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IMPROVING UPON STANDARD ESTIMATORS IN DISCRETE EXPONENTIAL FAMILIES WITH APPLICATIONS TO POISSON AND NEGATIVE BINOMIAL CASES

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Assume that $X_1, \cdots, X_p$ are independent random observations having discrete exponential densities $\rho_i(\theta_i)t_i(x_i)\theta_i^{x_i}$, $i = 1, \cdots, p$ respectively. A general technique of improving upon the uniform minimum variance unbiased estimator (UMVUE) of $(\theta_1, \cdots, \theta_p)$ is developed under possibly weighted squared error loss functions. It is shown that improved estimators can be constructed by solving a difference inequality.

Typical difference inequalities of a fairly general type are presented and solved. When specialized to Poisson and Negative binomial cases, broad classes of estimators are given that dominate the UMVUE. These results unify many known results in this rapidly diverging field, and some of them are new (especially those related to Negative Binomial distributions).

Improved estimators are also obtained for the problems in which some of the observations are from Poisson families and some from Negative Binomial families. For sum of squared errors loss, estimators which dominate the UMVUE in the discrete exponential families are also given explicitly.

1. Introduction. In Stein (1973, 1981), an identity (proven by integration by parts) was developed which has become a powerful tool in the problem of improving upon standard estimators. To be precise, assume that $\mathbf{X} = (X_1, \cdots, X_p)$ is a random vector which has a density $f(\mathbf{x} \mid \boldsymbol{\theta})$ with respect to a measure μ, and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$ is the unknown parameter that one tries to estimate based on $\mathbf{X}$. Under the loss function $L(\cdot, \cdot)$, the risk of the estimator $\boldsymbol{\delta}(\mathbf{X}) = (\delta_1(\mathbf{X}), \cdots, \delta_p(\mathbf{X}))$ for $\boldsymbol{\theta}$ is defined as $R(\boldsymbol{\theta}, \boldsymbol{\delta})$

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}) = \int L(\boldsymbol{\theta}, \boldsymbol{\delta}(\mathbf{x}))f(\mathbf{x} \mid \boldsymbol{\theta})\, d\mu(\mathbf{x}) = E_{\boldsymbol{\theta}}L(\boldsymbol{\theta}, \boldsymbol{\delta}(\mathbf{X}))$$

where $E_{\boldsymbol{\theta}}$ represents the expectation with respect to the density $f(\mathbf{x} \mid \boldsymbol{\theta})$. For a given estimator $\boldsymbol{\delta}^0$, the goal is to search for an estimator $\boldsymbol{\delta}^*(\mathbf{X})$, better than $\boldsymbol{\delta}^0(\mathbf{X})$, i.e., $R(\boldsymbol{\theta}, \boldsymbol{\delta}^*) \le R(\boldsymbol{\theta}, \boldsymbol{\delta}^0)$ for all $\boldsymbol{\theta}$ with strict inequality holding for some $\boldsymbol{\theta}$. Stein wrote $\boldsymbol{\delta}^*(\mathbf{X})$ as $\boldsymbol{\delta}^0(\mathbf{X}) + \boldsymbol{\Phi}(\mathbf{X})$, $\boldsymbol{\Phi}(\mathbf{X}) = (\Phi_1(\mathbf{X}), \cdots, \Phi_p(\mathbf{X}))$, and used his identity to obtain the representation

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^*) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = E_{\boldsymbol{\theta}}\mathscr{D}(\boldsymbol{\Phi}(\mathbf{X})) \tag{1.1}$$

where $\mathscr{D}(\boldsymbol{\Phi}(\mathbf{X}))$ is an expression that does not involve $\boldsymbol{\theta}$. Stein's idea was then to find $\boldsymbol{\Phi}(\mathbf{x})$ so that $\mathscr{D}(\boldsymbol{\Phi}(\mathbf{x})) < 0$, which clearly implies that the expression in (1.1) is negative for all $\boldsymbol{\theta}$ and hence $\boldsymbol{\delta}^*$ dominates $\boldsymbol{\delta}^0$ (i.e., $\boldsymbol{\delta}^*$ is better than $\boldsymbol{\delta}^0$). The expression $\mathscr{D}(\boldsymbol{\Phi}(\mathbf{x}))$ usually involves partial derivatives of Φ_i for continuous cases and partial differences of Φ_i for discrete cases. Brown (1979) also developed a general technique which relates the admissibility problems to differential inequalities. See also Brown (1971, 1975) and Berger (1976, a, b, c).

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In this paper, we develop a general technique for improving upon the uniform minimum variance unbiased estimator (UMVUE) of $\boldsymbol{\theta}$ when the observations X_i, $1 \le i \le p$, are independently obtained from discrete exponential families having densities

$$f(x_i \mid \theta_i) = \rho_i(\theta_i)t_i(x_i)\theta_i^{x_i}, \quad x_i = 0, 1, \cdots. \tag{1.2}$$

The loss functions considered are of the type

$$L_{\mathbf{m}}(\boldsymbol{\theta}, \boldsymbol{\delta}) = \sum_{i=1}^{p} \theta_i^{m_i}(\delta_i - \theta_i)^2, \tag{1.3}$$

where $\mathbf{m} = (m_1, \cdots, m_p)$ and m_i's are known integers. When $\mathbf{m} = (m, \cdots, m)$, $L_{\mathbf{m}}$ will be denoted by L_m^*. By solving a difference inequality $\mathscr{D}(\boldsymbol{\Phi}) \le 0$ of a general form (2.4), improved estimators are constructed by paralleling Stein's technique. Berger (1980) had obtained solutions to a differential inequality analogous to (2.4) and derived improved estimators over the standard one for continuous exponential families under L_m^*. See also Hudson (1978).

