also be done for the t-ordering.

DEFINITION 3.2. We call a cdf H with $H(0) = 0$ NBAFR (for "new-better-than-average-failure-rate") if

(3.3) $$t^{-1}\log \bar{H}(t) \leq \lim_{s\downarrow 0} s^{-1}\log \bar{H}(s), \quad t > 0.$$

If H is continuously differentiable in $[0, \varepsilon)$ for some $\varepsilon > 0$ and the failure rate $r(t) = h(t)/\bar{H}(t)$ exists for all t, (3.3) is equivalent to

(3.4) $$r(0) \leq t^{-1} \int_0^t r(s)\, ds, \quad t > 0,$$

thus accounting for the name NBAFR. As in (3.1), we can define

(3.5) $$\mathscr{F}(\text{NBAFR}) = \{F: \tilde{F} \text{ is NBAFR}\}$$
$$\mathscr{F}(\text{UNBAFR}) = \{F: F \text{ is unimodal and } \tilde{F} \text{ is NBAFR}\}.$$

With these definitions we have

THEOREM 3.2. *Let F be symmetric. Then*

(i) $$F \in \mathscr{F}(\text{NBAFR}) \Leftrightarrow F <_t \text{DE},$$

(ii) $$F \in \mathscr{F}(\text{UNBAFR}) \Leftrightarrow \text{Uniform} <_s F <_t \text{DE}.$$

PROOF. The proof is similar to that for Theorem 3.1. The s-ordering appears in part of (ii) because F is unimodal if and only if uniform $<_s F$.

Because the t-ordering is more general than the su-ordering, two conclusions are immediate from this theorem. They are that $\mathscr{F}(\text{NBU}) \subset \mathscr{F}(\text{NBAFR})$ and

{NBU} ⊂ {NBAFR}. It can be shown that although the NBAFR and NBUE classes both contain the NBU, neither contains the other.

4. Location estimates. We are now ready to obtain bounds for the AREs mentioned in the introduction. In this section all distributions F are assumed to be symmetric and absolutely continuous. Some formulas for the AREs are recalled in (4.1)–(4.6) below. The additional conditions on F required for their validity may be found in Lehmann (1983) and the references therein. These conditions, however, are satisfied by the uniform and double exponential, the only distributions for which we need to explicitly compute AREs here. The density and variance of F will always be written as $f(x)$ and σ_F^2 respectively, and Φ and ϕ are reserved for the standard normal cdf and density. In the formulas the distributions are taken to be centered at 0.

$$(4.1)\qquad e_{W,N}(F) = 12 \left\{ \int f^2(x)\,dx \right\}^2 \left\{ \int [f^2(x)/\phi\Phi^{-1}F(x)]\,dx \right\}^{-2};$$

$$(4.2)\qquad e_{\bar{X}_\alpha,\bar{X}}(F) = \sigma_F^2\sigma_\alpha^{-2}(F), \quad 0 < \alpha < 1/2;$$

$$(4.3)\qquad e_{\tilde{X},\bar{X}_\alpha}(F) = 4f^2(0)\sigma_\alpha^2(F), \quad 0 < \alpha < 1/2;$$

$$(4.4)\qquad e_{\tilde{X},W}(F) = 1/3\, f^2(0) \left\{ \int f^2(x)\,dx \right\}^{-2};$$

$$(4.5)\qquad e_{W,\bar{X}}(F) = 12\,\sigma_F^2 \left\{ \int f^2(x)\,dx \right\}^2;$$

where $\sigma_\alpha^2(F)$ is the asymptotic variance of $\bar{X}_\alpha$ given by

$$(4.6)\qquad \sigma_\alpha^2(F) = 2(1-2\alpha)^{-2} \left\{ \int_0^{F^{-1}(1-\alpha)} t^2 f(t)\,dt + \alpha[F^{-1}(1-\alpha)]^2 \right\}$$

and N is the normal scores estimator.

The results of this section are displayed in Table 4.1 where we also show the corresponding bounds for unrestricted F for comparison. Some of the latter bounds may be found in the papers of Hodges and Lehmann (1956, 1961) and Bickel (1965) while the rest are easy to deduce from the preceding formulas.

