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EDGEWORTH EXPANSIONS FOR THE POWER OF PERMUTATION TESTS

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A randomization model for a two-sample situation with additive treatment effects is considered. Edgeworth expansions for the power of the usual permutation test are derived, under some conditions on the unit errors, from previously obtained expansions under the null hypothesis of no treatment effect. A general error structure is considered and conditions for the validity of the expansions for both conditional and unconditional power are examined. The results are shown to generalise expansions obtained earlier by different methods for the special case of independent and identically distributed random variables.

1. Introduction. Permutation tests arise naturally from two different models. In the randomization model there is presumed to be a physical act of randomization which leads to a permutation distribution. We are interested in the difference in effects of two treatments, assumed to have additive effects, which have been applied at random to N experimental units, m receiving the first and $n = N - m$ the second. Associated with the ith unit is a random variable X_i, the unit error, which gives the observation expected if, say, the second treatment were applied to all units. Then we observe

$$Y_i = X_{R_i} + \theta, \quad i = 1, \cdots, m, \qquad Y_i = X_{R_i}, \quad i = m + 1, \cdots, N, \tag{1.1}$$

where $(R_1, \cdots, R_N)$ is a random permutation of $(1, \cdots, N)$ taking each possible value with probability $(N!)^{-1}$ and θ is the difference in effects of the first and second treatment. In the more usually described model it is assumed that the observations $Y_1, \cdots, Y_m$ and $Y_{m+1}, \cdots, Y_N$ are two samples from populations with distribution functions $F(x - \theta)$ and $F(x)$. If we assume $X_1, \cdots, X_N$ are independent, each with distribution function $F(x)$, then the first model incorporates the second.

Expansions for the power of the permutation test based on the statistic $\sum_{i=1}^{m} Y_i$ have been considered under the assumptions of the second model by Bickel and van Zwet (1978) and a similar treatment has been considered for the one sample case by Albers, Bickel and van Zwet (1976) and Albers (1974). In these papers the authors considered expansions for rank tests and obtained the results on permutation tests as a by-product. However, in the case of permutation tests it is essential to consider more general distributions for the unit errors, allowing a certain degree of dependency among them so as to incorporate more realistic models for them. In fact there is no point in considering randomization models if we make the further assumption that the errors are independent and identically distributed, for then randomization is irrelevant.

The results are obtained here by a different method from that of the forementioned articles, in that we condition on $X_1, \cdots, X_N$ and not on $Z_1, \cdots, Z_N$, the order statistics of $Y_1, \cdots, Y_N$. Thus only results from Robinson (1978) on equally probable permutations need to be used to obtain the power expansions. This simplifies this part of the proof and permits generalisation to models where it is not assumed that $X_1, \cdots, X_N$ are independently and identically distributed. We obtain an approximation to the conditional power of the permutation test under the contiguous alternative when θ is of order $N^{-1/2}$ on a certain set E where some conditions on the X_i hold. Under the assumption that the complement of E

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has probability of order $N^{-3/2}$, an expression for the unconditional power can be obtained and is the same as that given by Bickel and van Zwet (1978) in the case of independent and identically distributed $X_1, \cdots, X_N$. We can also obtain their result that the difference between the powers of the permutation test and Student's t test is of order $N^{-3/2}\log N$, directly. Moreover, the conditional power of the permutation test can be compared with that of the t test actually performed, namely that where tables of t with $N - 2$ degrees of freedom are used to obtain the critical point. A special case of this when $\theta = 0$ gives a comparison of significance levels for these tests.

The results of Section 2 are only of interest if the probability that E contains $X_1, \cdots, X_N$ is large enough. In Section 3, a model for $X_1, \cdots, X_N$ is introduced which is motivated by models proposed for randomization models. Then a generalisation of the methods of Albers (1974) enables us to obtain a bound on the probability of E for this class of random variables. This discussion of E is separated particularly so that these conditions are divorced from the actual expansions. It may be possible to obtain more general classes of random variables for which appropriate bounds on the probability of E can be found. It is also apparent that this approach could be used to generalise the one sample results of Albers (1974) and is of interest in considering the k-sample problem.