In discrete cases, improved estimators over the UMVUE have been found for some specific distributions. For Poisson families, Clevenson and Zidek (1975) obtained the results for L_{-1}^*, Peng (1975) for L_0^*, Tsui and Press (1978) for L_m^*, m a negative integer, and Tsui (1978 b) for $L_{\mathbf{m}}$, m_i's negative integers. For negative binomial distributions $NB(r_i, \theta_i)$ with density in (3.4), Hudson (1978) obtained an estimator dominating the UMVUE for $p \ge 4$ under L_0^* when all the r_i's are equal.

All the above results for discrete cases, except in Clevenson and Zidek (1975), were obtained by considering only a special difference inequality corresponding to the particular problem. (Clevenson and Zidek used an even more special feature of the Poisson distribution.)

Due to the generality of the difference inequality (2.4) considered in this paper, our proposed technique works for the general discrete exponential families. Not only does our technique generate many of the previous results when applied to their special cases, but also solves new concrete problems. For negative binomial families, with arbitrary known r_i's, our proposed estimators are shown to dominate the UMVUE under $L_{\mathbf{m}}$ for $p \ge 3$. Results are also given for the cases in which some of the observations are from Poisson families and some from negative binomial families. Also application to truncated Poisson distributions is direct, see section 4.2 (b). For arbitrary exponential families, we construct estimators that dominate the UMVUE for $p \ge 3$ under L_0^*. Note that all the previous results for this situation, by Hudson (1978) and Tsui (1979c) are based on very restrictive assumptions on $t_i(x_i)$.

2. The difference inequality and solution. Let $\mathbf{X} = (X_1, \cdots, X_p)$, X_i, $1 \le i \le p$, be p independent random variables having density $f_i(x_i \mid \theta_i)$ in (1.2) with $t_i(x_i) > 0$ if and only if $x_i = 0, 1, \cdots$. Then for any real valued function $\Phi(\mathbf{x})$ with $E_{\boldsymbol{\theta}} \mid \Phi(\mathbf{X}) \mid < \infty$, the following identity can be derived by using change of variables: If $\Phi(\mathbf{x}) = 0$ for $x_i < -m$, then

$$E_{\boldsymbol{\theta}}\theta_i^m\Phi(\mathbf{X}) = E_{\boldsymbol{\theta}}\{\Phi(\mathbf{X} - m\mathbf{e}_i)t_i(X_i - m)/t_i(X_i)\}. \tag{2.1}$$

The notation $\mathbf{e}_i$ denotes the ith coordinate vector whose ith component is one and the rest are zero. This identity, a generalization of those of Hudson (1974), and Tsui and Press (1978), was proved in Hwang (1979).

Now, under the loss function $L_{\mathbf{m}}$ as in (1.3), consider the problem of improving upon $\boldsymbol{\delta}^0(\mathbf{X})$ with ith component $\delta_i^0(\mathbf{X}) = t_i(X_i - 1)/t_i(X_i)$. It was shown in Roy and Mitra (1957) that $\delta_i^0(X_i)$ is the UMVUE of θ_i. Let $\boldsymbol{\delta}^*(\mathbf{X}) = \boldsymbol{\delta}^0(\mathbf{X}) + \boldsymbol{\Phi}(\mathbf{X})$ be a competitive estimator with finite risk for any $\boldsymbol{\theta}$. By using (2.1) it can be shown, as in Hwang (1979), that (1.1) is true with

$$\begin{aligned} &\mathscr{D}(\boldsymbol{\Phi}(\mathbf{x})) \\ &\quad = \sum_{i=1}^{p} \{[2t_i(x_i - m_i - 1)/t_i(x_i)]\Delta_i\Phi_i(\mathbf{x} - m_i\mathbf{e}_i) + [t_i(x_i - m_i)/t_i(x_i)]\Phi_i^2(\mathbf{x} - m_i\mathbf{e}_i)\}. \end{aligned} \tag{2.2}$$

Note that in the above and subsequent expressions, for any function $F(\mathbf{x})$, $\Delta_i F(\mathbf{x})$ denotes $F(\mathbf{x}) - \mathrm{F}(\mathbf{x} - \mathbf{e}_i)$.

We therefore have the following lemma.

LEMMA 2.1. *The estimator $\boldsymbol{\delta}^*(\mathbf{X})$ dominates $\boldsymbol{\delta}^0(\mathbf{X})$ if*

$$\mathscr{D}(\boldsymbol{\Phi}(\mathbf{x})) \le 0, \tag{2.3}$$

and form some $\mathbf{x}$, with $x_i \ge 0$ $i = 1, \cdots, p$, strict inequality holds in (2.3). □

In seeking for improved estimators, Lemma 2.1 will be used throughout this paper. The key to the problem is therefore the nontrivial solutions to (2.3). In this paper, we consider a slightly more general inequality,

$$\mathscr{D}(\boldsymbol{\psi}(\mathbf{x})) = \textstyle\sum_{i=1}^{p} v_i(x_i)\Delta_i\psi_i(\mathbf{x}) + w_i(\mathbf{x})\psi_i^2(\mathbf{x}) \le 0, \tag{2.4}$$

where v_i and w_i are nonnnegative and there exists α_i such that $v_i(x_i) > 0$ for $x_i \ge \alpha_i$. Inequality (2.4) relates to the problem of improving upon UMVUE as well as some other more general estimators; (see Hwang, 1979). All solutions to (2.4) will be of the forms

$$\psi_i(\mathbf{x}) = -c(\mathbf{x})h_i(x_i)/D, \quad i = 1, \cdots, p, \tag{2.5}$$

where

$$h_i(x_i) = \textstyle\sum_{k=\alpha_i}^{x_i} 1/v_i(k) \tag{2.6}$$

and

$$D = D(\mathbf{x}) = \textstyle\sum_{j=1}^{p} d_j(x_j). \tag{2.7}$$

Exact form of d_j would be specified in Theorem 2.1 and its corollaries. We will say $c(\mathbf{x})$ satisfies $\mathscr{C}(n, F(\mathbf{x}), \beta)$ for some function $F(\mathbf{x})$ and some numbers β and n, if both of the following conditions hold:

(i) $c(\mathbf{x}) \ne 0$ for some $\mathbf{x}$ and is nondecreasing in each coordinate,
(ii) $0 \le c(\mathbf{x}) \le n(F(\mathbf{x}) - \beta)^+$,

where $(a(\mathbf{x}))^+$ represents the positive part of $a(\mathbf{x})$. Throughout this paper, define, for any s_j, $\sum_{j=\alpha_1}^{\alpha_2} s_j = 0$ if $\alpha_2 < \alpha_1$. In particular, from (2.6) $h(x_i) = 0$ if $x_i < \alpha_i$. Further, we define $\#_\alpha(\mathbf{x})$, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_p)$ as the number of x_i's for which $x_i \ge \alpha_i$, and define $\#_\alpha^*(\mathbf{x}) = \#_\alpha(\mathbf{x})$ if $\boldsymbol{\alpha} = (\alpha, \cdots, \alpha)$. We are ready to state the main tool of this paper.

THEOREM 2.1. *Assume, for $i = 1, \cdots, p$, that* (i) *$d_i(\cdot)$ is a nondecreasing and nonnegative function, such that $d_i(x_i) > 0$ if $x_i \ge \alpha_i$,* (ii) *there exists a nonnegative integer β_i such that for any integer x_i,*

$$v_i(x_i)h_i(x_i - 1)\Delta_i\, d_i(x_i) \le \beta_i\, d_i(x_i - 1), \tag{2.8}$$

and (iii) *there exists a constant K such that*

$$\textstyle\sum_{i=1}^{p} w_i(\mathbf{x})h_i^2(x_i)/D \le K < \infty. \tag{2.9}$$

Then ψ, given componentwise in (2.5), *is a solution to* (2.4) *provided that $c(\mathbf{x})$ satisfies condition $\mathscr{C}(1/K, \#_\alpha(\mathbf{x}), \beta_{\max})$ where $\beta_{\max} = \max_{i \le i \le p} \beta_i$. Furthermore*

$$\mathscr{D}(\mathbf{x}) \le -c(\mathbf{x})(\#_\alpha(\mathbf{x}) - \beta_{\max} - Kc(\mathbf{x}))^+/D, \tag{2.10}$$

and strict inequality holds in (2.10) *for those $\mathbf{x}$ from which $h_i(x_i - 1)\Delta_i\, d_i(x_i) > 0$ for at least two i's.*

PROOF. See the Appendix. □

Condition (2.8) is not at all restrictive. Typically, the d_i that satisfies (2.8) has the form $h_i^{\beta_i}(x_i)$. The crucial condition (2.9) is then equivalent to the existence of K and β_i such that (2.9) is satisfied for $d_i(x_i) = h_i^{\beta_i}(x_i)$. If no matter how large β_i and K are, (2.9) is never satisfied, it is unlikely that Theorem 2.1 provides any solution. It is expected that all the admissible rules would correspond to such a situation.

We note that in the above theorem, nontrivial solutions are given only when $p > \beta_{\max}$. This is in accord with the fact that in some cases improvement is possible only for high dimension p, the well-known Stein's phenomenon (Stein, 1956).

The constant β_i will be chosen as small as possible for two reasons. First, nontrivial solutions then exist for smaller p. Second, $d_i(\cdot)$ will be smaller so that the improvement is larger (cf. (2.10), (2.2) and (1.1)). Of course, condition (2.9) will be harder to satisfy for such $d_i(x_i)$.

The following corollaries provide simple choices of d_i which will generate estimators of simple form and are more appealing.

COROLLARY 2.1.1. *If $\beta_i \geq 1$ is an integer and v_i is a nondecreasing function, then*

$$d_i(x) = h_i(x_i)h_i(x_i + 1) \cdots h_i(x_i + \beta_i - 1) \tag{2.11}$$

satisfies conditions (i) *and* (ii) *of Theorem* 2.1.

PROOF. By direct calculation and the fact

$$h_i(x_i + \beta_i - 1) - h_i(x_i - 1) \leq \frac{\beta_i}{v_i(x_i)}. \qquad \square$$

In applying Theorem 2.1 and Corollary 2.1.1 as well as the following corollaries, we assume first that $d_i(x_i)$ has the form $h_i^{\beta_i}(x_i)$, which is similar to but smaller than (2.11). We then choose β_i as small as possible so that (2.9) is satisfied for some K. If v_i is nondecreasing, then d_i in (2.11) will satisfy all the assumptions in Theorem 2.1, and will be used instead of $h_i^{\beta_i}(x_i)$. The idea is illustrated in the following example. The Poisson distribution with mean θ is denoted by $P_0(\theta)$. The loss L_0^* is used in Example 2.1. below.