We derive the bounds for the restricted classes in Table 4.1 in the order they appear. The first one is simple because of the work of van Zwet (1964).

THEOREM 4.1. *Let* $G = \mathscr{F}(\text{SU})$ *or* $\mathscr{F}(\text{IFR})$. *Then*

$$(4.7)\qquad \inf_{\mathscr{G}} e_{W,N}(F) = e_{W,N}(\text{Uniform}) = 0,$$

$$(4.8)\qquad \sup_{\mathscr{G}} e_{W,N}(F) = e_{W,N}(\text{DE}) = 3\pi/8.$$

PROOF. Since AREs are nonnegative and the uniform distribution is strongly unimodal, the first part of the theorem is a matter of verification (Hodges and Lehmann, 1961, first made this observation.) For the supremum, we use the fact,

TABLE 4.1
Bounds for AREs

ARE	Class	Infimum	Supremum
$e_{W,N}(F)$	$\mathscr{F}$(SU)	0	$3\pi/8 \simeq 1.18$
	$\mathscr{F}$(IFR)	0	$3\pi/8$
	unrestricted	0	$6/\pi \simeq 1.91$
$e_{\bar{X}_\alpha,\bar{X}}(F)$	$\mathscr{F}$(SU)	$(1+4\alpha)^{-1}$	$1 < k(\alpha)^* < 2$
	$\mathscr{F}$(IFRA)	$(1-2\alpha)^2$	$k(\alpha)$
	unrestricted	$(1-2\alpha)^2$	∞
$e_{\bar{X},\bar{X}_\alpha}(F)$	$\mathscr{F}$(SU)	$(1+4\alpha)/3$	$1 < 2k^{-1}(\alpha) < 2$
	$\mathscr{F}$(NBAFR)	0	$2k^{-1}(\alpha)$
	unrestricted	0	∞
$e_{\bar{X},W}(F)$	$\mathscr{F}$(SU)	1/3	4/3
	$\mathscr{F}$(UNBAFR)	1/3	4/3
	unrestricted	0	∞
$e_{\bar{X},N}(F)$	$\mathscr{F}$(SU)	0	$\pi/2 \simeq 1.57$
	$\mathscr{F}$(UIFR)	0	$\pi/2$
	unrestricted	0	∞
$e_{W,\bar{X}}(F)$	$\mathscr{F}$(SU)	0.864	< 3
	$\mathscr{F}$(UNBAFR)	0.864	≤ 6
	$\mathscr{F}$(NBAFR)	0.864	∞
	unrestricted	0.864	∞

* See (4.10).

proved by van Zwet (1964), that the ARE of W to N is order-preserving with respect to the s-ordering, i.e.

$$F <_s G \Rightarrow e_{W,N}(F) \leq e_{W,N}(G). \tag{4.9}$$

Together with Theorem 3.1, this completes the proof.

The next theorem is equally simple to prove using a result of Bickel and Lehmann (1975).

THEOREM 4.2. *Let* $\mathscr{G} = \mathscr{F}$(SU) *or* $\mathscr{F}$(IFRA), *and* $0 < \alpha < ½$. *Then*

$$\sup_{\mathscr{G}} e_{\bar{X}_\alpha,\bar{X}}(F) = e_{\bar{X}_\alpha,\bar{X}}(\mathrm{DE}) = k(\alpha) < 2$$

where

$$k(\alpha) = (1-2\alpha)^2\{1 - 2\alpha(1-\log(2\alpha))\}^{-1}. \tag{4.10}$$

Also,

$$\inf_{\mathscr{F}(\mathrm{SU})} e_{\bar{X}_\alpha,\bar{X}}(F) = e_{\bar{X}_\alpha,\bar{X}}(\mathrm{Uniform}) = (1+4\alpha)^{-1},$$

$$\inf_{\mathscr{F}(\mathrm{IFRA})} e_{\bar{X}_\alpha,\bar{X}}(F) = (1-2\alpha)^2.$$

PROOF. Bickel and Lehmann (1975) proved that Lawrence's (1975) r-ordering is preserved by the ARE of a pair of trimmed means. More precisely, they showed that if $0 \le \beta < \alpha \le 1/2$, then

$$F <_r G \Rightarrow e_{\bar{X}_\alpha, \bar{X}_\beta}(F) \le e_{\bar{X}_\alpha, \bar{X}_\beta}(G). \tag{4.11}$$

The first three results now follow easily from Theorem 3.1 and Lemma 4.1. The last result follows from Bickel (1965).

