The Annals of Statistics
1981, Vol. 9, No. 4, 893–898

SOME CLASSES OF OPTIMALITY CRITERIA AND OPTIMAL DESIGNS FOR COMPLETE TWO-WAY LAYOUTS

BY N. GAFFKE

RWTH Aachen, West-Germany

For a given class of linear models in standard form an optimal experimental design is to be chosen for estimating some linear functions of the unknown parameters. An optimality criterion is defined to be a real function of the covariance matrices of the Gauss-Markov estimators. Conditions which are imposed on the criteria are monotonicity, quasiconvexity or quasiconcavity, and invariance or order-invariance. A characterization of the D-criterion by order-invariance is included which strengthens a result of P. Whittle. In the main part of the paper optimal designs for the usual two-way layouts in ANOVA are computed for large classes of optimality criteria. Some related optimization problems are solved with the technique of majorization of vectors in the sense of Schur.

1. Introduction. Let Δ be a set of experimental designs for linear models

$$EY_d = X_d a, \qquad \text{Cov } Y_d = \sigma^2 I_n, \tag{1}$$

where Y_d is the n-dimensional vector of (real) observations, X_d is a known $n \times k$ matrix, $a \in \mathbb{R}^k$ is a vector of unknown parameters, $\sigma^2 > 0$ is known or unknown and I_n is the $n \times n$ identity matrix. Let K be a given $s \times k$ matrix of rank s, and assume that Ka is estimable under each $d \in \Delta$, i.e., for each $d \in \Delta$ there exists a matrix U_d with $K = U_d X_d$. Then for the BLUE $K\hat{a}_d$

$$V_d = \sigma^{-2} \text{ Cov } K\hat{a}_d = K(X_d^T X_d)^- K^T, \tag{2}$$

where A^T and A^- denote respectively the transpose and a generalized inverse of A. Let Ψ be a real function on the set $\mathscr{P}_s$ of all positive definite $s \times s$ matrices. A design $d^* \in \Delta$ is called Ψ-optimal in Δ for estimating Ka if

$$\Psi(V_{d^*}) = \min_{d \in \Delta} \Psi(V_d). \tag{3}$$

Ψ is called an optimality criterion. It is sometimes convenient to consider the inverses of V_d. Trivially (3) can be written as $\Phi(V_{d^*}^{-1}) = \min_{d \in \Delta} \Phi(V_d^{-1})$, where throughout this paper Φ denotes the function $\Phi(A) = \Psi(A^{-1})$, $A \in \mathscr{P}_s$ (for a given Ψ). Two optimality criteria Ψ and Ψ' are called equivalent if they represent the same order-relation on $\mathscr{P}_s$, the order-relation $\lhd$ represented by Ψ being defined by $A \lhd B$ iff $\Psi(A) < \Psi(B)$, $A, B \in \mathscr{P}_s$.

2. Classes of optimality criteria.

A. *Monotonicity.* For nonnegative definite $s \times s$ matrices A and B we write $A \leq B$ if $B - A$ is nonnegative definite. An optimality criterion is called increasing if $A \leq B$, $A \neq B$, $A, B \in \mathscr{P}_s$ imply $\Psi(A) \leq \Psi(B)$, and strictly increasing, if always $\Psi(A) < \Psi(B)$. Since for A, $B \in \mathscr{P}_s$ the relations $A \leq B$ and $B^{-1} \leq A^{-1}$ are equivalent, Ψ is (strictly) increasing iff Φ is (strictly) decreasing. If Ψ is an orthogonal invariant criterion

$$\Psi(A) = f(\lambda_1(A), \cdots, \lambda_s(A)), \qquad A \in \mathscr{P}_s, \tag{4}$$

where $\lambda_i(A)$ are the eigenvalues of A, and f is a symmetric function on $(0, \infty)^s$, then Ψ is

Received April, 1979; revised September, 1980.
AMS 1970 *subject classifications.* Primary 62K05; secondary 90C10.
Key words and phrases. Linear model, BLUE, optimal design, two-way layout, majorization.