EXAMPLE 2.1. Assume that X_i, $1 \leq i \leq p$, are independent $P_0(\theta_i)$ random variables. Hence $t_i(x_i) = (x_i!)^{-1}$ and the UMVUE of θ_i is $\delta_i^0(x_i) = x_i$. After dividing by 2 on both sides, (2.3) is equivalent to the inequality,

$$\frac{\mathscr{D}(\mathbf{\Phi}(\mathbf{x}))}{2} = \textstyle\sum_{i=1}^p x_i \Delta_i \Phi_i(\mathbf{x}) + \dfrac{\Phi_i^2(\mathbf{x})}{2} \leq 0. \tag{2.12}$$

To apply Theorem 2.1, take $v_i(x_i) = x_i$, $w_i(\mathbf{x}) = ½$ and hence $\alpha_i = 1$, $h_i(x_i) = \sum_{k=1}^{x_i} 1/k$. Let us first take d_i as $h_i^{\beta_i}(x_i)$. The smallest β_i so that (2.9) satisfies for such d_i is clearly 2. Theorem 2.1 and Corollary 2.1.1 therefore suggest taking $d_i(x_i) = h_i(x_i)h_i(x_i + 1)$ and imply that ψ_i as in (2.5) and (2.7) is a solution for any nondecreasing function $c(\mathbf{x})$, $0 \leq c(\mathbf{x}) \leq 2(\#_1^*(\mathbf{x}) - 2)^+$ and hence the estimator $\boldsymbol{\delta}^*(\mathbf{X}) = \boldsymbol{\delta}^0(\mathbf{X}) + \boldsymbol{\psi}(\mathbf{X})$ dominates $\boldsymbol{\delta}^0$ if $c(\mathbf{x}) \not\equiv 0$ and $p \geq 3$.

If $c(\mathbf{x})$ is taken to be $(\#_1^*(\mathbf{x}) - 2)^+$, the estimator is quite similar to the improved estimator proposed by Peng (1975). The only difference is that Peng used $d_i(x_i) = h_i^2(x_i)$. $\square$

The following corollaries provide other choices of $d_i(x_i)$. The proof of Corollary 2.1.2 is a simple application of mean value Theorem and that of Corollary 2.1.3 is a direct calculation. Both proofs are omitted.

COROLLARY 2.1.2. *The function $d_i(x_i) = h_i^{\beta_i}(x_i)$ satisfies condition* (i) *and* (ii) *of Theorem* 2.1, *provided* $0 \leq \beta_i \leq 1$. $\square$

COROLLARY 2.1.3. *If there exists a constant b_i for which $(v_i(x_i))^{-1} \leq b_i$ for all $x_i \geq \alpha_i$*

$+ 1$, *then* $d_i(x_i) = h_i^2(x_i) + b_i h_i(x_i)$ *satisfies condition* (i) *and* (ii) *of Theorem* 2.1 *with* $\beta_i = 2$. □

For $\beta_i > 2$, results analogous to Corollary 2.1.3 can be established by using the Mean Value Theorem.

3. Classes of improved estimators. In this section, we apply Theorem 2.1 and Corollaries in Section 2 to solve the difference inequality (2.3) (with $\mathscr{D}$ in (2.2)) in a way similar to Example 2.1. Improved estimators for this most general situation are therefore constructed.

To solve (2.2) and (2.3), it is equivalent to solve the following inequality

$$\sum_{i=1}^{p} \left\{ \left[\frac{t_i(x_i - m_i - 1)}{t_i(x_i)} \right] \Delta_i \psi_i(\mathbf{x}) + \left[\frac{t_i(x_i - m_i)}{2t_i(x_i)} \right] \psi_i^2(\mathbf{x}) \right\} \le 0 \tag{3.1}$$

where $\psi_i(\mathbf{x}) = \Phi_i(\mathbf{x} - m_i \mathbf{e}_i)$. Let us take $\alpha_i = (m_i + 1)^+$, and

$$h_i(x_i) = \sum_{k=\alpha_i}^{x_i} \frac{t_i(k)}{t_i(k - m_i - 1)}. \tag{3.2}$$

Let η_{ij} be the Kronecker constant, i.e., $\eta_{ij} = 1$ if $i = j$ and $\eta_{ij} = 0$ if $i \neq j$. We then have the following theorem.

THEOREM 3.1. *Suppose that* $d_i(x_i)$*'s are one of the forms given in the corollaries in Section* 2 *and there exists* K_i *such that* $t_i(x_i - m_i) h_i^2(x_i)/t_i(x_i) \le K_i d_i(x_i)$ *for all* $x_i \ge 0$. *If* $p \ge 2$ *and* $p \ge \beta_{\max}$, *then* $\boldsymbol{\delta}^0 + \mathbf{\Phi}$ *dominates* $\boldsymbol{\delta}^0$ *under* L_m, *where, for some positive numbers* $a_1, \cdots, a_p$,

$$\Phi_i(\mathbf{X}) = -c(\mathbf{X} + m_i \mathbf{e}_i) h_i(X_i + m_i) / \sum_{j=1}^{p} a_j d_j(X_j + m_i \eta_{ij}) \tag{3.3}$$

and $c(\mathbf{x})$ *satisfies condition* $\mathscr{C}(2(\max K_i/a_i)^{-1}, \#_\alpha(\mathbf{x}), \beta_{\max})$.