2. Edgeworth expansions for power. In the sequel we will use ε, c, C, B as positive generic constants which may vary on each occasion. Let $Y_1, \cdots, Y_N$ be given by (1.1) and let $Z_1, \cdots, Z_N$ be their ordered values. Let $\bar{X} = N^{-1} \sum_{i=1}^N X_i$ and define $\bar{Y}$ and $\bar{Z}$ similarly. Put $p = m/N$, $q = 1 - p$. Let $E(X)$ be the set where

$$\sum_{i=1}^N (X_i - \bar{X})^2 > CN \tag{2.1}$$

$$\sum_{i=1}^N |X_i - \bar{X}|^5 < CN \tag{2.2}$$

$$\sum_{i=1}^N \sin^2\{\psi + \tfrac{1}{2}t(X_i - \bar{X})(pq \sum_{i=1}^N (X_i - \bar{X})^2)^{-1/2}\} > CN \tag{2.3}$$

for all $|\psi| < \frac{1}{2}\pi$, $cN^{1/2} < |t| < CN^{3/2}$, and define $E(Y)$ and $E(Z)$ similarly. Let E be the set where (2.1) to (2.3) hold and in addition

$$\sum_{i=1}^N (X_i - \bar{X})^4 - N^{-1}\{\sum_{i=1}^N (X_i - \bar{X})^2\}^2 > CN \tag{2.4}$$

and

$$P\{E(Y) \mid X\} \geq 1 - BN^{-3/2}. \tag{2.5}$$

Define $T_\theta = \sum_{i=1}^m (Y_i - \bar{Y})/\{pq \sum_{i=1}^N (Y_i - \bar{Y})^2\}^{1/2}$. Let $\xi_\alpha^*(Z)$ be the level α critical point of the permutation test based on T_θ. On $E(Z)$, as in Bickel and van Zwet (1978, page 962) we have, provided

$$\varepsilon < \alpha < 1 - \varepsilon \tag{2.6}$$

and

$$\varepsilon < p < 1 - \varepsilon, \tag{2.7}$$

that

$$|\xi_\alpha^*(Z) - \zeta_\alpha(Z)| \leq CN^{-3/2} \tag{2.8}$$

where

$$\zeta_\alpha(Z) = u_\alpha + (u_\alpha^2 - 1)\beta_3(Z) + u_\alpha/2N + (u_\alpha^3 - 3u_\alpha)\beta_4(Z) - (2u_\alpha^3 - 5u_\alpha)\beta_3^2(Z), \tag{2.9}$$

in which

$$\beta_3(Z) = \tfrac{1}{6}(q - p)(pq)^{-1/2} \sum_{i=1}^N (Z_i - \bar{Z})^3/\{\sum_{i=1}^N (Z_i - \bar{Z})^2\}^{3/2} \tag{2.10}$$

$$\beta_4(Z) = \frac{1-6pq}{24pq}\left[\sum_{i=1}^N (Z_i - \bar{Z})^4 / \{\sum_{i=1}^N (Z_i - \bar{Z})^2\}^2 - 3/N\right] - (4N)^{-1} \tag{2.11}$$

and $u_\alpha = \Phi^{-1}(1-\alpha)$, where Φ is the standard normal distribution function.

We can now obtain an approximation to the conditional power of the permutation test under the contiguous alternative when θ is of order $N^{-1/2}$ on the set E.

THEOREM 1. *Suppose* (2.6), (2.7) *and*

$$0 \le \theta \le CN^{-1/2} \tag{2.12}$$

hold. Then on E,

$$|P\{T_\theta \ge \xi_\alpha^*(Z) \mid X\} - \tilde{\pi}_\theta(X)| \le BN^{-3/2}\log N \tag{2.13}$$

where

$$\begin{aligned}\tilde{\pi}_\theta(X) = 1 &- \Phi(u_\alpha - A_\theta) - A_\theta\phi(u_\alpha - A_\theta)\{\beta_3(X)(2u_\alpha - A_\theta) \\ &+ \beta_4(X)(3u_\alpha^2 - 3u_\alpha A_\theta + A_\theta^2 - 3) + \tfrac{1}{2}N^{-1}(1 + 2u_\alpha^2 - u_\alpha A_\theta) + \tfrac{1}{2}\beta_3^2(X) \\ &\cdot(A_\theta^4 - 5u_\alpha A_\theta^3 - 8A_\theta^2 + 8u_\alpha^2 A_\theta^2 - 4u_\alpha^2 A_\theta + 24u_\alpha A_\theta - 20u_\alpha^2 + 10)\},\end{aligned} \tag{2.14}$$

for

$$A_\theta = \theta N\{(pq)^{-1}\sum_{i=1}^N (X_i - \bar{X})^2\}^{-1/2}. \tag{2.15}$$