LEMMA 4.1. *Let $0 < \alpha < 1/2$ and $k(\alpha)$ be defined in* (4.10). *Then $k(\alpha)$ is strictly increasing in* $(0, 1/2)$ *and* $\lim_{\alpha \to 0} k(\alpha) = 1$, $\lim_{\alpha \to 1/2} k(\alpha) = 2$.

PROOF. Elementary calculus.

To obtain bounds on the next three AREs in Table 4.1, we need some facts on the t-ordering. These are given below culminating in Theorems 4.3 and 4.4.

LEMMA 4.2. *Suppose $\psi(x)$ is symmetric about* 0 *and nondecreasing on* $(0, \infty)$. *Let F and G be symmetric about* 0 *and* $f(0) = g(0) < \infty$. *Then*

$$F <_t G \Rightarrow E_F \psi(X) \le E_G \psi(X).$$

PROOF. It follows from the definition of the t-ordering that the assumptions in the lemma imply that $\tilde{F}(x) \ge \tilde{G}(x)$ for all $x > 0$. The result is now a consequence of a well known property of stochastically ordered distributions.

LEMMA 4.3. *For $i = 1, 2$, let H_i be a distribution with support on* $[0, \infty)$ *such that $h_i(x) \downarrow$ and $h_i(0) < \infty$. Suppose further that H_2 is stochastically larger than H_1. Then*

$$\int h_1^n(x)\, dx \ge \int h_2^n(x)\, dx, \quad n = 1, 2, \cdots. \tag{4.12}$$

PROOF. Fix $j, k \in \{0, 1, 2, \cdots\}$ and let $q(x) = h_1^j(x) h_2^k(x)$, and $\psi(x) = q(0) - q(x)$. Since the latter is nondecreasing we have by stochastic ordering that $E_1 \psi(X) \le E_2 \psi(X)$ which is equivalent to

$$\int h_1^j h_2^{k+1} \le \int h_1^{j+1} h_2^k \quad \text{for all} \quad j, k = 0, 1, 2, \cdots.$$

THEOREM 4.3. *Let δ be any estimator with asymptotic variance given by $\tau_F^2 = E_F \psi(X)$ for some $\psi(x)$ symmetric about* 0 *and nondecreasing on* $(0, \infty)$. *Then*

$$F <_t G \Rightarrow e_{\tilde{X}, \delta}(F) \le e_{\tilde{X}, \delta}(G). \tag{4.13}$$

PROOF. Since $e_{\tilde{X}, \delta}(F) = 4f^2(0)\tau_F^2$ and we may choose $f(0) = g(0)$, the result follows from Lemma 4.2.

Many L- and M-estimators satisfy the conditions of this theorem, including the mean, trimmed mean, and Huber's M-estimator.

COROLLARY 4.1. *Let $\mathscr{G} = \mathscr{F}$(SU) or $\mathscr{F}$(NBAFR), and $k(\alpha)$ be defined in*

(4.10). *Then for* $0 < \alpha < \frac{1}{2}$,

$$\sup_{\mathscr{G}} e_{\tilde{X},\bar{X}_\alpha}(F) = e_{\tilde{X},\bar{X}_\alpha}(\mathrm{DE}) = 2k^{-1}(\alpha) < 2,$$

$$\inf_{\mathscr{F}(\mathrm{SU})} e_{\tilde{X},\bar{X}_\alpha}(F) = e_{\tilde{X},\bar{X}_\alpha}(\mathrm{Uniform}) = (1 + 4\alpha)/3$$

and

$$\inf_{\mathscr{F}(\mathrm{NBAFR})} e_{\tilde{X},\bar{X}_\alpha}(F) = 0.$$

PROOF. The first two results are direct consequences of Theorems 4.3, 3.1 and 3.2 and Lemma 4.1. To see that the infimum over $\mathscr{F}$(NBAFR) is 0, we observe from (4.3) that the ARE vanishes whenever $f(0) = 0$. Such densities are clearly contained in $\mathscr{F}$(NBAFR).

