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MINIMAX CONFIDENCE SETS FOR THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION

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For the problem of estimating a p-variate normal mean, the existence of confidence procedures which dominate the usual one, a sphere centered at the observations, has long been known. However, no explicit procedure has yet been shown to dominate. For $p \geq 4$, we prove that if the usual confidence sphere is recentered at the positive-part James Stein estimator, then the resulting confidence set has uniformly higher coverage probability, and hence is a minimax confidence set. Moreover, the increase in coverage probability can be quite substantial. Numerical evidence is presented to support this claim.

1. Introduction. The problem of improving upon the usual point estimator of a multivariate normal mean has received enormous attention in the literature during the past 15 years. The companion problem, that of set estimation, has received comparatively little attention, however. This is partially due to the increased difficulty of the set estimation problem, and also because many of the techniques developed for point estimation (notably integration by parts) do not readily carry over to the set estimation problem.

If X is one observation from a p-variate normal distribution with mean θ and identity covariance matrix, the confidence set

$$C_X^0 = \{\theta : \| \theta - X \|^2 \leq c^2\}, \tag{1.1}$$

a sphere centered at X, has probability $1 - \alpha$ of covering the true value of θ if c^2 satisfies $P(\chi_p^2 \leq c^2) = 1 - \alpha$. C_X^0 enjoys many optimality properties; for example, it is unbiased and best translation invariant. It is also minimax, which means that among all procedures with coverage probability at least $1 - \alpha$, C_X^0 minimizes the maximum expected volume.

A natural question that arises is whether C_X^0 is a unique minimax set estimator, or do others exist. If so, then since the coverage probability of C_X^0 is constant for all θ, there would be room to increase coverage probability without increasing volume. This question was first posed by Stein (1962), who developed heuristic arguments that show that improved set estimators can be developed. Later, Brown (1966) and Joshi (1967) independently demonstrated the existence of a dominating (minimax) procedure for $p \geq 3$. Joshi proved that the set

$$C^J = \{\theta : \| \theta - \delta^J(X)\|^2 \leq c^2\}, \tag{1.2}$$

where $\delta^J(X) = \{1 - a/(b + X'X)\}X$, has higher coverage probability than C_X^0 if a is sufficiently small and b is sufficiently large. Olshen (1977) simulated the coverage probability of C^J for selected a, b and $|\theta|$. The results indicated that large gains can be achieved. Morris (1977) also simulated coverage probabilities for certain generalized Bayes estimators and again the results were good.

Two other important works are those of Faith (1976) and Berger (1980). Faith derives confidence sets from Bayes credible sets and shows, for $p = 3$ or 5, that these sets have

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smaller volume and higher coverage probability than C_X^0 for all $|\theta|$ except an interval of middle values. Berger also proceeds in a Bayesian fashion, but also considers the posterior covariance matrix, and constructs confidence ellipsoids. These sets are shown to have uniformly smaller volume, and to dominate in coverage probability for sufficiently large $|\theta|$.

Casella (1980) extended the method of Faith and derived exact formulas for confidence sets centered at the James Stein or positive-part James Stein estimator. No analytical results were presented, but the computed coverage probabilities show that substantial improvement is possible. The most recent work on this subject to appear (although the research was done in 1974) is that of Stein (1981). Using a heuristic argument, a variable radius confidence sphere is developed which is conjectured to have confidence coefficient $1 - \alpha$.

Thus far, however, no one has exhibited a procedure which can be proven to dominate C_X^0 for all θ. Our main result is that for $p \geq 4$,

$$C_{\delta^+} = \{\theta : \| \theta - \delta^+(X)\|^2 \leq c^2\}, \tag{1.3}$$

where $\delta^+(X) = (1 - a/X'X)^+X$ ("+" denotes positive part) has, for a specified range of values of a, higher coverage probability than C_X^0 for all θ. Since the volume of C_{δ^+} is the same as that of C_X^0, it follows that C_{δ^+} is minimax set estimator of θ.

It is unfortunate that the dominance result could not be obtained for $p = 3$, but, as will be seen in Section 3, the integrand we must deal with changes drastically as p moves from 3 to 4, and is exceedingly difficult to deal with when $p = 3$. However, the results for $p \geq 4$ are surprisingly good. Even though our upper bound on a does not reach $p - 2$, the coverage probabilities of C_{δ^+} are a substantial improvement over C_X^0, and are virtually equal to those obtained with $a = p - 2$.

The proof of the dominance of C_{δ^+} over C_X^0 proceeds in the following way. First we establish that, for $|\theta| < c$, $P_\theta(C_{\delta^+}) \geq P_\theta(C_X^0)$. Then, since $\lim_{|\theta| \to \infty} P_\theta(C_{\delta^+}) = P_\theta(C_X^0)$, a sufficient condition for the minimaxity of C_{δ^+} is that $(\partial/\partial|\theta|)P_\theta(C_{\delta^+}) \leq 0$ for $|\theta| > c$. The major portion of this paper is dedicated to establishing this result. To obtain a workable expression for $(\partial/\partial|\theta|)P_\theta(C_{\delta^+})$ we ultimately employ an integration by parts (in a manner analogous to the point estimation problem) but, since our integrand is an indicator function this gives us a Dirac delta function in the integrand. (A Dirac delta function, say $\Delta_m(t)$, is defined by $\Delta_m(t) = 0$ if $t \neq m$ and $\Delta_m(m) = \infty$.) By a spherical transformation, the p-dimensional integral is reduced to a two-dimensional one. Then, by evaluating the delta function, a one-dimensional representation is obtained. The condition on the constant a is then derived (i.e., $0 \leq a \leq a_0$) which guarantees that the integrand is everywhere negative, and hence that $P_\theta(C_{\delta^+})$ is decreasing in $|\theta|$ for $|\theta| > c$.

In Section 2 we make this argument rigorous by expressing $(\partial/\partial|\theta|)P_\theta(C_{\delta^+})$ as the limit of a sequence of integrals with differentiable integrands, and obtain the necessary representation. In Section 3 we apply these results to the set (1.3) and determine a_0 such that $P_\theta(C_{\delta^+}) \geq P_\theta(C_X^0)$ if $a \leq a_0$. Section 4 contains some comments and generalizations. Technical lemmas, needed in Section 2, are in the Appendix.

2. Representations of coverage probabilities and their derivatives. In this section we show how to represent coverage probabilities and their derivatives (with respect to $|\theta|$) as the limit of a sequence of integrals with differentiable integrands. By first applying integration by parts and then taking limits we obtain workable expressions for these quantities.