PROOF. $\sum_{i=1}^N (Y_i - \bar{Y})^2 = 0$ implies that $\sum_{i=1}^N (X_i - \bar{X})^2 = Npq\theta^2$ so (2.1) and (2.12) ensure that T_θ is well defined on $E(X)$. Now

$$T_\theta = (T_0 + A_\theta)(1 + 2A_\theta N^{-1}T_0 + A_\theta^2 N^{-1})^{-1/2}, \tag{2.16}$$

where T_0 is T_θ at $\theta = 0$. If $0 < A_\theta < N^{1/2}$,

$$x(t) = (t + A_\theta)(1 + 2A_\theta tN^{-1} + A_\theta^2 N^{-1})^{-1/2}$$

is an increasing function of t for $t > -\tfrac{1}{2}NA_\theta^{-1} - \tfrac{1}{2}A_\theta$, since $x'(t) \ge 0$ there and it may be inverted to yield

$$t = x\{1 - A_\theta^2 N^{-1}(1 - x^2N^{-1})\}^{1/2} - A_\theta(1 - x^2N^{-1}). \tag{2.17}$$

Also for $0 < A_\theta < N^{1/2}$, $-\tfrac{1}{2}NA_\theta^{-1} - \tfrac{1}{2}A_\theta < -N^{-1/2}$, so $|t| \le N^{1/2}$ lies in the above domain, since $-\tfrac{1}{2}NA_\theta^{-1} - \tfrac{1}{2}A_\theta$ has a maximum of $-N^{-1/2}$ at $A_\theta = N^{1/2}$.

Now from (2.1) and (2.12), $A_\theta < C < N^{1/2}$ for large enough N and since $T_0 \le N^{1/2}$, from (2.16) and (2.17), we have for all $|v| \le N^{1/2}$,

$$P(T_\theta \le v \mid X) = P(T_0 \le v^* \mid X)$$

where

$$v^* = v\{1 - A_\theta^2 N^{-1}(1 - v^2N^{-1})\}^{1/2} - A_\theta(1 - v^2N^{-1}).$$

So from Robinson (1978), for $|v| \le N^{1/2}$,

$$|P(T_\theta \le v \mid X) - G(v^*, X)| < BN^{-3/2} \tag{2.18}$$

where

$$G(v, X) = \Phi(v) - \phi(v)\{\tfrac{1}{2}N^{-1}v + H_2(v)\beta_3(X) + H_3(v)\beta_4(X) + \tfrac{1}{2}H_5(v)\beta_3^2(X)\}, \tag{2.19}$$

where $\beta_3(X)$, $\beta_4(X)$ are defined as in (2.10) and (2.11) with X replacing Z, $\phi(v) = \Phi'(v)$ and $\phi^{(i)}(v) = (-1)^i H_i(v)\phi(v)$.

Now $A_\theta < C$, so for $|v| < \log N$, it is easy to show that

$$|v^* - (v - A_\theta) - N^{-1}(A_\theta v^2 - \tfrac{1}{2}A_\theta^2 v)| < CN^{-3/2}.$$

So for $|v| < \log N$,

$$|G(v^*, X) - G_\theta(v, X)| < CN^{-3/2} \tag{2.20}$$

where

$$G_\theta(v, X) = G(v - A_\theta, X) + N^{-1}(A_\theta v^2 - \tfrac{1}{2}A_\theta^2 v)\phi(v - A_\theta). \tag{2.21}$$

Also for $|v| > \log N$ and large enough N, $v^* > \frac{1}{2}\log N$, so then also (2.20) is true.

It can be shown, although we omit the proof here since it is relatively straightforward, that if (2.6), (2.7) and (2.12) hold, then on E

$$P\{|\zeta_\alpha(Z) - \zeta_\alpha(X)| < BN^{-3/2}\log N | X\} > 1 - BN^{-3/2}. \tag{2.22}$$

Thus, on E, from the definitions of E and $E(Y)$ and (2.8)

$$|P\{T_\theta \geq \xi_\alpha^*(Z) | X\} - P\{T_\theta \geq \zeta_\alpha(X) + R(Z) | X\}| \leq BN^{-3/2},$$

where $R(Z)$ is a quantity such that on E,

$$P\{|R(Z)| \leq BN^{-3/2}\log N | X\} \geq 1 - BN^{-3/2}.$$

From this, together with (2.18) and (2.20), we have on E,

$$\begin{aligned} P\{T_\theta \geq \xi_\alpha^*(Z) | X\} &= 1 - G_\theta(\zeta_\alpha(X) + O(N^{-3/2}\log N), X) + O(N^{-3/2}) \\ &= \tilde{\pi}_\theta(X) + O(N^{-3/2}\log N), \end{aligned}$$

where $\tilde{\pi}_\theta(X)$ is obtained by a direct substitution of (2.9) into (2.21) and a straightforward expansion. □

The theorem is of interest only if

$$P(E) > 1 - BN^{-3/2}. \tag{2.23}$$

This condition is studied in detail in Section 3. We can also obtain a result on the unconditional power.