PROOF. Since $d_i(x_i)$ satisfies (2.8), so does $a_i d_i(x_i)$. By Theorem 2.1 and the corollaries in Section 2, $\psi_i(\mathbf{x}) = \Phi_i(\mathbf{x} - m_i \mathbf{e}_i)$ is a solution to (3.1). Lemma 2.1 then completes the proof. □

A natural criterion for selecting $c(\mathbf{x})$ and a_i is to minimize the upper bound in (2.10) for each $\mathbf{x}$ so that the improvement of the proposed estimator over $\boldsymbol{\delta}^0$ might be maximized. This leads to the choice that $c(\mathbf{x}) = (\#_\alpha(\mathbf{x}) - \beta_{\max})^+$ and $a_i = K_i$.

According to Theorem 3.1, the choice of d_i and h_i (and hence β_i and K_i) is independent of the other coordinates $j \neq i$. Furthermore, the rule (3.3) to combine these functions h_i and d_i to get the improved estimators is the same no matter what estimation problems are considered. Therefore to describe the improved estimators, all we need to do is to specify the "elements" h, α, β, and K. We specialize our results to Poisson distributions and Negative Binomial distributions in what follows.

The Negative Binomial distribution NB(r, θ) considered has the density

$$f(x \mid \theta) = C(r + x - 1, r - 1)\, \theta^x (1 - \theta)^r, \quad x = 0, 1, \cdots \tag{3.4}$$

where $C(n_1, n_2) = n_1!/\{(n_1 - n_2)! n_2!\}$ and r is a known positive integer. The density is clearly a special case of (1.2), with $t(x) = C(r + x - 1, r - 1)$. Tables 1 and 2 specify the "elements" h, β, K, and the recommended d, which satisfy Theorem 3.1 for Poisson and Negative Binomial distributions. Note in either case, the UMVUE is admissible for $p = 1$ under squared error loss; see Hodges and Lehman (1975) for the Poisson case, and Blackwell and Girshick (1954, p. 307), for the other. However, for higher p, the improved estimators over the UMVUE are obtained.

Improved estimators can be constructed for a variety of problems by applying Theorem 3.1 and Tables 1 and 2. As an illustration, assume that X_1, X_2 and X_3 are independent

TABLE 1
Specification of the functions d that satisfy Theorem 3.1 when $X \sim Po(\theta)$ and $\delta^0(X) = X$.

	α	β	$h(x)$	$d(x)$	K
$m = 0$	1	2	$\sum_{k=1}^{x} (1/k)$	$h(x)h(x+1)$	1
$m < 0$	0	1	$(x+1)\cdots(x-m)/(-m)$	$h(x)$	$1/(-m)$

TABLE 2
Specification of the functions d that satisfy Theorem 3.1 when $X \sim NB(r, \theta)$ and $\delta^0(X) = X/(X + r - 1)$

	α	β	$h(x)$
$m = 0$	1	2	$\sum_{k=1}^{x} (r-1+k)/k$
$m = -1$	0	2	$x + 1$
$m < -1$	0	2	$\sum_{k=0}^{x} \frac{(k+1)\cdots(k-m-1)}{(k+r)\cdots(k-m+r-2)}$
$m > 0$	$m + 1$	2	$\sum_{k=m+1}^{x} \frac{(r-1+k-m)\cdots(r-1+k)}{(k-m)\cdots k}$

	$d(x)$	K
$m = 0$	$h^2(x) + (1+r)h(x)/2$	1
$m = -1$	$h^2(x) + h(x)$	r
$m < -1$	$h^2(x) + h(x)$	$C(r-m-1, r-1)$
$m > 0$	$h^2(x) + \frac{(r+1)\cdots(m+r+1)}{(m+2)!} h(x)$	1

random variables having the distributions $P_0(\theta_1)$, NB(2, θ_2), and NB(5, θ_3) respectively. Consider the problem of improving the UMVUE $\boldsymbol{\delta}^0(X_1, X_2, X_3) = (X_1, X_2/(X_2+1), X_3/(X_3+4))$ of $\boldsymbol{\theta}$ under the loss function $L_{\mathbf{m}}$, $\mathbf{m} = (0, -1, 2)$. Since $m_1 = 0$, from Table 1, $\alpha_1 = 1$, $\beta_1 = 2$, $h_1(x_1) = \sum_{k=1}^{x_1} (1/k)$, $d_1(x_1) = h_1(x_1)h_1(x_1+1)$, and $K_1 = 1$. From Table 2 and the fact that $m_2 = -1$ and $r_2 = 2$, we have $\alpha_2 = 0$, $\beta_2 = 2$, $h_2(x_2) = x_2 + 1$, $d_2(x_2) = h_2^2(x_2) + h_2(x_2)$, and $K_2 = 2$. Finally, $m_3 = 2$, $r_3 = 5$ and from Table 2, $\alpha_3 = 3$, $\beta_3 = 2$,

$$h_3(x_3) = \sum_{k=3}^{x_3} \frac{(k+2)(k+3)(k+4)}{(k-2)(k-1)k}, \qquad d_3(x_3) = h_3^2(x_3) + \frac{6\cdot 7\cdot 8}{4!} h_3(x_3),$$

and $K_3 = 1$. Since $p = 3 > 2 = \beta_{\max}$, $\boldsymbol{\delta}^0 + \boldsymbol{\Phi}$ dominates $\boldsymbol{\delta}^0$ with $\boldsymbol{\Phi}$ being given componentwise in (3.3), if $c(\mathbf{x})$ and a_i satisfy the assumptions in Theorem 3.1. The recommended choice for (a_1, a_2, a_3) is (1, 2, 1) and that for $c(\mathbf{x})$ is $(\#_\alpha(\mathbf{x}) - 2)^+$.