The results of this section are not restricted to the positive part James-Stein estimator, but are valid for a more general form of estimator. Hence, we now consider estimators of the form

$$\delta(X) = \gamma(|X|)X, \tag{2.1}$$

and confidence sets,

$$C_\delta = \{\theta : |\theta - \delta(X)|^2 \leq c^2\}, \tag{2.2}$$

where $\gamma(r)$ satisfies the following conditions.

CONDITIONS. (i) $\gamma(r) \geq 0$.
(ii) $\gamma(r)$ is nondecreasing.
(iii) $\gamma(r)$ is strictly increasing for all r such that $\gamma(r) > 0$. (Hence, assume $\gamma(r) > 0$ for $r > r_0$.)
(iv) $\gamma(r)$ is continuous and differentiable for $r > r_0$.
(v) Both $\gamma(r)$ and $\gamma'(r)$ can be bounded above by some polynomial.

We note that the positive-part James-Stein estimator satisfies these conditions, but the ordinary James-Stein estimator does not.

In evaluating the coverage probability of C_δ it is easier to work with the θ section

$$C_\theta = \{X : \| \theta - \delta(X)\|^2 \leq c^2\}. \tag{2.3}$$

Since $X \in C_\theta$ if and only if $\theta \in C_\delta$, it follows that $P_\theta(C_\theta) = P_\theta(C_\delta)$.

Before proceeding to the representation theorems, we present the following theorems, which establish the superiority of C_θ for $|\theta| < c$.

THEOREM 2.1. *For $\delta(X) = \gamma(|X|)X$, where γ satisfies $0 \leq \gamma(|X|) \leq 1$, $P_\theta(C_\theta) \geq P_\theta(C_\theta^0)$ for all $|\theta| \leq c$.*

PROOF. This result seems to be fairly well known, and is given here only for completeness. The theorem follows by establishing that, for $|\theta| \leq c$, $C_\theta^0 \subset C_\theta$ or, equivalently, $X \in C_\theta^0$ implies $\gamma(|X|)X \in C_\theta^0$. Since $|\theta| \leq c$, $0 \in C_\theta^0$ and hence if $X \in C_\theta^0$ then $\gamma(|X|)X \in C_\theta^0$ by convexity of C_θ^0, since $0 \leq \gamma(|X|) \leq 1$. □

In light of this theorem, we only need be concerned with the case $|\theta| > c$. We proceed with the following lemma, which shows how $P_\theta(C_\theta)$ and $(\partial/\partial|\theta|)P_\theta(C_\theta)$ can be expressed as the limits of a sequence of integrals with differentiable integrands.

LEMMA 2.1. *Let Φ_n denote a univariate normal cdf with mean 0 and variance n^{-2}. Define*

$$B_n(\theta) = \int_{\mathbb{R}^p} \Phi_n\{c^2 - |\gamma(|X|)X - \theta|^2\} f_\theta(X)\, dX,$$

where $f_\theta(X)$ is a multivariate normal density with mean θ and identity covariance matrix, and γ satisfies Conditions (i)–(v). Then

$$\text{(a) } \lim_{n\to\infty} B_n(\theta) = P_\theta(C_\theta), \qquad \text{(b) } \lim_{n\to\infty} \frac{\partial}{\partial|\theta|} B_n(\theta) = \frac{\partial}{\partial|\theta|} P_\theta(C_\theta) \quad \text{for } |\theta| > c.$$

PROOF. It follows from Condition (iii) that $t\gamma(t)$ is strictly increasing, which implies that $P_\theta\{X: |\gamma(|X|)X - \theta|^2 = c^2\} = 0$. Then, by the fact that if $t \neq 0$, $\Phi_n(t) \to I(t)$ where $I(t) = 1$ if $t > 0$ and 0 otherwise, (a) clearly follows by the Bounded Convergence Theorem. To proceed with the proof of (b), first transform to spherical coordinates to obtain

$$P_\theta(C_\theta) = K \int_0^{\beta_0} \int_{r_-}^{r_+} r^{p-1} \sin^{p-2}\beta \exp\{-(r^2 - 2r|\theta|\cos\beta + |\theta|^2)/2\}\, dr\, d\beta \tag{2.4}$$

and

$$B_n(\theta) = K \int_0^{\beta_0} \int_0^{\infty} \Phi_n(\omega) r^{p-1} \sin^{p-2}\beta \exp\{-(r^2 - 2r|\theta|\cos\beta + |\theta|^2)/2\}\, dr\, d\beta, \tag{2.5}$$

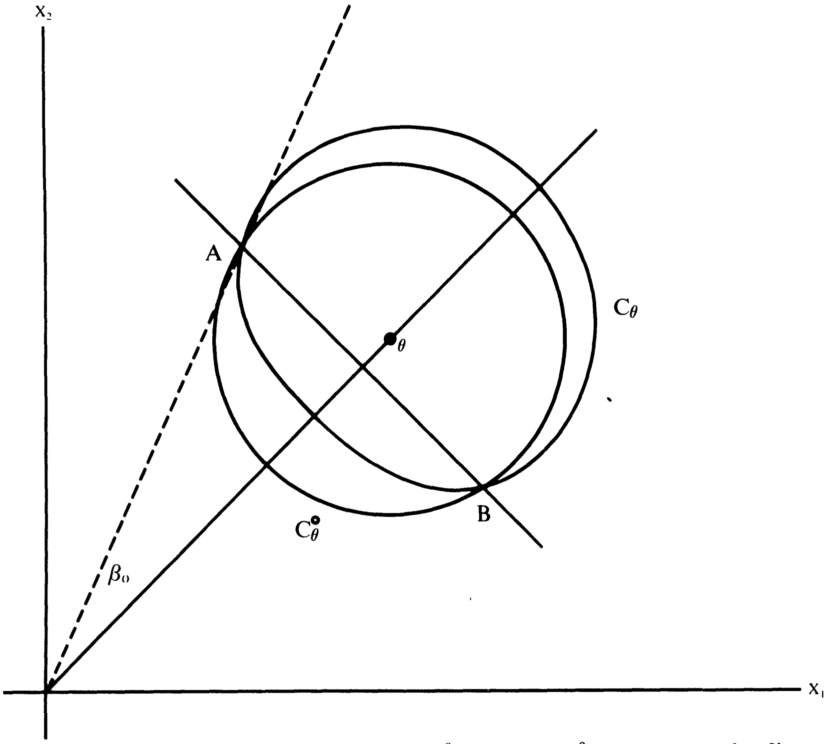

FIG. 1. *Two-dimensional representation of C_θ and C_θ^0 for $|\theta| > c$. C_θ^0 is the sphere of radius c centered at θ. As β (the angle between X and θ) varies, $r_+(\beta)$ traces the locus of points above the line AB, while $r_-(\beta)$ traces out those below. C_θ and C_θ^0 intersect at points A and B, where β_0 satisfies* $\sin\beta_0 = c/|\theta|$ *and* $r_+(\beta_0) = |\theta|\cos\beta_0$.