COROLLARY 1. *If* (2.23) *holds then*

$$|P\{T_\theta \geq \xi_\alpha^*(Z)\} - E\tilde{\pi}_\theta(X)| < BN^{-3/2}\log N. \tag{2.24}$$

The result for the unconditional power of the permutation test in the case $X_1, \cdots, X_N$ independent, identically distributed, as obtained by Bickel and van Zwet (1978, page 980, equation (6.34)), is readily derived from (2.24).

We also remark that an expression for the power of the two sample Student's test follows via a conditional argument from Theorem 1, since the two sample Student statistic, T_S, and T_θ are related by the identity

$$T_\theta = T_S\{1 + (T_S^2 - 2)/N\}^{-1/2}.$$

In the independent identically distributed case the difference between the powers of Student's test and the permutation test is $O(N^{-3/2}\log N)$.

It is of interest to compare the conditional power of the permutation test with that of the t-test performed in practice where the critical point is taken as the upper 100α percentage point of the t-distribution with $N - 2$ degrees of freedom. If the latter power is denoted by $\pi^*_{S,\theta}(X)$, we have

$$\tilde{\pi}_\theta(X) - \pi^*_{S,\theta}(X) = -(u_\alpha^2 - 1)\beta_3(X) - (u_\alpha^3 - 3u_\alpha)\{\beta_4(X) + \tfrac{1}{4}N^{-1}\}$$
$$- \tfrac{1}{2}\beta_3^2(X)[u_\alpha^5 - 10u_\alpha^3 + 15u_\alpha - (u_\alpha^2 - 1)\{2A_\theta^3 - 6A_\theta^2 u_\alpha + 5A_\theta(u_\alpha^2 - 1)\}].$$

For given observations $y_1, \cdots, y_N$, we could calculate this with $X_i = y_i - \theta$, $i = 1, \cdots, m$, $X_i = y_i$, $i = m + 1, \cdots, N$, but for θ satisfying (2.12), we have from Theorem 1 that it is equivalent, to $O(N^{-3/2} \log N)$, to calculating it replacing X_i with y_i. Further, this gives the difference in the significance levels of the tests when $\theta = 0$ and so $A_\theta = 0$. If $\beta_3(X)$ and $\beta_4(X) + \tfrac{1}{4}N^{-1}$ are small, and so skewness and kurtosis are small, the difference in conditional power is small, but as is expected, with a highly skew or kurtic distribution the errors are non-negligible. It is interesting to note that with equal sample sizes $\beta_3(X) = 0$ and the difference in power is the same, to $O(N^{-3/2}\log N)$, for all θ satisfying (2.12).

3. The probability for E. The results of the previous section have no practical value unless (2.23) can be shown to hold for a wide class of variables $X_1, \cdots, X_N$. The methods of Albers, Bickel and van Zwet (1976) and Bickel and van Zwet (1978), who obtain closely related results for the case of $X_1, \cdots, X_N$ being independent, identically distributed random variables, do not appear to extend to more general random variables. However, in the case $\theta = 0$ in particular, we need to obtain this more general result in order for our approximations to the critical point of the permutation test to be of use. It may be possible to obtain the result for some general class of dependent random variables; however, we propose a particular model for the errors which incorporates some degree of dependency and nonstationarity. This model is motivated by the context of randomized agricultural experiments where the plot error is considered as an independent random error and a "soil" error (see for example Neyman, Iwaszkiewicz and Kolodziejczyk, 1935). Let

$$X_i = V_i + W_i \tag{3.1}$$

where $V_1, \cdots, V_N$ are independent random variables, independent of $W_1, \cdots, W_N$ about whose joint distribution we will make only mild assumptions. We will show that, subject to certain smoothness and moment assumptions on V_i and some "boundedness" condition on the W_i, $P(E)$ is sufficiently large for this class of variables.