Although the estimator $\delta = W$ does not satisfy the conditions of Theorem 4.3, it satisfies (4.13) for unimodal F. This is proved in the next theorem.

THEOREM 4.4. *For unimodal F and G,*

$$F <_t G \Rightarrow e_{\tilde{X},W}(F) \le e_{\tilde{X},W}(G).$$

Let $\mathscr{G} = \mathscr{F}$(SU) *or* $\mathscr{F}$(UNBAFR). *Then*

$$\inf_{\mathscr{G}} e_{\tilde{X},W}(F) = \tfrac{1}{3}, \quad \sup_{\mathscr{G}} e_{\tilde{X},W}(F) = \tfrac{4}{3},$$

the bounds being attained at the uniform and DE *respectively.*

PROOF. The proof of the first part parallels that of Theorem 4.3, the only difference being that Lemma 4.3 is used in place of Lemma 4.2. The bounds are inferred from Theorem 3.2 and the fact that no unimodal distribution t-precedes the uniform.

These results make it easy to deduce bounds for the ARE of $\tilde{X}$ to N.

THEOREM 4.5. *For unimodal F, G,*

$$F <_s G \Rightarrow e_{\tilde{X},N}(F) \le e_{\tilde{X},N}(G).$$

Further if $\mathscr{G} = \mathscr{F}$(SU) *or* $\mathscr{F}$(UIFR), *then*

$$\inf_{\mathscr{G}} e_{\tilde{X},N}(F) = e_{\tilde{X},N}(\mathrm{Uniform}) = 0,$$

$$\sup_{\mathscr{G}} e_{\tilde{X},N}(F) = e_{\tilde{X},N}(\mathrm{DE}) = \tfrac{1}{2}\pi.$$

PROOF. This follows from the relation $e_{\tilde{X},N} = e_{\tilde{X},W} \cdot e_{W,N}$, (4.9) and Theorem 4.4, recalling that the s-ordering is a special case of the t-ordering.

The AREs considered so far have all been shown to be order-preserving relative to at least one of the orderings. The ARE of W to $\bar{X}$, in contrast, is not. To see this, note that the uniform distribution s-precedes any distribution with a concave density, and recall the Hodges and Lehmann (1956) result which says that the infimum of $e_{W,\bar{X}}(F)$ over the class of all symmetric F is attained by an F with a concave parabolic density. This shows that $e_{W,\bar{X}}$ does not preserve the s-ordering, and hence also the other more general orderings.

THEOREM 4.6.

$$\inf_{\mathscr{F}(\mathrm{SU})} e_{W,\bar{X}}(F) = 0.864, \quad \sup_{\mathscr{F}(\mathrm{SU})} e_{W,\bar{X}}(F) < 3$$

$$\sup_{\mathscr{F}(\mathrm{UNBAFR})} e_{W,\bar{X}}(F) \le 6, \quad \sup_{\mathscr{F}(\mathrm{IFR})} e_{W,\bar{X}}(F) = \infty.$$

PROOF. The infimum of 0.864 follows from the above mentioned result of Hodges and Lehmann (1956) since any distribution with a concave density is log-concave and hence strongly unimodal. To get upper bounds, we use the relation

$$e_{W,\bar{X}}(F) = 1.5 e_{\tilde{X},\bar{X}}(F * F) \tag{4.14}$$

where $F * F$ is the two-fold convolution of F. We consider the classes separately.