(strictly) increasing iff f is (strictly) increasing with respect to the componentwise partial ordering on $(0, \infty)^s$ (see e.g. Marshall and Olkin (1979, page 475)).

B. *Convexity.* An optimality criterion Ψ is called quasiconvex, if for all $A, B \in \mathscr{P}_s$, $A \neq B$, and $\alpha \in (0, 1)$

$$\Psi(\alpha A + (1 - \alpha)B) \leq \max(\Psi(A), \Psi(B)), \tag{5}$$

and Ψ is called strictly quasiconvex if there is always strict inequality in (5). Ψ is called (strictly) quasiconcave if $-\Psi$ is (strictly) quasiconvex. From the well-known inequality

$$(\alpha A + (1 - \alpha)B)^{-1} \leq \alpha A^{-1} + (1 - \alpha)B^{-1}, \qquad A, B \in \mathscr{P}_s, \alpha \in (0, 1)$$

it follows that for an increasing criterion Ψ quasiconvexity of Ψ implies quasiconvexity of Φ, but the converse is not true. For a Ψ in (4) quasiconvexity of Ψ is equivalent with quasiconvexity of f, which can be seen as in Kiefer (1974, page 862).

C. *Invariance.* Let G be a group of nonsingular $s \times s$ matrices with respect to matrix multiplication. We say that an optimality criterion Ψ is G-order-invariant if $\Psi(A) < \Psi(B)$ always implies $\Psi(LAL^T) < \Psi(LBL^T)$ for all $A, B \in \mathscr{P}_s$, $L \in G$. This means that Ψ-optimality of a design d^* for estimating Ka always implies Ψ-optimality of d^* for estimating LKa, $L \in G$ (see equation (2), (3) of Section 1). For a compact group G there is no difference between G-order-invariance and G-invariance:

LEMMA 1. *Let G be a compact group of nonsingular $s \times s$ matrices. If Ψ is measurable and G-order-invariant, then Ψ is G-invariant.*

PROOF. First replace Ψ by an equivalent criterion Ψ' which is bounded (and measurable). Let μ be the right, normalized Haar measure on G (cf. Halmos (1961, Chapter XI)). Then defining

$$\Psi''(A) = \int_G \Psi'(LAL^T)\, d\mu(L), \qquad A \in \mathscr{P}_s,$$

we obtain an equivalent criterion Ψ'' which is G-invariant. Hence Ψ must be G-invariant. □

Order invariance with respect to the largest group $\mathscr{R}_s$ of all nonsingular $s \times s$ matrices characterizes the D-criterion (or its converse). The following theorem strengthens a result of Whittle (1973, Theorem 2).

THEOREM 1. *If $\Psi \not\equiv$ const. is measurable and $\mathscr{R}_s$-order-invariant, then either Ψ or $-\Psi$ is equivalent to the D-criterion.*

PROOF. Considering the subgroup of all orthogonal matrices, Lemma 1 shows that Ψ is of type (4). Furthermore f is such that

$$f(x_1, \cdots, x_s) < f(y_1, \cdots, y_s) \Rightarrow f(z_1 x_1, \cdots, z_s x_s) < f(z_1 y_1, \cdots, z_s y_s) \tag{6}$$

$$\text{for all} \quad x, y, z \in (0, \infty)^s.$$

Let $x, y \in (0, \infty)^s$ be given with $\prod_{i=1}^s x_i = \prod_{i=1}^s y_i$. Putting

$$\beta_i = \beta_1 \prod_{j=1}^{i-1} x_j y_{j+1}^{-1}, \qquad 2 \leq i \leq s,$$

$\beta_1 > 0$ arbitrary, we have $\beta_i x_i = \beta_{\pi(i)} y_{\pi(i)}$, $1 \leq i \leq s$, where $\pi(i) = i + 1 \pmod s$. Hence $f(\beta_1 x_1, \cdots, \beta_s x_s) = f(\beta_1 y_1, \cdots, \beta_s y_s)$, and by (6) $f(x) = f(y)$. So $f(x) = g(\prod_{i=1}^s x_i)$ for some real function g on $(0, \infty)$, and $\Psi(A) = g(\det A)$, $A \in \mathscr{P}_s$. Clearly g is measurable. It remains to show that g is either strictly increasing or strictly decreasing. The function $h(r) = g(e^r)$, $r \in \mathbb{R}$, is such that

$$h(r) < h(t) \Rightarrow h(r+u) < h(t+u) \qquad \text{for all} \quad r, t, u \in \mathbb{R}. \tag{7}$$