Put $EW_i = \mu_i$, $E\,|\,W_i - \mu_i\,|^j = \omega_{j,i}$, $EV_i = 0$ and $E\,|\,V_i\,|^j = \nu_{j,i}$. Let $\bar{\mu} = N^{-1} \sum_{i=1}^N \mu_i$. Consider the following conditions

$$\textstyle\sum_{i=1}^N \omega_{15,i} \le CN, \tag{3.2}$$

$$\textstyle\sum_{i=1}^N \nu_{15,i} \le CN, \tag{3.3}$$

$$\textstyle\sum_{i=1}^N |\,\mu_i - \bar{\mu}\,|^{15} \le CN, \tag{3.4}$$

(3.5) There exist positive constants δ, η, c, C such that for at least δN indices i there is an interval χ_i of length η such that on χ_i, V_i has density f_i and $c < f_i < C$,

$$P(\textstyle\sum_{i=1}^N |\,W_i - \bar{W}|^5 < CN) > 1 - BN^{-3/2}, \tag{3.6}$$

$$P(|\textstyle\sum_{i=1}^N W_i\,| < CN) > 1 - BN^{-3}, \tag{3.7}$$

$$P\{\textstyle\sum_{i=1}^N (W_i - \bar{W})^2 < CN\} > 1 - BN^{-3}. \tag{3.8}$$

A remark concerning the conditions (3.1) to (3.8) is perhaps warranted. Clearly some restraint must be placed on the W_i, and those imposed by these conditions are fairly general. For example, they will be satisfied if the W_i are themselves bounded random variables. Further examples based on simple interaction of neighbouring plots in an agricultural context are not difficult to construct.

We proceed by conditioning on $W_1, \cdots, W_N$, thereby reducing the problem to the case where $X_1, \cdots, X_N$ are independent. The following lemmas establish bounds on the probability that each of the conditions (2.1) to (2.5) hold. The required large probability for the set E for the more general case follows by taking expectations of these results. The proofs of the lemmas are straightforward but technically involved and are omitted. Details can be found in John (1981).

We remark that the condition (3.5) is really only required in the proof of Lemma 2, but is used in the other lemma because it implies certain non-negligibility conditions on the second moments of the V_i, which would otherwise have to be assumed.

LEMMA 1. *Suppose* (3.3) *and* (3.5) *hold. Then on the set where* $\sum_{i=1}^N (W_i - \bar{W})^2 < CN$, $\sum_{i=1}^N |W_i - \bar{W}|^5 < CN$ *and* $|\sum_{i=1}^N W_i| < CN$, *there exist constants* c, C, B *such that*

$$P\{c < N^{-1} \textstyle\sum_{i=1}^N (X_i - \bar{X})^2 < C | W\} \geq 1 - BN^{-4}\{N + \sum_{i=1}^N (W_i - \bar{W})^{12}\} \tag{3.9}$$

$$P(\textstyle\sum_{i=1}^N |X_i| < CN | W) > 1 - BN^{-3} \tag{3.10}$$

$$P(\textstyle\sum_{i=1}^N |X_i - \bar{X}|^5 < CN | W) > 1 - BN^{-5/2}(N + \sum_{i=1}^N |W_i - \bar{W}|^{15}) \tag{3.11}$$

$$\begin{aligned} P[\textstyle\sum_{i=1}^N (X_i - \bar{X})^4 - N^{-1}\{\sum_{i=1}^N (X_i - \bar{X})^2\}^2 > CN | W] \\ > 1 - BN^{-5/2}\{N + \textstyle\sum_{i=1}^N (W_i - \bar{W})^{12}\}. \quad \square \end{aligned} \tag{3.12}$$

LEMMA 2. *Suppose* (2.7), (3.3), (3.5) *hold and that* $|\theta| < K$. *Let* $R_1, \cdots, R_N$ *be a uniform random permutation of* $(1, \cdots, N)$ *independent of* $X_1, \cdots, X_N$. *Let* E_1 *be the set where for all* $|\psi| \leq \pi/2$ *and all* $cN^{1/2} < |t| < CN^{3/2}$,

$$\begin{aligned} &\textstyle\sum_{i=1}^m \sin^2\{\psi + \frac{1}{2}t(X_{R_i} - \bar{X} + q\theta)(pqS_\theta^2)^{-1/2}\} \\ &\qquad + \textstyle\sum_{i=m+1}^N \sin^2\{\psi + \frac{1}{2}t(X_{R_i} - \bar{X} - p\theta)(pqS_\theta^2)^{-1/2}\} > cN, \end{aligned}$$

where $S_\theta^2 = \sum_{i=1}^m (X_{R_i} - \bar{X} + q\theta)^2 + \sum_{i=m+1}^N (X_{R_i} - \bar{X} - p\theta)^2$. *Then on the set where* $\sum_{i=1}^N (W_i - \bar{W})^2 < CN$ *and* $|\sum_{i=1}^N W_i| < CN$ *there exists a constant* B *such that*

$$P(E_1 | W) > 1 - BN^{-4}\{N + \textstyle\sum_{i=1}^N (W_i - \bar{W})^{12}\}. \quad \square \tag{3.13}$$

We now have the following theorem.