In constructing Table 1, d is chosen according to Corollary 2.1.1 for $m = 0$ and Corollary 2.1.2 for $m < 0$. In Table 2, Corollary 2.1.3 is the guide to choose d. For the Poisson distribution and $m > 0$, we fail to construct the function d so that it satisfies (2.8) and (2.9) no matter how large β we choose. In fact, it is conjectured in Hwang (1979) that $\boldsymbol{\delta}^0(\mathbf{x}) = \mathbf{x}$ is admissible under L_m^*, $m > 0$ for any p.

For the case in which all the independent observations $X_1, \cdots, X_p$ have Poisson distributions and the loss function is $L_{\mathbf{m}}$, $m_i < 0$ for all i, Tsui (1978b) obtained a class of estimators that dominates $\boldsymbol{\delta}^0(\mathbf{X}) = \mathbf{X}$. Part of his improved estimators can be derived from Theorem 3.1 and Table 1 (with $a_i = -m_i$). Tsui's results, however, allow a larger upper bound on $c(\mathbf{x})$ for the case in which not all m_i's are identical.

TABLE 3
Specification of $\boldsymbol{\delta}^P$ as in (3.5).

L_m^*	$c(\mathbf{x})$ satisfies condition	$h(x)$	$d(x)$	p
$m=0$	$\mathscr{C}(2, \#_1^*(\mathbf{x}), 2)$	$\sum_{k=1}^{x}(1/k)$	$h(x)h(x+1)$	$p \geq 3$
$m<0$	$\mathscr{C}(-2m, p, 1)$	$(x+1)\cdots(x-m)$	$h(x)$	$p \geq 2$

TABLE 4
Specification of $\boldsymbol{\delta}^{NB}$ as in (3.6), $p \geq 3$.

L_m^*	$c(x)$ satisfies condition	b_i
$m=0$	$\mathscr{C}C(2, \#_1^*(\mathbf{x}), 2)$	$(1+r_i)/2$
$m=-1$	$\mathscr{C}(n_1, p, 2)$	1
$m<-1$	$\mathscr{C}(n_2, p, 2)$	1
$m>0$	$\mathscr{C}(2, \#_{m+1}^*(\mathbf{x}), 2)$	$(r_i+1)\cdots(m+r_i+1)/(m+2)!$

Theorem 3.1, Tables 1 and 2 show that Stein's effect does not rely on the symmetry of the problems. (i.e. the m_i's are all equal and the X_i's have the same family of distributions.) This phenomenon was first pointed out in Berger (1980) for the continuous case. Tsui (1978b) also had an example in the Poisson case.

Of course, the most practical situations arise when the loss function is L_m^*, and the independent observations come from one family of distributions. For these cases, we summarize the improved estimators in Tables 3 and 4. The results follow mainly from Theorem 3.1 and Tables 1 and 2. When $\alpha_i = 0$, $i = 1, \cdots, p$, (by Lemma 3.1 in Hwang, 1979) Theorem 3.1 is also true if $\mathscr{C}(2(\max K_i/a_i)^{-1}, \#_\alpha(\mathbf{x}), \beta_{\max})$ is replaced by $\mathscr{C}(2(\max K_i/a_i)^{-1}, p, \beta_{\max})$. The natural choice of a_i for this symmetric case is $a_i = 1$, $1 \leq i \leq p$. With this choice of a_i, the estimator based on independent Poisson observations that dominates $\boldsymbol{\delta}^0(\mathbf{X}) = \mathbf{X}$ under L_m^* is $\boldsymbol{\delta}^P = (\delta_1^P, \cdots, \delta_p^P)$ where

$$\delta_i^P(\mathbf{x}) = x_i - \frac{c(\mathbf{x} + m\mathbf{e}_i)h(x_i + m)}{\sum_{j=1}^{p} d(x_j + m\eta_{ij})}. \tag{3.5}$$

The specification of $c(\mathbf{x})$, $h(x)$, $d(x)$, and p are given in Table 3.

Under L_0^*, the relation between present work and Peng's result was discussed in Example 2.1.

Under L_{-1}^*, the improved estimators in Clevenson and Zidek (1975) are special forms of $\boldsymbol{\delta}^P$ with $c(\mathbf{x})$ depending on $\mathbf{x}$ only through $\sum_{i=1}^{p} x_i$. Note that Tsui's improved estimators (1978b), when restricting to L_{-1}^* are the same as those in Clevenson and Zidek (1975) and hence are, again, the special forms of $\boldsymbol{\delta}^P$. Larger class of improved estimators are also obtained in Tsui and Press (1978).

Under L_m^*, $m < 0$, part of the improved estimators obtained in Tsui and Press (1978) are included in the class of $\boldsymbol{\delta}^P$'s, with $c(\mathbf{x})$ depending on $\mathbf{x}$ only through $\sum_{i=1}^{p} x_i$. They also obtained other estimators which are not included here, having similar form as $\boldsymbol{\delta}^P$.

Table 4, similar to Table 3, is established for the case in which X_i has NB(r_i, θ_i), $i = 1, \cdots, p$, $p \geq 3$. We take again $a_i = 1$. The estimators that dominate $\boldsymbol{\delta}^0(\mathbf{X})$, with $\delta_i^0(x_i) = x_i/(x_i + r_i - 1)$, are therefore of the form $\boldsymbol{\delta}^{\mathrm{NB}} = (\delta_1^{\mathrm{NB}}, \cdots, \delta_p^{\mathrm{NB}})$, where

$$\delta_i^{\mathrm{NB}}(\mathbf{x}) = \delta_i^0(x_i) - \frac{c(\mathbf{x} + m\mathbf{e}_i)h_i(x_i + m)}{\sum_{j=1}^{p}\{h_j^2(x_j + m\eta_{ij}) + b_j h_j(x_j + m\eta_{ij})\}}. \tag{3.6}$$

The functions h_i's are the same as in Table 2 with r substituted by r_i and hence are not included in Table 4. Note we define $n_1 = \min 2/r_i$ and $n_2 = \min 2/C(r_i - m - 1, r_i - 1)$.