where $r = |X|$, β is the angle between X and θ,

$$K = (2\pi)^{-(p-2)/2}\prod_{i=1}^{p-3}\left\{\int_0^\pi \sin^i t\,dt\right\},$$

and

$$\omega = \omega(r, \beta, \theta) = c^2 - |\theta|^2\sin^2\beta - \{\gamma(r)r - |\theta|\cos\beta\}^2. \tag{2.6}$$

The limits of integration of (2.4) satisfy $\sin\beta_0 = c/|\theta|$, $0 < \beta_0 < \pi/2$, and r_+ and r_- ($r_- < r_+$) are the roots of

$$r^2\gamma^2(r) - 2r|\theta|\gamma(r)\cos\beta + |\theta|^2 - c^2 = 0, \tag{2.7}$$

which is equivalent to

$$r\gamma(r) = |\theta|\cos\beta \pm (c^2 - |\theta|^2\sin^2\beta)^{1/2}. \tag{2.8}$$

Notice that, for $0 < \beta < \beta_0$ we have $c^2 > |\theta|^2\sin^2\beta$ and, moreover, $|\theta|^2\cos^2\beta > c^2 - |\theta|^2\sin^2\beta$ if $|\theta| > c$. Condition (iii) on γ guarantees the uniqueness of the solutions of (2.8), and we have

$$r_\pm\gamma(r_\pm) = |\theta|\cos\beta \pm (c^2 - |\theta|^2\sin^2\beta)^{1/2} > |\theta| - c > 0. \tag{2.9}$$

We also note that we are dealing with $\gamma(r)$ only for r satisfying $\gamma(r) > 0$.

For fixed θ, as β varies, $r_+(\beta)$ and $r_-(\beta)$ define the boundary of $C_\theta(\delta)$. Figure 1 illustrates the set C_θ in two dimensions for $\delta = \delta^+$. There is a flattening C_θ on the side near the origin, and an expansion away from the origin. Both C_θ and C_θ^0 are symmetric across the ray through θ.

Since β_0, r_+, and r_- are all differentiable functions of $|\theta|$, expression (2.4) shows that $P_\theta(C_\theta)$ is differentiable with respect to $|\theta|$. For ease of notation define

$$h(r, \beta) = r^{p-1}\sin^{p-2}\beta \exp\{-\tfrac{1}{2}(r^2 - 2r|\theta|\cos\beta + |\theta|^2)\}. \tag{2.10}$$

Differentiating with respect to $|\theta|$ we obtain

$$\begin{aligned}\frac{\partial}{\partial|\theta|}P_\theta(C_\theta) = &-|\theta|\,P_\theta(C_\theta) + K\int_0^{\beta_0}\int_{r_-}^{r_+} r\cos\beta h(r, \beta)\,dr\,d\beta \\ &+ K\int_0^{\beta_0} h(r_+, \beta)\frac{\partial r_-}{\partial|\theta|}\,d\beta - K\int_0^{\beta_0} h(r_-, \beta)\frac{\partial r_-}{\partial|\theta|}\,d\beta\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\frac{\partial}{\partial|\theta|}B_n(\theta) = &-|\theta|\,B_n(\theta) + K\int_0^{\pi}\int_0^{\infty}\Phi_n(\omega) r\cos\beta\, h(r, \beta)\,dr\,d\beta \\ &+ K\int_0^{\pi}\int_0^{\infty}\left\{\frac{\partial}{\partial|\theta|}\Phi_n(\omega)\right\}h(r, \beta)\,dr\,d\beta.\end{aligned} \tag{2.12}$$

Note that in calculating $(\partial/\partial|\theta|)P_\theta(C_\theta)$, the term containing $\partial\beta_0/\partial|\theta|$ is zero. From part (*a*), $-|\theta|\ B_n(\theta) \to -|\theta|\ P_\theta(C_\theta)$ and, from the Dominated Convergence Theorem, the second term in (2.12) converges to the second term in (2.11). Thus it only remains to show that

$$\begin{aligned}\lim_{n\to\infty}B_n^*(\theta) &\stackrel{\text{def}}{=} \lim_{n\to\infty}K\int_0^{\pi}\int_0^{\infty}\left[\frac{\partial}{\partial|\theta|}\Phi(\omega)\right]h(r, \beta)\,dr\,d\beta \\ &= A^*(\theta) \stackrel{\text{def}}{=} K\int_0^{\beta_0}\left[h(r_+, \beta)\frac{\partial r_+}{\partial|\theta|} - h(r_-, \beta)\frac{\partial r_-}{\partial|\theta|}\right]d\beta.\end{aligned} \tag{2.13}$$

From Lemma 4 in the Appendix, we have

$$\lim_{n\to\infty}B_n^*(\theta) = K\int_0^{\beta_0}\lim_{n\to\infty}\int_0^{\infty} -2\varphi_n(\omega)\{|\theta| - r\gamma(r)\cos\beta\}h(r, \beta)\,dr\,d\beta, \tag{2.14}$$

where $\varphi_n(t) = (d/dt)\Phi_n(t)$. Note that $\omega = \omega(r, \beta, |\theta|)$ can be written

$$\omega(r, \beta, |\theta|) = \{r\gamma(r) - r_+\gamma(r_+)\}\{r\gamma(r) - r_-\gamma(r_-)\}.$$