THEOREM 2. *If conditions* (3.2) *to* (3.8) *hold and if*

$$|\theta| < K, \tag{3.14}$$

then

$$P(E) \geq 1 - BN^{-3/2}. \tag{3.15}$$

PROOF. Let $\Lambda = \{W: \sum_{i=1}^N |W_i - \bar{W}|^5 < CN\}$, then from Lemma 1, on Λ

$$P(\textstyle\sum_{i=1}^N |X_i - \bar{X}|^5 < CN | W) \geq 1 - BN^{-5/2}\{\sum_{i=1}^N E(|V_i|^{15}) + \sum_{i=1}^N |W_i - \bar{W}|^{15}\}. \tag{3.16}$$

Applying the C_r inequality and using (3.2) and (3.4) we have

$$EI_\Lambda(W) \textstyle\sum_{i=1}^N |W_i - \bar{W}|^{15} \leq CN \tag{3.17}$$

where I_Λ is the characteristic function of the set Λ. So taking the expectation in (3.16)

$$\begin{aligned} &P(\textstyle\sum_{i=1}^N |X_i - \bar{X}|^5 < CN) \\ &\qquad \geq 1 - BN^{-5/2}\{\textstyle\sum_{i=1}^N \nu_{15,i} + EI_\Lambda(W) \sum_{i=1}^N |W_i - \bar{W}|^{15}\} - \{1 - P(\Lambda)\} \geq 1 - BN^{-3/2} \end{aligned}$$

from (3.3), (3.6) and (3.17).

Similarly we can show that the sets on which (2.4) and (2.1) hold have probabilities $1 - O(N^{-3/2})$ and $1 - O(N^{-3})$, respectively, and in the same way we see that the set on which (2.3) holds has probability $1 - O(N^{-3})$.

It remains to show that the set on which (2.5) holds has probability $1 - O(N^{-3/2})$. Now

$$\sum_{i=1}^{N} | Y_i - \bar{Y} |^5 \leq 2^4 \{\sum_{i=1}^{N} | X_i - \bar{X} |^5 + Npq(q^4 + p^4) | \theta |^5\},$$

so since (3.14) holds

$$P_\theta(\sum_{i=1}^{N} | Y_i - \bar{Y} |^5 < CN | X) \geq P_\theta(\sum_{i=1}^{N} | X_i - \bar{X} |^5 < cN | X)$$

where the constant in the left hand inequality is larger than 2^4 times the constant in (2.2). Thus, the set where $P_\theta(\sum_{i=1}^{N} | Y_i - \bar{Y} |^5 < CN | X) > 1 - BN^{-3/2}$ contains the set where $\sum_{i=1}^{N} | X_i - \bar{X} |^5 < cN$. So

$$P\{P_\theta(\sum_{i=1}^{N} | Y_i - Y |^5 < CN | X) > 1 - BN^{-3/2}\} > 1 - B'N^{-3/2}.$$

If $E'(Y)$ is the set where (2.3) holds with Y_i replacing X_i, then again beginning with Lemma 2 and arguing as above we have $P_\theta(E'(Y)^c) \leq BN^{-3}$. Then, using a Markov inequality

$$P[| P_\theta\{E'(Y)^c | X\} - P_\theta\{E'(Y)^c\} | \geq CN^{-3/2}] \leq BN^{-3/2}$$

and it follows that

$$P[P_\theta\{E'(Y) | X\} \geq 1 - CN^{-3/2}] \geq 1 - BN^{-3/2}.$$

Finally, $\sum_{i=1}^{N} (Y_i - \bar{Y})^2 \geq cN$ on the set where a similar inequality holds for the X_i, for θ less than a chosen constant, so

$$P[P_\theta\{\sum_{i=1}^{N} (Y_i - \bar{Y})^2 \geq cN | X\} = 1] \geq 1 - O(N^{-3}). \quad \square$$

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