Consider the sets $S_> = \{r \in \mathbb{R} : h(r) > h(0)\}$ and $S_< = \{r \in \mathbb{R} : h(r) < h(0)\}$. It can easily be seen from (7) that $S_> + S_> \subset S_>$, $\mathbb{Q}_+ S_> \subset S_>$, and $S_< = -S_>$ ($\mathbb{Q}_+$ = set of all positive rationals). $S_>$ must have positive Lebesgue-measure, because otherwise $h(r) = h(0)$ a.e. and by (7) h would be a constant. $\lambda(S_>) > 0$ implies that the interior of $S_> \supset S_> + S_>$ is nonempty (cf. Hewitt and Stromberg (1975, page 144, Exercise (10.44))). This implies $S_> = (0, \infty)$ or $S_> = (-\infty, 0)$, from which the assertion follows. □

3. Optimal designs. We consider the usual two-way layout in ANOVA

$$EY_{ij\nu} = a_{ij}, \qquad 1 \le \nu \le n_{ij}, \quad 1 \le i \le u, \quad 1 \le j \le v,$$

where the a_{ij} are unknown mean effects (cf. Scheffé (1959, pages 106 ff.)). The observations $Y_{ij\nu}$ are assumed to be uncorrelated with equal variance $\sigma^2 > 0$. The $u \times v$ matrix of the nonnegative integers n_{ij} is called an experimental design. Writing the model in the form (1) of Section 1, one obtains

$$X_d^T X_d = \operatorname{diag}(n_{11}, n_{12}, \cdots, n_{uv}). \tag{8}$$

Let K be an $s \times uv$ matrix, and suppose that one wishes to estimate Ka, where $a = (a_{11}, a_{12}, \cdots, a_{uv})^T \in \mathbb{R}^{uv}$. Assume that

(9) K has orthonormal rows and $K^T K$ has equal diagonal elements.

Then (9) is satisfied, for example, when K is given by an orthonormal basis of the main effects or of the interactions (for definitions see Scheffé (1959, page 93, equation (4.19))). As can easily be seen, Ka is estimable under $d = (n_{ij})$ iff d is complete, i.e., $n_{ij} \in \mathbb{N} = \{1, 2, \cdots\}$ for all i, j. So for a given integer $n \ge uv$ let

$$\Delta = \Delta_{u,v,n} = \{d = (n_{ij}) : n_{ij} \in \mathbb{N},\ 1 \le i \le u,\ 1 \le j \le v,\ \textstyle\sum_{i=1}^{u} \sum_{j=1}^{v} n_{ij} = n\}.$$

By $1_{(r \times m)}$ we denote the $r \times m$ matrix all elements of which are one.

THEOREM 2. *Let K satisfy* (9), *let Ψ be an increasing, orthogonal invariant criterion, and let Φ be quasiconvex. If n is divisible by uv, then the equireplicate design $d^* = (n/uv)1_{(u \times v)}$ is Ψ-optimal in Δ for estimating Ka.*

PROOF. The theorem is a consequence of Proposition 1′ in Kiefer (1975), applied on the matrices V_d^{-1}, $d \in \Delta$, where V_d is defined by (2). For checking the assumptions of this proposition the inequality

$$V_d^{-1} \le K X_d^T X_d K^T \qquad \text{(with equality for } d = d^*)$$

is useful (cf. Gaffke and Krafft (1977) and also Kiefer (1978). □

We turn to the special case of estimating the main effects $\alpha_i(a) = \overline{a_{i\cdot}} - \overline{a_{\cdot\cdot}}$ of the first factor. A representation according to (9) leads to