Applying Lemma 3 in the Appendix to the inner integral in (2.14) yields

$$\begin{aligned}\lim_{n\to\infty}B_n^*(\theta) = K\int_0^{\beta_0}\lim_{n\to\infty}\Bigg[&\int_0^{\infty}\frac{-2\varphi_n(r - r_+)\{|\theta| - r\gamma(r)\cos\beta\}h(r, \beta)\,dr}{\ell(r_+)|r_+\gamma(r_+) - r_-\gamma(r_-)|} \\ &+ \int_0^{\infty}\frac{-2\varphi_n(r - r_-)\{|\theta| - r\gamma(r)\cos\beta\}h(r, \beta)\,dr}{\ell(r_-)|\,r_+\gamma(r_+) - r_-\gamma(r_-)|}\Bigg]d\beta,\end{aligned} \tag{2.15}$$

where $\ell(r) = (d/dr)r\gamma(r) = r\gamma'(r) + \gamma(r) > 0$. Taking the limit in (2.15) yields

$$\begin{aligned}\lim_{n\to\infty}B_n^*(\theta) = K\int_0^{\beta_0} -2\Bigg[&\frac{\{|\theta| - r_+\gamma(r_+)\cos\beta\}h(r_+, \beta)}{\ell(r_+)|r_+\gamma(r_+) - r_-\gamma(r_-)|} \\ &+ \frac{\{|\theta| - r_-\gamma(r_-)\cos\beta\}h(r_-, \beta)}{\ell(r_-)|r_+\gamma(r_+) - r_-\gamma(r_-)|}\Bigg]d\beta.\end{aligned} \tag{2.16}$$

Lastly, it remains to be shown that (2.16) is equal to $A^*(\theta)$ as defined in (2.13). From (2.9) we have that

$$\frac{\partial r_\pm}{\partial|\theta|}=\frac{2\{|\theta|-r_\pm\gamma(r_\pm)\cos\beta\}}{\{2|\theta|\cos\beta-2r_\pm\gamma(r_\pm)\}\ell(r_\pm)},\tag{2.17}$$

and also

$$2|\theta|\cos\beta-2r_\pm\gamma(r_\pm)=\mp\{r_+\gamma(r_+)-r_-\gamma(r_-)\}.\tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16) shows that $A^*(\theta)$ is equal to expression (2.16), and hence the lemma is proved. □

We now come to the main theorem of this section, which gives a representation for the derivative of $P_\theta\{C_\theta(\delta)\}$.

THEOREM 2.2. *For γ satisfying Conditions* (i)–(v) *and* $|\theta|>c$,

$$\begin{aligned}\frac{\partial}{\partial|\theta|}P_\theta(C_\theta)&=K\int_0^{\beta_0}\left[\begin{matrix}\cos\beta\{1-\gamma'(r_+)r_+-\gamma(r_+)\}\\-\sin^2\beta\{1-\gamma(r_+)\}\{(c^2/|\theta|^2)-\sin^2\beta\}^{-1/2}\end{matrix}\right]\frac{h(r_+,\beta)\,d\beta}{\{r_+\gamma'(r_+)+\gamma(r_+)\}}\\&\quad-K\int_0^{\beta_0}\left[\begin{matrix}\cos\beta\{1-\gamma'(r_-)r_--\gamma(r_-)\}\\+\sin^2\beta\{1-\gamma(r_-)\}\{(c^2/|\theta|^2)-\sin^2\beta\}^{-1/2}\end{matrix}\right]\frac{h(r_-,\beta)\,d\beta}{\{r_-\gamma'(r_-)+\gamma(r_-)\}}\end{aligned}\tag{2.19}$$

where $\lambda(r)=1-r\gamma'(r)-\gamma(r)$.

REMARK. It is interesting to note that if $\gamma(r)=1-a/r$, then $1-r\gamma'(r)-\gamma(r)=0$, and it is then easily established that (2.19) is negative. This implies that the coverage probability of the confidence set centered at $\delta(X)=(1-a/|X|)^+X$ is decreasing in $|\theta|$ for $|\theta|>c$. However, it is also true that for this estimator, $\lim_{|\theta|\to\infty}P_\theta(C_\delta)<P_\theta(C_X^0)$, so this confidence set cannot dominate the usual one. More generally, if we define $\delta^\epsilon(X)=(1-a/|X|^{1+\epsilon})^+X$, then $\lim_{|\theta|\to\infty}P_\theta(C_{\delta^\epsilon})=P_\theta(C_X^0)$ if and only if $\epsilon>0$.

PROOF. From Lemma 2.1, it is clear that $P_\theta(C_\theta)$ and $B_n(\theta)$ depend on θ only through $|\theta|$. Hence, without loss of generality, let $\theta=(|\theta|,0,\cdots,0)$. Then

$$\begin{aligned}\frac{\partial}{\partial|\theta|}B_n(\theta)&=\int_{\mathbb{R}^p}\frac{\partial}{\partial|\theta|}\Phi_n\{\omega(X,\theta)\}f_\theta(X)\,dX\\&=\int_{\mathbb{R}^p}2\varphi_n\{\omega(X,\theta)\}\{\gamma(|X|)X_1-|\theta|\}f_\theta(x)\,dX\\&\quad+\int_{\mathbb{R}^p}\Phi_n\{\omega(X,\theta)\}(X_1-|\theta|)f_\theta(X)\,dX,\end{aligned}$$

where $\omega(X,\theta)=c^2-|\gamma(|X|)X-\theta|^2$. We now apply integration by parts to the second integral, which allows us to replace $\Phi_n\{\omega(X,\theta)\}(X_1-|\theta|)$ by $(\partial/\partial X_1)\Phi_n\{\omega(X,\theta)\}$. After performing the differentiation, collecting terms and writing $X_1=X'\theta/|\theta|$, we obtain

$$\begin{aligned}\frac{\partial}{\partial|\theta|}B_n(\theta)&=\frac{2}{|\theta|}\int_{\mathbb{R}^p}\varphi_n\{\omega(X,\theta)\}\\&\quad\cdot\left[\{1-\gamma(|X|)\}\{\gamma(|X|)X'\theta-|\theta|^2\}-\frac{X'\theta\gamma'(|X|)}{|X|}\{\gamma(|X|)X-\theta\}'X\right]f_\theta(X)\,dX.\end{aligned}$$

Once again we transform to spherical coordinates and obtain

$$\frac{\partial}{\partial|\theta|} B_n(\theta) = \frac{2}{|\theta|} K \int_0^{\pi/2} \int_0^{\infty} \varphi_n\{\omega(r, \beta, \theta)\} H(r, \beta) h(r, \beta)\, dr\, d\beta,$$

where

$$H(r, \beta) = \gamma(r) r |\theta| \cos\beta\{1 - \gamma(r) - \gamma'(r) r\} + \gamma'(r) r |\theta|^2 \cos^2\beta - |\theta|^2\{1 - \gamma(r)\}.$$