(a) <u>$F \in \mathscr{F}(\mathrm{SU})$.</u> The definition of strong unimodality implies that $F * F \in \mathscr{F}(\mathrm{SU})$. It follows from Corollary 4.1 (letting $\alpha \to 0$) that the RHS of (4.14) is ≤ 3, with the value 3 attained if and only if there is $F \in \mathscr{F}(\mathrm{SU})$ such that $F * F = \mathrm{DE}$. The only F satisfying the latter condition is the cdf of the product of two independent normals (Nyquist, Rice and Riordan, 1954), and its density is a Bessel function that is unbounded at the origin. Since every strongly unimodal distribution that is nondegenerate has an absolutely continuous density in its support, this F cannot belong to $\mathscr{F}(\mathrm{SU})$. We conclude that there does not exist an $F \in \mathscr{F}(\mathrm{SU})$ such that the LHS of (4.14) is 3. In fact, since the set of strongly unimodal distributions is closed under weak limits (Ibragimov, 1956), the sup of the LHS of (4.14) over $\mathscr{F}(\mathrm{SU})$ is strictly less than 3. The exact value of this sup is not known.

(b) <u>$F \in \mathscr{F}(\mathrm{IFR})$.</u> To see that the sup is infinite over this class, recall first that any distribution with a U-shaped density belongs to $\mathscr{F}(\mathrm{IFR})$. Now pick one such that $f(0) = 0$. Moving the two halves of this density away from each other preserves its membership in $\mathscr{F}(\mathrm{IFR})$ (since it remains U-shaped), but increases the ARE. The latter fact may be seen from (4.5) since this operation increases σ_F^2 but keeps the integral constant. Therefore we have a sequence $F_n \in \mathscr{F}(\mathrm{IFR})$ such that $e_{W,\bar{X}}(F_n) \to \infty$.

(c) <u>$F \in \mathscr{F}(\mathrm{UNBAFR})$.</u> The upper bound for such F follows from the inequalities

$$e_{W,\bar{X}}(F) = 12\sigma_F^2\left(\int f^2\right)^2 \le 12\sigma_F^2 f^2(0) = 3e_{\tilde{X},\bar{X}}(F) \le 6,$$

where the last inequality comes from Corollary 4.1. This completes the proof of the theorem.

5. Kolmogorov-type tests. In this section we obtain certain efficiency bounds for pairs of Kolmogorov-type two-sample tests for location shifts. The test statistics are denoted by

K: Kolmogorov-Smirnov	W: Wilcoxon
V: Cramer-von Mises	T: Terry-Fisher-Yates.

Wieand (1976) found expressions for the approximate Bahadur efficiency (or limiting Pitman efficiency) of pairs of these tests. They turn out to be very similar to the expressions for the AREs considered in the preceding section. Bounds for these Bahadur efficiencies are given in Sinha and Wieand (1977) for distributions F subject to the conditions:

(SW) $F(x)$ is strictly increasing on $(-\infty, \infty)$, and $f(x) = F'(x)$ has two bounded derivatives.

Here we obtain the corresponding bounds for (i) the subclass $\mathscr{G}$ where F is additionally strongly unimodal (but not necessarily symmetric), and (ii) the slightly smaller

$$\mathscr{G}^* = \{F \in \mathscr{G}\colon \text{median}(F) = \text{mode}(F)\}.$$

This latter class of course contains the subclass $\mathscr{G}^{**}$ of symmetric strongly unimodal distributions. It will be apparent in the proofs that the same bounds are obtained whether we use $\mathscr{G}^*$ or $\mathscr{G}^{**}$. We quote the following formulas for approximate Bahadur efficiency from Wieand (1976).

$$e_{K,V}(F) = 4\pi^{-2}\sup f^2(x) \Big/ \int f^3(x)\,dx \tag{5.1}$$

$$e_{K,W}(F) = \frac{1}{3}\sup f^2(x)\left\{\int f^2(x)\,dx\right\}^{-2} \tag{5.2}$$

$$e_{W,T}(F) = 12\left\{\int f^2(x)\,dx\right\}^2\left\{\int \Phi^{-1\prime}(F(x))f^2(x)\,dx\right\}^{-2}. \tag{5.3}$$

As in Section 4, we will show that the bounds on these formulas are attained at the uniform and double exponential. The difficulty that these distributions do not satisfy conditions (SW) is overcome the usual way by embedding them in the larger family of densities

$$f(x, b) \propto \exp\{-\tfrac{1}{2}\,|x|^{2/(1+b)}\}, \quad -1 < b < 1, \tag{5.4}$$

and taking limits as $b \to -1$ and $+1$ respectively. Clearly these densities are strongly unimodal and satisfy (SW). The results are displayed in Table 5.1.