$$V_d = U C_d^+ U^T \qquad \text{and} \qquad V_d^{-1} = U C_d U^T, \tag{10}$$

where U is a given $(u-1) \times u$ matrix with orthonormal rows and orthogonal to $1_{(u \times 1)}$, and

$$C_d = H_d - (\operatorname{tr} H_d)^{-1} H_d 1_{(u \times u)} H_d, \tag{11}$$

where H_d denotes the $u \times u$ diagonal matrix with the harmonic means $h_{di} = (v^{-1} \sum_{j=1}^{v} n_{ij}^{-1})^{-1}$ in the diagonal. For the Moore-Penrose inverse of C_d

$$C_d^+ = E_u H_d^{-1} E_u, \qquad \text{where} \quad E_u = I_u - \frac{1}{u} 1_{(u \times u)}. \tag{12}$$

Clearly for $d \in \Delta$ the matrices C_d (and C_d^+) are elements of the set $\mathscr{B}_{u,0}$ of all nonnegative definite $u \times u$ matrices with zero row sums and rank $u - 1$. For a given optimality criterion Ψ on $\mathscr{P}_{u-1}$ let $\hat{\Psi}(B) = \Psi(UBU^T)$, $B \in \mathscr{B}_{u,0}$, and similarly $\hat{\Phi}(B)$. Trivially by (10) a design $d^* \in \Delta$ is Ψ-optimal iff it minimizes $\hat{\Psi}(C_d^+)$ $(= \hat{\Phi}(C_d))$ over $d \in \Delta$, which we will call now $\hat{\Psi}$ -optimality (for estimating the main effects of factor 1). The conditions on Ψ and Φ of Section 2 carry over to $\hat{\Psi}$ and $\hat{\Phi}$ in an obvious way. Theorem 2 can now be strengthened slightly. By Π_u we denote the group of all $u \times u$ permutation matrices.

THEOREM 2′. *Let $\hat{\Psi}$ be increasing and Π_u-invariant, and let $\hat{\Phi}$ be quasiconvex. If n is divisible by uv, then the equireplicate design d^* is $\hat{\Psi}$ -optimal in Δ (for estimating the main effects), and it is the unique $\hat{\Psi}$ -optimal design in Δ, if $\hat{\Psi}$ is strictly increasing.*

The proof is an application of Proposition 1 of Kiefer (1975). □

For the "nonregular" case where n is not divisible by uv we can use the concept of majorization and weak majorization of vectors, cf. e.g. Marshall and Olkin (1979). Recall that for $x, y \in \mathbb{R}^k$

(13) $\quad x \prec y \quad$ iff x is an element of the convex hull of $\{Qy : Q \in \pi_k\}$,

(14) $\quad x \prec_w y \quad$ iff there exists a $z \in \mathbb{R}^k$ such that $x \le z \prec y$.

The proofs of the following simple lemmas are omitted.

LEMMA 2. *Let $x \in \mathbb{R}^k$ have integer components. Choose an $x^* \in \mathbb{R}^k$ with integer components such that $\sum_{i=1}^k x_i^* = \sum_{i=1}^k x_i$ and $|x_i^* - x_j^*| \le 1$ for all i, j. Then $x^* \prec x$.*

LEMMA 3. *For given positive real numbers c, γ with $c \le k\gamma$ let*

$$M_{c,\gamma} = \{x \in \mathbb{R}^k : 0 \le x_i \le \gamma,\ 1 \le i \le k,\ \textstyle\sum_{i=1}^k x_i = c\}.$$

The set of extreme points of $M_{c,\gamma}$ is equal to

$$\{x \in M_{c,\gamma} : 0 < x_i < \gamma \quad \text{for at most one} \quad i \in \{1, \cdots, k\}\}.$$

Hence for $c, \gamma \in \mathbb{N}$ all extreme points of $M_{c,\gamma}$ have integer components.