Hudson (1978) obtained an estimator which dominates δ^0 under the loss function L_0^* only for $p \geq 4$ and identical r_i's. Hudson's improved estimator δ^H has a correction term as given componentwise in (3.6) with $b_i = m = 0$ and $c(\mathbf{x}) = (\#_1^*(\mathbf{x}) - 3)^+$, an estimator not included in our class of improved estimators δ^{NB}. Our δ^{NB}, requiring $b_i = (1 + r_i)/2$, looks more complicated, but is designed for more general cases (including the situation that $p = 3$, r_i's not all equal and $m \neq 0$.) Meanwhile, for $m = 0$, Table 4 allows us to choose $c(\mathbf{x}) = (\#_1^*(\mathbf{x}) - 2)^+$ which, according to Hwang (1979) is more appropriate than Hudson's choice.

For the loss function L_0^*, a very general result is established. The fact that $f(x_i \mid \theta_i)$ as in (1.2) is a discrete density function for some $\theta_i^* > 0$, implies that

$$\frac{t_i(k)}{t_i(k-1)} \leq M_i \tag{3.7}$$

for some M_i and all positive integer k. This follows from the application of ratio test to the series $\sum_{k=0}^{\infty} t_i(k)(\theta_i^*)^k < \infty$. Therefore Corollary 2.1.3 is applicable, which, together with Theorem 3.1, implies the following theorem. Let $h_i^0(x_i)$ be as in (3.2) with $m_i = 0$ and hence $\alpha_i = 1$.

THEOREM 3.2. *If* $p \geq 3$, δ^0 *is inadmissible under* L_0^*: Let $c(\mathbf{x})$ *satisfy condition* $\mathscr{C}(2, \#_1^*(\mathbf{x}), 2)$ *and* M_i *be such that* (3.7) *holds. Then* $\delta^0(\mathbf{X})$ *is dominated by* $\delta^0(\mathbf{X}) + \mathbf{\Phi}^*(\mathbf{X})$ *where* $\mathbf{\Phi}^* = (\Phi_1^*, \cdots, \Phi_p^*)$,

$$\Phi_i^*(\mathbf{x}) = -c(\mathbf{x}) h_i^0(x_i) / \sum_{j=1}^{p} \{[h_j^0(x_j)]^2 + M_j h_j^0(x_j)\}. \tag{3.8}$$ □

From Peng (1975), it is known that the dimension requirement $p \geq 3$ is the weakest possible. In Hudson (1978), and Tsui (1979c), under fairly restrictive conditions (e.g. $t_i = t$ and $t(x)/t(x-1)$ is increasing, etc.), the UMVUE was shown to be inadmissible. Earlier, Brown (1966) had proved the inadmissibility of the best invariant estimator of a location parameter under very general loss function if $p \geq 3$. However, the problem we address here is not invariant in any natural sense.

In the case of simultaneously estimating the means of independent binomial populations, Johnson (1971) has shown that Stein's phenomenon (1956) (i.e., the surprising fact that there exists an inadmissible estimator with each coordinate being componentwise admissible), does not appear. The fact that the sample space has infinitely many points seems to be crucial to the existence of Stein's phenomenon.

4. Comments and generalizations.

4.1. *Comments.* One advantage in providing a whole class of estimators is that they might include some admissible estimators. (A little modification of $h(x)$ and $d(x)$ might be needed.) In two cases, admissible estimators are successfully found through a similar procedure. First, in the estimation of the mean of a multivariate normal population, with known covariance matrix, broad class of improved estimators that include the one in James and Stein (1961) was obtained in Baranchick (1964). Later Strawderman (1971) and Berger (1976b) generalized Baranchick's (1964) class and succeeded in showing the admissibility of some of their estimators. Second, under L_{-1}^* and $p \geq 2$, admissible estimators that dominate UMVUE were first obtained in Clevenson and Zidek (1975) which are the improved estimators developed in Table 3 with appropriate choice of $c(\mathbf{x})$. Our class of improved estimators are broad in Baranchick's (1964) spirit and might be useful in the search of admissible improved estimators for other situations.

The improved estimator could estimate the parameter by negative number which is outside the parameter space. We can (and should) replace the negative estimate by zero which clearly makes the improved estimator even better.

It is difficult to calculate analytically the improvement in risk when the improved estimators are used. For Poisson families, numerical studies were performed in Tsui and Press (1978), Tsui (1978a, 1979d), and Clevenson and Zidek (1975). In these cases, significant reduction of risk is gained by using the improved estimators. Our class of improved estimators for Poisson families include those estimators (or similar ones), presented by the authors mentioned above. Therefore, with recommended choice of $c(\mathbf{x})$, our estimators are expected to perform well in other different cases. Of course, further numerical study is needed to confirm this especially for Negative Binomial families.

The estimators presented in this paper are all shrinking the UMVUE toward the origin and therefore are expected to perform the best when the unknown parameters are close to zero. For the case when the parameters are expected to be very large, the improved estimators will be close to the UMVUE with high probability and little gain is expected. In such a case an improved estimator which shrinks toward some nonzero point is desirable. Some authors (Tsui, 1978a, 1979c; Hudson and Tsui, 1981) have succeeded in this direction for Poisson observations under loss function L_0^*. For more general cases, improved estimators shrinking toward some prefixed point (other than the origin) can be obtained by modifying Theorem 2.1 and will be presented in a forthcoming paper.