The method of proof consists in identifying the formulas in this section with those of the preceding one and then using the bounds derived there. Because symmetry is not assumed here, some preliminary work is necessary in each case before this program can be carried out.

TABLE 5.1
Bounds for approximate Bahadur efficiencies

	Class	Infimum	Supremum
$e_{K,V}$	$\mathscr{G}$	$4\pi^{-2} \simeq 0.405$	$12\pi^{-2} \simeq 1.22$
	unrestricted†	$4\pi^{-2}$	∞
$e_{K,W}$	$\mathscr{G}$	1/3	4/3
	unrestricted†	1/3	∞
$e_{W,T}$	$\mathscr{G}^*$	0	$3\pi/8 \simeq 1.18$
	unrestricted†	0	$6/\pi \simeq 1.91$
$e_{K,T}$	$\mathscr{G}^*$	0	$\pi/2 \simeq 1.57$
	unrestricted†	0	∞

† subject to conditions (SW).

THEOREM 5.1. *Let $F \in \mathscr{G}$. Then*

$$\inf e_{K,V}(F) = 4\pi^{-2}, \quad \sup e_{K,V}(F) = 12\pi^{-2}.$$

PROOF. The inf follows easily from Sinha and Wieand (1977). For the sup, let $F \in \mathscr{G}$ and assume without loss of generality that the mode of F is at the origin and $f(0) = 1$. Let $d = F(0)$ and define the densities

$$\begin{aligned} f_1(x) &= d^{-1}f(x), & x < 0 \\ f_2(x) &= (1-d)^{-1}f(x), & x > 0 \\ g_1(x) &= d^{-1}\exp(x/d), & x < 0 \\ g_2(x) &= (1-d)^{-1}\exp(-x/(1-d)), & x > 0. \end{aligned}$$

If $d = 0$, we leave f_1 and g_1 undefined, and similarly for f_2 and g_2 if $d = 1$. Since f_i, $i = 1, 2$ are strongly unimodal, we can apply Lemma 4.3 to obtain

$$\int f^3(x)\,dx = d^3 \int f_1^3 + (1-d)^3 \int f_2^3 \geq d^3 \int g_1^3 + (1-d)^3 \int g_2^3 = \frac{1}{3}.$$

The result now follows from (5.1).

THEOREM 5.2. *Let $F \in \mathscr{G}$. Then*

$$\inf e_{K,W}(F) = 1/3, \quad \sup e_{K,W}(F) = 4/3.$$

PROOF. The proof is analogous to that for Theorem 5.1 with the lower bound again following from Sinha and Wieand's (1977) results.

THEOREM 5.3. *Let $F \in \mathscr{G}^*$. Then*

$$\inf e_{W,T}(F) = 0, \quad \sup e_{W,T}(F) = 3\pi/8.$$

PROOF. Assume that F has its median at the origin. We "symmetrize" the two halves of its density by defining the densities $f_1(x) = f(-|x|)$, $f_2(x) = f(|x|)$

for $-\infty < x < \infty$. Then the formulas in (4.1) and (5.3) are identical for $F = F_i$, $i = 1, 2$. Therefore Theorem 4.1 yields

$$0 \le e_{W,T}(F_i) \le 3\pi/8, \quad i = 1, 2. \tag{5.5}$$

Writing $\int f^2 = \frac{1}{2}\{\int f_1^2 + \int f_2^2\}$, etc., it is easy to see that (5.5) in fact holds for the original F too. Taking F as in (5.4) and letting $b \to \pm 1$ shows that the bounds are sharp.

THEOREM 5.4. *Let $F \in \mathscr{S}^*$. Then*

$$\inf e_{K,T}(F) = 0, \quad \sup e_{K,T}(F) = \pi/2.$$

PROOF. Follows from the preceding theorems and the relation $e_{K,T} = e_{K,W} \cdot e_{W,T}$.

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