Now we define

$$\Delta^* = \Delta^*_{u,v,n} = \{d^* = (n^*_{ij}) \in \Delta : |n^*_{ij} - n^*_{hk}| \le 1,$$
$$|n^*_{i\cdot} - n^*_{h\cdot}| \le 1, \quad \text{for all} \quad i, h = 1, \cdots, u, \quad j, k = 1, \cdots, v\},$$

where $n_{i\cdot} = \sum_{j=1}^v n_{ij}$,

$\Delta' = \Delta'_{u,v,n} = \{d' = (n'_{ij}) \in \Delta$: there exists an $i_0 \in \{1, \cdots, u\}$ with $|n'_{i_0 j} - n'_{i_0 k}| \le 1$ for all $j, k = 1, \cdots, v$, for each fixed $i \ne i_0$ the n'_{ij}, $j = 1, \cdots, v$, are equal, and $|n'_{ij} - n'_{hj}| \le 1$ for all $i \ne i_0$, $h \ne i_0\}$,

and for the "semiregular" case that n is divisible by v let

$\Delta^{**} = \Delta^{**}_{u,v,n} = \{d^{**} = (n^{**}_{ij}) \in \Delta$: for each fixed i the n^{**}_{ij}, $j = 1, \cdots, v$, are equal, and $|n^{**}_{ij} - n^{**}_{hj}| \le 1$ for all $i, h = 1, \cdots, u\}$.

In the following theorem "$\hat{\Psi}$-optimality" always means $\hat{\Psi}$-optimality in $\Delta = \Delta_{u,v,n}$.

THEOREM 3. *Let $\hat{\Psi}$ be increasing and Π_u-invariant, and $u \ge 3$.*

(a) *Let $\hat{\Psi}$ be quasiconvex. Then every $d^* \in \Delta^*$ is $\hat{\Psi}$-optimal. If $\hat{\Psi}$ is strictly quasiconvex, then Δ^* equals the set of all $\hat{\Psi}$-optimal designs.*

(b) *Let $\hat{\Psi}$ be quasiconcave and $\hat{\Phi}$* be quasiconvex.

(α) *Then there exists a $d' \in \Delta'$ which is $\hat{\Psi}$-optimal. If $\hat{\Psi}$ is strictly quasiconcave, then Δ' contains the set of all $\hat{\Psi}$-optimal designs.*

(β) *Let n be divisible by v. Then every $d^{**} \in \Delta^{**}$ is $\hat{\Psi}$-optimal. If $\hat{\Psi}$ is strictly quasiconcave, then Δ^{**} equals the set of all $\hat{\Psi}$-optimal designs.*

PROOF. It has been shown in Gaffke and Krafft (1979, Theorem 1 and Remark (ii)), that the set

$$\hat{\Delta} = \hat{\Delta}_{u,v,n} = \{d = (n_{ij}) \in \Delta : |n_{ij} - n_{ik}| \le 1 \quad \text{for all} \quad i = 1, \cdots, u, \quad j, k = 1, \cdots, v\}$$

is a complete set in the following sense. For each $d_1 \in \Delta \backslash \hat{\Delta}$ there exists a $d \in \hat{\Delta}$ such that $C_d^+ \le C_{d_1}^+$ and $C_d^+ \ne C_{d_1}^+$, and hence $\hat{\Psi}(C_d^+) \le \hat{\Psi}(C_{d_1}^+)$with strict inequality if $\hat{\Psi}$ is strictly increasing (what is satisfied if $\hat{\Psi}$ is strictly quasiconvex or strictly quasiconcave). Let $\bar{h}_d = (\bar{h}_{d1}, \cdots, \bar{h}_{du})$ for $d \in \hat{\Delta}$ be the diagonal of $\bar{H}_d^{-1}$ (see equation (12)). Writing the totals $n_{i\cdot}$ as $p_i v + q_i$, $p_i \in \mathbb{N}$, $q_i \in \{0, 1, \cdots, v\}$, we have

$$\bar{h}_{di} = \bar{h}_i(p_i, q_i) = v^{-1}(v - q_i)p_i^{-1} + v^{-1}q_i(p_i + 1)^{-1}. \tag{15}$$

So we are concerned with an optimization problem over the set of integer vectors $p = (p_1, \cdots, p_u)$ and $q = (q_1, \cdots, q_u)$ satisfying

$$v \textstyle\sum_{i=1}^u p_i + \sum_{i=1}^u q_i = n, \qquad p_i \in \mathbb{N}, \quad q_i \in \{0, 1, \cdots, v\}. \tag{16}$$

Note that for given $n_{i\cdot}$ the p_i and q_i are not always unique, because $q_i = v$ is admitted. This will simplify some computations below.