4.2. Generalizations. There are many other possible generalizations of the results of this research. Two of them are discussed here.

(a) All the results can be easily extended to the loss functions of the form $L_{\mathbf{m}}^{\mathbf{n}}(\boldsymbol{\theta},\boldsymbol{\delta}) = \sum_{i=1}^{p} n_i\theta_i^{m_i}(\theta_i-\delta_i)^2$, where $n_1, \cdots, n_p$ are some positive constants. We refer to Tsui (1979a) for the motivation of such a loss function.

There are two ways to deal with loss function $L_{\mathbf{m}}^{\mathbf{n}}$:

(1) Include these constants $n_1, \cdots, n_p$, in the difference inequality and solve it. Clearly, nontrivial solutions to the difference inequality can be obtained by using theorems in Section 2, if and only if the difference inequality corresponding to the loss function L_m can be solved.

A theorem similar to Theorem 3.1 can thus be established for $L_{\mathbf{m}}^{\mathbf{n}}$. In fact, if α_i, β_i, k_i, h_i, and d_i are as specified in the assumptions of Theorem 3.1, then $\boldsymbol{\delta}^0 + \boldsymbol{\Phi}$ dominates $\boldsymbol{\delta}^0$ under $L_{\mathbf{m}}^{\mathbf{n}}$ provided that Φ_i is as in (3.3) with h_i being replaced by h_i/n_i, and $\mathscr{C}(2(\max k_i/a_i)^{-1}, \#\alpha(\mathbf{x}), \beta_{\max})$ by $\mathscr{C}(2[\max k_i/(a_in_i)]^{-1}, \#\alpha(\mathbf{x}), \beta_{\max})$.

(2) Apply the results of Berger (1977), in which the problem is decomposed into p subproblems under the loss functions $\sum_{i=1}^{j}\theta_i^{m_i}(\theta_i-a_i)^2, j = 1, \cdots, p$. Improved estimators can be found for the original problems, once improved estimators are found under at least one of the subproblems.

(b) All the distributions of $X_1, \cdots, X_p$ considered in this work were assumed to be as in (1.2) with $t_i(x_i) > 0$ if and only if $x_i = 0, 1, \cdots$. For the case $t_i(x_i) > 0$ if and only if $x_i = a_i$, $a_i + 1, \cdots$, for some integer a_i, a simple transformation

$$X_i' = X_i - a_i$$

will make our results applicable to the equivalent estimation problem based on $X_1', \cdots$, X_p'. One particular example of interest is the truncated Poisson distributions.

APPENDIX

PROOF OF THEOREM 2.1. Assume that $\#_\alpha(\mathbf{x}) > \beta_{\max}$, since otherwise the theorem is trivial. The monotonicity of $c(\mathbf{x})$ with respect to each coordinate implies that

$$\Delta_i\psi_i(\mathbf{x}) \le c(\mathbf{x})\Delta_i\left[\frac{-h_i(x_i)}{D}\right].$$

Then, with $D_i = D(\mathbf{x}-\mathbf{e}_i)$ and $D' = \sum_{i=1}^{p} d_i(x_i-1)$,

$$\text{(A.1)} \qquad \Delta_i\left(\frac{-h_i(x_i)}{D}\right) = \frac{-\Delta_i h_i(x_i)}{D} + \frac{h_i(x_i-1)\Delta_i D}{DD_i}$$

and hence

$$\text{(A.2)}\qquad \begin{aligned}\textstyle\sum_{i=1}^{p} v_i(\mathbf{x}_i)\Delta_i\psi_i(\mathbf{x}) &\le \frac{c(\mathbf{x})}{D}\left(-\#_\alpha(\mathbf{x}) + \textstyle\sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i D}{D_i}\right)\\ &\le \frac{c(\mathbf{x})}{D}\left(-\#_\alpha(\mathbf{x}) + \textstyle\sum_{i=1}^{p}\frac{v_i(x_i)h_i(x_i-1)\Delta_i d_i(x_i)}{D'}\right).\end{aligned}$$

In the last transition, the inequality is actually strict for those $\mathbf{x}$ for which $c(\mathbf{x}) \neq 0$ and two of the x_i's satisfying $h_i(x_i - 1)\Delta_i d_i(x_i) > 0$. It follows, from (A.2) and (2.8), that

$$\text{(A.3)}\qquad \textstyle\sum_{i=1}^{p} v_i(x_i)\Delta_i\psi_i(\mathbf{x}) \le \frac{c(\mathbf{x})}{D}(\beta_{\max} - \#_\alpha(\mathbf{x})).$$

By (2.9), it is clear that $\sum_{i=1}^{p} w_i(\mathbf{x})\psi_i^2(\mathbf{x}) \le Kc^2(\mathbf{x})/D$, which, together with (A.3), implies that

$$\mathscr{D}(\mathbf{x}) \le \frac{c(\mathbf{x})}{D}(Kc(\mathbf{x}) + \beta_{\max} - \#_\alpha(\mathbf{x})).$$

Since by condition $\mathscr{C}(1/K, \#_\alpha(\mathbf{x}), \beta_{\max})$,

$$c(\mathbf{x})(Kc(\mathbf{x}) + \beta_{\max} - \#_\alpha(\mathbf{x})) = -c(\mathbf{x})(\#_\alpha(\mathbf{x}) - \beta_{\max} - Kc(\mathbf{x}))^+ \le 0,$$

the theorem is established. □

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