By first applying Lemma 4 and then Lemma 3 in the Appendix, we obtain

$$\frac{\partial}{\partial|\theta|} P_\theta(C_\theta) = \lim_{n\to\infty} \frac{\partial}{\partial|\theta|} B_n(\theta)$$

$$(2.20) \qquad = \frac{2K}{|\theta|} \int_0^{\beta_0} \left\{ \frac{H(r_+, \beta) h(r_+, \beta)}{|r_+\gamma'(r_+) + \gamma(r_+)|} + \frac{H(r_-, \beta) h(r_-, \beta)}{|r_-\gamma'(r_-) + \gamma(r_-)|} \right\} \cdot |r_+\gamma(r_+) - r_-\gamma(r_-)|^{-1}\, d\beta.$$

From (2.9) we have that $r_+\gamma(r_+) - r_-\gamma(r_-) = 2(c^2 - |\theta|^2 \sin^2\beta)^{1/2}$, and a simple calculation shows that

$$H(r_\pm, \beta) = -\{1 - \gamma(r_\pm)\}\, |\theta|^2 \sin^2\beta \pm |\theta| \cos\beta (c^2 - |\theta|^2 \sin^2\beta)^{1/2} \{1 - \gamma'(r_\pm) r_\pm - \gamma(r_\pm)\}.$$

Substitution of these two expressions in (2.20) yields expression (2.19), and the theorem is proved. □

The expression given in Theorem 2.1 is still rather difficult to handle; however, we are mainly concerned with the performance of the positive part James-Stein estimator. The derivative of the coverage probability simplifies considerably for this estimator, and is given in the following corollary.

COROLLARY 2.1. *For $\gamma(|X|) = (1 - a/|X|^2)^+$, where a is a constant, and $|\theta| > c$,*

$$(2.21) \qquad \frac{\partial}{\partial|\theta|} P_\theta(C_\theta) = -K \int_0^{\beta_0} \frac{a \sin^2\beta}{\{(c^2/|\theta|^2) - \sin^2\beta\}^{1/2}} \left\{ \frac{h(r_+, \beta)}{r_+^2 + a} + \frac{h(r_-, \beta)}{r_-^2 + a} \right\} d\beta - K \int_0^{\beta_0} a \cos\beta \left\{ \frac{h(r_+, \beta)}{r_+^2 + a} - \frac{h(r_-, \beta)}{r_-^2 + a} \right\} d\beta.$$

PROOF. Notice that, as mentioned in Lemma 2.1, we only need be concerned with the region where $\gamma(\cdot) > 0$. Hence, we can let $\gamma(|X|) = (1 - a/|X|^2)$. Substitution in Theorem 2.2 gives the result. □

As mentioned before, the ultimate goal is to show that $(\partial/\partial|\theta|)P_\theta(C_\theta)$ is negative for $|\theta| > c$, which implies dominance over the usual confidence set. This problem has been reduced to showing that the expression given in (2.21) is negative but, alas, that task is still formidable. It is possible, however, to find conditions under which the integrand in (2.21) is negative and hence, produce confidence sets that dominate the usual one. In the next section we find bounds on the constant a that guarantee that the integrand in (2.21) is negative, and we examine the size of possible improvement.

3. Minimax confidence sets. In this section we concentrate on the positive-part James-Stein estimator $\delta^+(X) = (1 - a/|X|^2)^+ X$, and determine values of the constant a for which the sphere centered at $\delta^+(X)$ has higher coverage probability, for all θ, than that centered at X. Our technique is to show that the integrand of $(\partial/\partial|\theta|)P_\theta(C_\theta)$ is negative for $|\theta| > c$, which shows that $P_\theta(C_\theta)$ decreases to its value at infinity. Since $\lim_{|\theta|\to\infty} P_\theta(C_\theta)$

TABLE 1
Coverage probabilities for the set C_{δ^+} where $a = a_o$ and c^2 satisfies $P(\chi_p^2 \le c^2) = 0.9$

$\lvert\theta\rvert$	p							
	3*	5	7	9	11	13	15	25
0	.9064	.9600	.9842	.9933	.9970	.9986	.9994	.9999
2	.9047	.9506	.9776	.9896	.9951	.9976	.9988	.9999
4	.9009	.9232	.9519	.9731	.9859	.9924	.9959	.9999
6	.9003	.9109	.9278	.9451	.9606	.9732	.9827	.9988
8	.9002	.9063	.9166	.9283	.9402	.9516	.9618	.9925
10	.9001	.9040	.9109	.9191	.9278	.9366	.9452	.9794
15	.9000	.9018	.9050	.9089	.9133	.9180	.9230	.9481
20	.9000	.9010	.9028	.9051	.9077	.9105	.9135	.9303
25	.9000	.9006	.9018	.9033	.9050	.9068	.9088	.9204
50	.9000	.9002	.9005	.9008	.9013	.9017	.9028	.9055
100	.9000	.9000	.9001	.9002	.9003	.9004	.9006	.9014
500	.9000	.9000	.9000	.9000	.9000	.9000	.9000	.9001
1000	.9000	.9000	.9000	.9000	.9000	.9000	.9000	.9001

* *This column is not covered by Theorem* 3.1. *The value of a_o used is a_o* = .076. [See the discussion following (3.8).]

TABLE 2
Coverage probabilities for the set C_{δ^+} where $a = p - 2$ and c^2 satisfies $P(\chi_p^2 \le c^2) = 0.9$

$\lvert\theta\rvert$	p							
	3	5	7	9	11	13	15	25
0	.9565	.9879	.9959	.9985	.9994	.9998	.9999	.9999
2	.9458	.9809	.9926	.9972	.9989	.9995	.9998	.9999
4	.9062	.9343	.9622	.9808	.9949	.9977	.9989	.9999
6	.9026	.9162	.9337	.9510	.9661	.9780	.9866	.9993
8	.9014	.9093	.9202	.9323	.9443	.9556	.9657	.9943
10	.9009	.9060	.9133	.9218	.9307	.9397	.9484	.9819
15	.9004	.9027	.9061	.9102	.9147	.9196	.9247	.9502
20	.9002	.9015	.9035	.9059	.9085	.9114	.9145	.9317
25	.9001	.9010	.9022	.9038	.9055	.9075	.9095	.9214
50	.9000	.9002	.9006	.9010	.9014	.9019	.9024	.9058
100	.9000	.9001	.9001	.9002	.9004	.9005	.9006	.9015
500	.9000	.9000	.9000	.9000	.9000	.9000	.9000	.9001
1000	.9000	.9000	.9000	.9000	.9000	.9000	.9000	.9000

$= P_\theta(C_\theta^0)$, the dominance will be established for $\lvert\theta\rvert > c$, and the dominance for $\lvert\theta\rvert \le c$ follows from Theorem 2.1.