(a) Assume that $\hat{\Psi}$ is quasiconvex. Let $d \in \hat{\Delta} \backslash \Delta^*$ be given. We prove that one can find a $d^* \in \Delta^*$ such that $\bar{h}_{d^*} \prec_w \bar{h}_d$ and $\bar{h}_{d^*} \ne Q\bar{h}_d$ for all $Q \in \Pi_u$. Then it follows from (14) and (12) that

$$C_{d^*}^+ \le \textstyle\sum_Q \alpha_Q Q C_d^+ Q^T \qquad \text{for some} \quad a_Q \ge 0$$

with $\sum_Q \alpha_Q = 1$, and $C_{d^*}^+ \ne Q C_d^+ Q^T$ for all $Q \in \Pi_u$. Hence $\hat{\Psi}(C_{d^*}^+) \le \hat{\Psi}(C_d^+)$ with strict inequality, if $\hat{\psi}$ is strictly quasiconvex. Since $\hat{\psi}(C_{d^*}^+)$ is constant on Δ^*, the assertion will follow. We may assume that $p_1 \le p_2 \le \cdots \le p_u$, and if $p_i = p_{i+1}$ then $q_i \le q_{i+1}$. Then by (15) $\bar{h}_1 \ge \bar{h}_2 \ge \cdots \ge \bar{h}_u$. We distinguish three cases:

$$\text{(i) } p_u - p_1 = 0; \qquad \text{(ii) } p_u - p_1 = 1; \qquad \text{(iii) } p_u - p_1 \ge 2.$$

Case (i). Choosing by Lemma 2 a $q^* \prec q$ we obtain a $d^* \in \Delta^*$ with $\bar{h}_{d^*} \prec \bar{h}_d$.

Case (ii). Let $p_1 = \cdots = p_m = \beta$ and $p_{m+1} = \cdots = p_u = \beta + 1$, $m \in \{1, \cdots, u - 1\}$. Then for $r \in \{1, \cdots, u\}$:

$$\sum_{i=1}^r \bar{h}_i(p_i, q_i) = \begin{cases} r\beta^{-1} - (v\beta(\beta+1))^{-1} \sum_{i=1}^r q_i & \text{for} \quad r \le m, \\ m\beta^{-1} + (r-m)(\beta+1)^{-1} - (v\beta(\beta+1))^{-1} \sum_{i=1}^m q_i & \\ \qquad - (v(\beta+1)(\beta+2))^{-1} \sum_{i=m+1}^r q_i & \text{for} \quad r > m. \end{cases}$$

By increasing $q_1, \cdots, q_m$ and decreasing $q_{m+1}, \cdots, q_u$ one obtains a $\tilde{q}$ with $\sum_{i=1}^u \tilde{q}_i = \sum_{i=1}^u q_i$ and either $\tilde{q}_1 = \cdots = \tilde{q}_m = v$ or $\tilde{q}_{m+1} = \cdots = \tilde{q}_u = 0$, and $\sum_{i=1}^r \bar{h}_i(p_i, \tilde{q}_i) \le \sum_{i=1}^r \bar{h}_i(p_i, q_i)$, $1 \le r \le u$. Hence $\bar{h}_{\tilde{d}} = \bar{h}(p, \tilde{q}) \prec_w \bar{h}(p, q) = \bar{h}_d$, and (i) can be applied on $\tilde{d}$.

Case (iii). Choose $\tilde{p}_1 = p_1 + 1$, $\tilde{p}_u = p_u - 1$, and keep the other p_i and the q_i fixed. One easily verifies that in $\mathbb{R}^2$:

$$(\bar{h}_1(\tilde{p}_1, q_1), \bar{h}_u(\tilde{p}_u, q_u)) \prec_w (\bar{h}_1(p_1, q_1), \bar{h}_u(p_u, q_u)).$$

Clearly this implies $\bar{h}(\tilde{p}, q) \prec_w \bar{h}(p, q)$ in $\mathbb{R}^u$. Proceeding in this way one obtains a $d_0 \in \hat{\Delta}$ which satisfies (i) or (ii) and $\bar{h}_{d_0} \prec_w \bar{h}_d$.