By bounding the integrand, rather than the integral, the bounds obtained on the constant a are smaller than necessary for dominance of $P_\theta(C_\theta^0)$. In fact, the upper bounds on a are less than $p - 2$, which was demonstrated numerically by Casella (1980) to yield a dominating procedure for $p \ge 3$. It is also unfortunate that the result is not established for $p = 3$; however, in this case the integrand becomes extremely difficult to handle.

For $p \ge 4$, the upper bounds obtained do provide substantial improvement in coverage probability over the usual confidence set. Table 1 gives coverage probabilities for these sets for $1 - \alpha = .9$, and it can be seen that the coverage probability can be as high as .99 at $\lvert\theta\rvert = 0$ if $p \ge 9$. Moreover, even though the upper bounds are smaller than $p - 2$, the coverage probabilities we obtain are almost as high. Table 2 gives coverage probabilities for $a = p - 2$ and $1 - \alpha = .9$, and comparison with Table 1 shows that the probabilities are virtually identical.

TABLE 3
Selected values of a_o

p	α = .10	α = .05	p	α = .10	α = .05
4	0.54	0.50	15	9.50	9.29
5	1.28	1.21	16	10.34	10.12
6	2.06	1.97	17	11.18	10.96
7	2.86	2.75	18	12.02	11.79
8	3.67	3.55	19	12.86	12.63
9	4.49	4.35	20	13.71	13.47
10	5.32	5.17	21	14.55	14.30
11	6.15	5.99	22	15.40	15.14
12	6.98	6.81	23	16.24	15.98
13	7.82	7.63	24	17.09	16.82
14	8.66	8.46	25	17.94	17.67

It should also be mentioned that the choice $a = 2(p - 2)$ does not produce a confidence set which dominates the usual one. (Recall that this is the largest value of a for which $\delta^+(X)$ is minimax.) This statement is based on numerical evidence using the formula of Casella (1980). Calculations show that for moderate values of $|\theta|$, the coverage probability of this set falls below that of the usual one.

The main result of this section, the sufficient condition for dominance of C_X^0 by C_{δ^+}, is given in the following theorem.

THEOREM 3.1. *For fixed c^2 and $p \geq 4$, define a_0 as the unique solution to*

$$\left\{\frac{c + (c^2 + a_0)^{1/2}}{\sqrt{a_0}}\right\}^{p-3} e^{-c\sqrt{a_0}} = 1. \tag{3.1}$$

Then for all $0 < a \leq a_0$ the sphere centered at $\delta^+(X) = (1 - a/|X|^2)^+X$ has higher coverage probability than the sphere centered at X, i.e., C_{δ^+} is a minimax confidence set.

REMARK. Values of a_0 for $p = 4(1)25$ and c^2 corresponding to nominal 90% and 95% procedures have been calculated. These values are presented in Table 3. It is evident that these values are smaller than $p - 2$; however, as previously noted, the coverage probabilities are very close to those of $a = p - 2$.

PROOF. It is sufficient to show that expression (2.21) is negative for all $|\theta| > c$, which is implied by

$$\left\{\frac{\sin^2\beta}{(c^2 - |\theta|^2 \sin^2\beta)^{1/2}} + \cos\beta\right\}\frac{h(r_+, \beta)}{r_+^2 + a} + \left\{\frac{\sin^2\beta}{(c^2 - |\theta|^2 \sin^2\beta)^{1/2}} - \cos\beta\right\}\frac{h(r_-, \beta)}{r_-^2 + a} \geq 0, \qquad \forall\beta \leq \beta_0, \quad |\theta| > c. \tag{3.2}$$

Notice that the function $\sin^2\beta(c^2 - |\theta|^2 \sin^2\beta)^{-1/2} - \cos\beta$ is increasing in β. Define $\beta_1 < \beta_0$ as the unique root of this function. Then for $\beta > \beta_1$ all the terms in (3.2) are positive and, hence, we need only concentrate on $\beta < \beta_1$.

If $\beta < \beta_1$, expression (3.2) is true if and only if

$$\left\{\frac{\cos\beta(c^2 - |\theta|^2 \sin^2\beta)^{1/2} + \sin^2\beta}{\cos\beta(c^2 - |\theta|^2 \sin^2\beta)^{1/2} - \sin^2\beta}\right\}\frac{h(r_+, \beta)}{h(r_-, \beta)}\left\{\frac{r_-^2 + a}{r_+^2 + a}\right\} \geq 1, \qquad \forall\beta < \beta_1, \quad |\theta| > c. \tag{3.3}$$

The term in braces is ≥ 1 if $\beta < \beta_1$, and can be replaced by 1. Now recalling the definition of $h(r, \beta)$, we can write

$$\frac{h(r_+, \beta)}{h(r_-, \beta)} = \frac{r_+^{p-1} \sin^{p-2}\beta \exp\{-(r_+^2 - 2r_+|\theta|\cos\beta + |\theta|^2)/2\}}{r_-^{p-1} \sin^{p-2}\beta \exp\{-(r_-^2 - 2r_-|\theta|\cos\beta + |\theta|^2)/2\}}$$

$$= \left(\frac{r_+}{r_-}\right)^{p-1} \exp[-\{r_+^2 - 2(r_+ - r_-)|\theta|\cos\beta - r_-^2\}/2].$$

Now, from (2.9), $r_+\gamma(r_+) + r_-\gamma(r_-) = 2|\theta|\cos\beta$ and, since $\gamma(r) = 1 - (a/r^2)$,

$$r_+^2 - 2(r_+ - r_-)|\theta|\cos\beta - r_-^2 = r_+^2 - (r_+ - r_-)\{r_+ - (a/r_+) + r_- - (a/r_-)\} - r_-^2$$

$$= a\left(\frac{r_+}{r_-} - \frac{r_-}{r_+}\right).$$

Hence, we can write

$$\frac{h(r_+, \beta)}{h(r_-, \beta)} = t^{p-1}\exp\{-a(t - t^{-1})/2\}, \tag{3.4}$$

where $t = r_+/r_-$. Furthermore,

$$\frac{r_+^2(r_-^2 + a)}{r_-^2(r_+^2 + a)} = \frac{1 + a/r_-^2}{1 + a/r_+^2} \geq 1, \qquad \forall\beta, \quad |\theta| > c, \tag{3.5}$$

since $r_-^2 \leq r_+^2$. Thus, combining (3.4) and (3.5), a sufficient condition for the derivative of $P_\theta(C_\theta)$ to be negative is that

$$\rho(t) = t^{p-3}\exp\{-a(t - t^{-1})/2\} \geq 1, \qquad 0 < \beta < \beta_0, \quad |\theta| > c. \tag{3.6}$$

It is straightforward to establish that $\rho(1) = 1$ and for $t > 1$ $\rho(t)$ increases to a unique maximum, then decreases to 0 as $t \to \infty$. Since $t = r_+/r_- \geq 1$, (3.6) will be established if we can show that $\rho(t^*) \geq 1$, where

$$t^* = \sup_{\beta<\beta_0, |\theta|>c}(r_+/r_-).$$

For fixed $|\theta|$, t is decreasing in β, hence

$$\sup_{\beta<\beta_0} t = \frac{r_+}{r_-}\bigg|_{\beta=0} = \frac{|\theta| + c + \{(|\theta| + c)^2 + 4a\}^{1/2}}{|\theta| - c + \{(|\theta| - c)^2 + 4a\}^{1/2}},$$

where the last equality follows from (2.9). Also, differentiation will establish that $\sup_\beta t$ is decreasing in $|\theta|$ for $|\theta| > c$, and hence, by substituting $|\theta| = c$,

$$t^* = \sup_{\beta<\beta_0, |\theta|>c} t = \frac{c + (c^2 + a)^{1/2}}{\sqrt{a}}.$$

A little algebra will verify that $t - t^{-1} = 2c/\sqrt{a}$, and hence (3.6) will hold if

$$\rho^*(a) = \left[\frac{c + (c^2 + a)^{1/2}}{\sqrt{a}}\right]^{p-3} e^{-c\sqrt{a}} \geq 1. \tag{3.7}$$

Since this function is decreasing in a, if we define a_0 as the unique solution to $\rho^*(a) = 1$, (3.7) is true for all $a \leq a_0$, and the theorem is proved. □

The coarsest inequality used in this proof is bounding $(1 + a/r_-^2)/(1 + a/r_+^2)$ by 1 since, for fixed β, this function is decreasing in $|\theta|$. This means that this function would increase the integrand at $|\theta| = c$, $\beta = 0$. However, it seems rather difficult to bound the integrand independent of $|\theta|$ if this function is left in. What is most unfortunate is that this bound decreased the exponent of t to $p - 3$, so the theorem does not cover the case $p = 3$.

The inequality used on the term in braces in (3.3) probably did not lose very much. This is because at $\beta = 0$ this term is equal to 1.

The technique used here, that of bounding the integrand, does not seem to be powerful enough to cover the case $p = 3$. Although it has not been proved, we believe that when $p = 3$ the minimum value of expression (3.3) is obtained at $|\theta| = c$, $\beta = 0$. This gives the inequality

$$\frac{2e^{-c\sqrt{a}}\{c + (c^2 + a)^{1/2}\}^2}{a + \{c + (c^2 + a)^{1/2}\}^2} \geq 1, \tag{3.8}$$

which we can think of as a necessary condition for the integrand of $(\partial/\partial|\theta|)P_\theta\{C_\theta(\delta^+)\}$ to be negative. If c^2 is taken to be the 90% cutoff point of a χ_3^2, then $a_0 = .076$. At $|\theta| = 0$, the coverage probability of this set is .906, a minimal improvement. Thus, while the minimax confidence sets developed here yield substantial improvements for $p \geq 4$, a much more difficult technique (bounding the integral) is required to get substantial improvement when $p = 3$.

4. Comments and generalizations. It has been shown that by merely recentering the usual confidence set at a positive-part James-Stein estimator, the coverage probability of the usual confidence set can be uniformly improved, with substantial gains for some values of the parameter. Although our upper bound, a_0, is smaller than necessary, we have seen that a larger upper bound ($a = p - 2$) will not substantially increase coverage probabilities. Also, since there is numerical evidence that $a = 2(p - 2)$ is too large, it seems that there is not room for much improvement over the bound a_0.

We have restricted consideration to recentered confidence sets, and have not dealt with more complicated forms such as those of Faith (1976) or Berger (1980). It has been shown that these sets can have reduced volume while maintaining a dominating coverage probability over most of the parameter space. However, these sets are conceptually more difficult to deal with. The recentered sets are easy to visualize, and can also yield confidence intervals for the individual components of the parameter.

Finally, we note two straightforward generalizations of our results. Let $X \sim N(\theta, \Sigma)$, Σ known, and let

$$\delta_{\theta_0}^+(X) = \theta_0 + \{1 - a/(X - \theta_0)'\Sigma^{-1}(X - \theta_0)\}^+(X - \theta_0),$$

where θ_0 is a prior guess at θ. Centering the estimator at a prior guess can be a great benefit, since the region of maximum improvement in coverage probability will be near θ_0. For this situation, the usual confidence set is the ellipsoid

$$C_X^0 = \{\theta: (X - \theta)'\Sigma^{-1}(X - \theta) \leq c^2\},$$

and the recentered set is

$$C_{\delta_{\theta_0}^+} = \{\theta: [\delta_{\theta_0}^+(X) - \theta]'\Sigma^{-1}[\delta_{\theta_0}^+(X) - \theta] \leq c^2\}.$$

By applying the transformation $Y = \Sigma^{-1/2}(X - \theta_0)$, this set-up is transformed into that of Section 3, and hence it follows that $C_{\delta^+\theta_0}$ dominates C_X^0 for all θ.

Another straightforward generalization is to a wider class of loss functions. If we measure the loss of the confidence procedure C by

$$L(C, \theta) = \omega_1 \text{ Volume}(C) - \omega_2 I_C(\theta),$$

where $I_C(\theta) = 1$ if $\theta \in C$ and 0 otherwise, and ω_1 and ω_2 are any two known positive weights, it then follows that C_{δ^+} dominates C_X^0 with respect to $L(C, \theta)$ for $a \leq a_0$.

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APPENDIX

Let φ_n be a normal density with mean 0 and variance n^{-2}. Let f be a continuous, integrable function. The proof of the following three lemmas can be established using the continuity of f and the Dominated Convergence Theorem. They are omitted here.