(b) Assume that $\hat{\Psi}$ is quasiconcave and $\hat{\Phi}$ is quasiconvex. Let $\tilde{d} \in \hat{\Delta} \backslash \Delta'$ be given. Since by (15) and (12) the function $\hat{\Psi}(C_d^+)$ is (strictly) quasiconcave with respect to q for fixed p, we can find by Lemma 3 a $d \in \hat{\Delta}$ with $q_i = 0$ except for at most one $i = i_0$, and $\hat{\psi}(C_d^+) \leq \hat{\Psi}(C_{\hat{d}}^+)$ (with strict inequality if $\hat{\Psi}$ is strictly quasiconcave). By (16) $q_{i_0} = n - v \operatorname{int}(n/v)$. The matrix function $(E_u H^{-1} E_u)^+$ is strictly concave with respect to H (cf. Marshall and Olkin (1979, page 469, *E.7.h.*)), and hence by (11), (12) $\hat{\Phi}(C_d)$ is a (strictly) quasiconvex and symmetric function of the diagonal $h_d = (h_{d1}, \cdots, h_{du})$ of H_d. Observing that $h_{di} = p_i$ if $q_i = 0$, the assertion follows from Lemma 2 (applied on the p_i, $i \neq i_0$, for α), and on all p_i for β. □

REMARK. Of course, part (bα) of the theorem is rather unsatisfactory. For the D-criterion the set of D-optimal designs is given by

$$\Delta_D = \Delta' \cap \{d = (n_{ij}) \in \Delta : | n_{ij} - n_{hk} | \leq 1 \quad \text{for all} \quad i, h = 1, \cdots, u, \quad j, k = 1, \cdots, v\},$$

cf. Gaffke and Krafft (1979, Theorem 2). But for a general $\hat{\Psi}$ (which satisfies the assumptions of part (b), a proof using weak majorization as above fails, because $\sum_{i=1}^{u} h_{di}$ attains its maximum exactly at those unbalanced designs with $n_{ij} = 1$ for $i \neq i_0$ (if $v \mid n$). However, no example is known to the author where $d \in \Delta_D$ are not $\hat{\Psi}$-optimal.

REFERENCES

GAFFKE, N. and KRAFFT, O. (1977). Optimum properties of Latin square designs and a matrix inequality. *Math. Operationsforsch. Statist.* **8** 345–350.

GAFFKE, N. and KRAFFT, O. (1979). Optimum designs in complete two-way layouts. *J. Statist. Planning and Inference* **3** 119–126.

HALMOS, P. R. (1961). *Measure Theory.* 7th printing. Van Nostrand, Princeton.

HEWITT, E. and STROMBERG, K. (1975). *Real and Abstract Analysis.* 3rd printing. *Springer*, New York.

KIEFER, J. (1974). General equivalence theory for optimum designs (Approximate theory). *Ann. Statist.* **2** 849–879.

KIEFER, J. (1975). Construction and optimality of generalized Youden designs. In *A Survey of Statistical Design and Linear Models* (J. N. Srivastava, ed.). 333–353. North-Holland, Amsterdam.

KIEFER, J. (1978). Random number assignment in simulation experiment, comment. *J. Amer. Statist. Assoc.* **73** 523–524.

MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications.* Academic, New York.

SCHEFFÉ, H. (1959). *The Analysis of Variance.* Wiley, New York.

WHITTLE, P. (1973). Some general points in the theory of optimal experimental design. *J. Roy. Statist. Soc. B* **35** 123–130.

INSTITUT FÜR STATISTIK
UND WIRTSCHAFTSMATHEMATIK
RWTH AACHEN
WULLNERSTRASSE 3
D-5100 AACHEN, WEST GERMANY