LEMMA 1. *For any constant a and $\epsilon > 0$,*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(t-a)f(t)\,dt = \lim_{n\to\infty} \int_{\{t:|t-a|<\epsilon\}} \varphi_n(t-a)f(t)\,dt = f(a).$$

LEMMA 2. *Let $h:(c,d)\to \mathbb{R}$ be a strictly monotone function that satisfies $h(a)=0$ for some $a \in (c,d)$. Then*

$$\lim_{n\to\infty} \int_c^d \varphi_n[h(t)]f(t)\,dt = \lim_{n\to\infty} \int_c^d \frac{\varphi_n(t-a)f(t)\,dt}{|h'(a)|} = \frac{f(a)}{|h'(a)|}.$$

LEMMA 3. *Let $h(t)$ be any strictly monotone differentiable function. Then for any constants a and b,*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n\{[h(t)-h(a)][h(t)-h(b)]\}f(t)\,dt$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\varphi_n(t-a)f(t)\,dt}{|h'(a)|\,|h(b)-h(a)|} + \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\varphi_n(t-b)f(t)dt}{|h'(b)|\,|h(b)-h(a)|}$$

$$= \left[\frac{f(a)}{|h'(a)|} + \frac{f(b)}{|h'(b)|}\right]|h(b)-h(a)|^{-1}.$$

LEMMA 4. *For $B_n^*(\theta)$ as defined in* (2.13), *using the notation of Section* 2, *if $|\theta| > c$ then*

$$\lim_{n\to\infty} B_n^*(\theta) = K \int_0^{\beta_0} \left\{\lim_{n\to\infty} \int_0^{\infty} -2\varphi_n(\omega)[|\theta| - r\gamma(r)\cos\beta]h(r,\beta)\,dr\right\} d\beta. \tag{A.1}$$

The proof of this lemma is greatly complicated by the fact that the function in braces in (A.1) has a singularity at $\beta = \beta_0$. This function is, however, integrable over $(0, \beta_0)$, which is the key to the proof. The detailed proof is quite lengthy, and only an outline is given here. The interested reader is referred to Hwang and Casella (1981).

OUTLINE OF PROOF. From (2.13), $B_n^*(\theta)$ is defined as

$$B_n^*(\theta) = K \int_0^{\pi} \int_0^{\infty} \left[\frac{\partial}{\partial|\theta|}\Phi(\omega)\right] h(r,\beta)\,dr\,d\beta$$

$$= K \int_0^{\pi} \int_0^{\infty} -2\varphi_n(\omega)[|\theta| - r\gamma(r)\cos\beta]h(r,\beta)\,dr\,d\beta.$$

We first show that, as $n\to\infty$ the value of the integral over the region $\{\beta:\beta>\beta_0\}$ goes to zero. It follows quickly that the integral over the region $\{\beta:\pi/2<\beta<\pi\}$ goes to zero since $\varphi_n(\omega) \le n(2\pi)^{-1/2}\exp\{-[n(|\theta|^2 - c^2)]^2/2\} \to 0$ as $n\to\infty$. Thus we only need consider $\{\beta:$

$\beta_0 < \beta < \pi/2\}$. For β in this region, $|\theta|^2 \sin^2\beta \geq c^2$ and hence

$$\begin{aligned}\varphi_n(\omega) &= \frac{n}{(2\pi)^{1/2}} \exp\{-n^2\omega^2/2\} \\ &\leq \frac{n}{(2\pi)^{1/2}} \exp\left\{-\frac{n^2}{2}[[r\gamma(r) - |\theta|\cos\beta]^4 + (|\theta|^2\sin^2\beta - c^2)^2]\right\} \\ &\overset{\text{def}}{=} \varphi_n{}^*(r, \beta),\end{aligned}$$

where ω is defined in (2.6). Now define

$$h^*(r) = \sup_\beta\{[|\theta| - r\gamma(r)\cos\beta]h(r, \beta)\} \tag{A.2}$$

which is clearly integrable over $(0, \infty)$. We then have

$$\begin{aligned} K\int_{\beta_0}^{\pi/2}\int_0^\infty &|-2\varphi_n(\omega)[|\theta| - r\gamma(r)\cos\beta]|\, h(r, \beta)\, dr\, d\beta \\ &\leq 2K\int_{\beta_0}^{\pi/2}\int_0^\infty \varphi_n^*(r, \beta)h^*(r)\, dr\, d\beta \qquad (A.3)\\ &= 2K\int_0^\infty\left\{\int_{\beta_0}^{\pi/2}\varphi_n^*(r, \beta)\, d\beta\right\}h^*(r)\, dr,\end{aligned}$$

where the last equality follows from Fubini's Theorem. It can be shown that there exists $M(|\theta|)$, independent of n and r, such that

$$\int_{\beta_0}^{\pi/2}\varphi_n^*(r, \beta)\, d\beta \leq M(|\theta|) \quad \forall r.$$

The Dominated Convergence Theorem is then applied to expression (A.3) to obtain

$$\lim_{n\to\infty}\int_0^\infty\left\{\int_{\beta_0}^{\pi/2}\varphi_n^*(r, \beta)\, d\beta\right\}h^*(r)\, dr = \int_0^\infty\left\{\lim_{n\to\infty}\int_{\beta_0}^{\pi/2}\varphi_n^*(r, \beta)\, d\beta\right\}h^*(r)\, dr. \tag{A.4}$$

Furthermore, it can be shown that

$$\lim_{n\to\infty}\int_{\beta_0}^{\pi/2}\varphi_n^*(r, \beta)\, d\beta = 0 \quad \forall r. \tag{A.5}$$

Hence, it has been demonstrated that

$$\lim_{n\to\infty} B_n^*(\theta) = K\lim_{n\to\infty}\int_0^{\beta_0}\int_0^\infty - 2\varphi_n(\omega)[|\theta| - r\gamma(r)\cos\beta]h(r, \beta)\, dr\, d\beta, \tag{A.6}$$

and it remains to show that the limit can be passed through the first integral. Using the fact that $h^*(r)$ is uniformly bounded by some constant, by the Dominated Convergence Theorem it will be sufficient to demonstrate that

$$\left|\int_0^\infty \varphi_n[\omega(r, \beta)]\, dr\right| < M_1 + M_2(|\theta|, \beta), \tag{A.7}$$

where $\int_0^{\beta_0} M_2(|\theta|, \beta)\, d\beta < \infty$. This can be established if we choose $M_2(|\theta|, \beta)$ to be a constant multiplied by $(c^2 - |\theta|^2\sin^2\beta)^{-1/2}$, which is integrable over $(0, \beta_0)$. □

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