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ON NONPARAMETRIC MEASURES OF DEPENDENCE FOR RANDOM VARIABLES

BY B. SCHWEIZER AND E. F. WOLFF

University of Massachusetts and Beaver College

In 1959 A. Rényi proposed a set of axioms for a measure of dependence for pairs of random variables. In the same year A. Sklar introduced the general notion of a copula. This is a function which links an n-dimensional distribution function to its one-dimensional margins and is itself a continuous distribution function on the unit n-cube, with uniform margins. We show that the copula of a pair of random variables X, Y is invariant under a.s. strictly increasing transformations of X and Y, and that any property of the joint distribution function of X and Y which is invariant under such transformations is solely a function of their copula. Exploiting these facts, we use copulas to define several natural nonparametric measures of dependence for pairs of random variables. We show that these measures satisfy reasonable modifications of Rényi's conditions and compare them to various known measures of dependence, e.g., the correlation coefficient and Spearman's ρ.

1. Introduction. Let X, Y be random variables with continuous distribution functions F, G and joint distribution function H. Many measures of dependence for the pair (X, Y), which are symmetric in X and Y, have been proposed and studied in the literature. Among the most familiar of these are Pearson's correlation coefficient r, Spearman's ρ and Kendall's τ. These are given, respectively, by

$$r(X, Y) = \frac{1}{D(X)D(Y)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x, y) - F(x)G(y)]\, dx\, dy, \tag{1}$$

$$\rho(X, Y) = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x, y) - F(x)G(y)]\, dF(x)\, dG(y), \tag{2}$$

$$\tau(X, Y) = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x, y)\, dH(x, y) - 1, \tag{3}$$

where D stands for standard deviation (Hoeffding (1948), Kruskal (1958) and Lehmann (1966)).

If in the above integrals one makes the substitution $u = F(x)$, $v = G(y)$, i.e., if one employs the probability transform (Whitt (1976)), then one obtains

$$r(X, Y) = \frac{1}{D(X)D(Y)} \int_{0}^{1} \int_{0}^{1} [C(u, v) - uv]\, dF^{-1}(u)\, dG^{-1}(v), \tag{4}$$

$$\rho(X, Y) = 12 \int_{0}^{1} \int_{0}^{1} [C(u, v) - uv]\, du\, dv, \tag{5}$$

$$\tau(X, Y) = 4 \int_{0}^{1} \int_{0}^{1} C(u, v)\, dC(u, v) - 1, \tag{6}$$

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where F^{-1}, G^{-1} and the usual inverses of F, G, respectively, and C is the function given by

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)). \tag{7}$$

On analyzing the effects of this transformation, several things become apparent. First, integrals over the plane are transformed into integrals over the unit square. Second, the nonparametric measures ρ and τ are distinguished from the measure r in that they are functions of C alone. Third, the integrand in (5) is simply the signed volume between the surfaces $z = C(u, v)$ and $z = uv$. Since X and Y are independent if and only if $C(u, v) = uv$, these observations suggest that any suitably normalized measure of distance between the surfaces $z = C(u, v)$ and $z = uv$, e.g., any L_p-distance, should yield a symmetric, nonparametric measure of dependence. The principal purpose of this paper is to show that this is indeed the case. Specifically we shall study the L_1, L_2 and L_∞ distances, which we denote by $\sigma(X, Y)$, $\gamma(X, Y)$, and $\kappa(X, Y)$, respectively. These are given by

$$\sigma(X, Y) = 12 \int_0^1 \int_0^1 |C(u, v) - uv| \, du \, dv, \tag{8}$$

$$\gamma(X, Y) = \left(90 \int_0^1 \int_0^1 [C(u, v) - uv]^2 \, du \, dv\right)^{1/2}, \tag{9}$$

$$\kappa(X, Y) = 4 \sup_{u,v \in [0,1]} |C(u, v) - uv|, \tag{10}$$

or, when expressed in terms of the distributions F, G, H, by

$$\sigma(X, Y) = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |H(x, y) - F(x)G(y)| \, dF(x) \, dG(y), \tag{11}$$

$$\gamma(X, Y) = \left(90 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x, y) - F(x)G(y)]^2 \, dF(x) \, dG(y)\right)^{1/2}, \tag{12}$$

$$\kappa(X, Y) = 4 \sup_{x,y \in R} |H(x, y) - F(x)G(y)|. \tag{13}$$

The quantity $\gamma(X, Y)$ was introduced by R. Blum, J. Kiefer and M. Rosenblatt in 1961 as a distribution-free statistic to test for the independence of X and Y.

We shall show that, when evaluated according to a (suitably modified) set of criteria introduced by A. Rényi in 1959, 1970, these measures possess many pleasant properties; we shall compare and contrast these measures with various known measures of dependence; and we shall present arguments which show that the functions C which appear in the displays (4)–(10) are of intrinsic interest.

2. Copulas. The function C in Kruskal (1958) is a copula as defined by A. Sklar in 1959 and studied further in Schweizer and Sklar (1974). The copula concept is a convenient hub for our investigations.

Definition 1. A (two-dimensional) copula is a mapping C from the unit square $[0, 1] \times [0, 1]$ onto the unit interval $[0, 1]$ satisfying the conditions:

(a) $C(u, 0) = C(0, u) = 0$ and $C(u, 1) = C(1, u) = u$, for every $u \in [0, 1]$.

(b) $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geqq 0$, for all $u_1, u_2, v_1, v_2, \in [0, 1]$ such that $u_1 \leqq u_2$ and $v_1 \leqq v_2$. It readily follows that any copula C satisfies the Lipschitz condition

$$|C(u_1, v_1) - C(u_2, v_2)| \leqq |u_1 - u_2| + |v_1 - v_2|, \tag{14}$$

whence C is continuous, and that

$$\max(u + v - 1, 0) \leqq C(u, v) \leqq \min(u, v), \tag{15}$$

for all $u, v \in [0, 1]$. Moreover, $\max(u + v - 1, 0)$ and $\min(u, v)$ are themselves copulas.

Note. In the sequel we will use the symbols C^-, C^0, C^+ to denote the copulas whose values for any u, v in $[0, 1]$ are given by $\max(u + v - 1, 0)$, uv, $\min(u, v)$, respectively.

The graph of any copula is a surface over the unit square which is bounded above by the surface $z = C^+(u, v)$ and below by the surface $z = C^-(u, v)$. The hyperbolic paraboloid $z = uv$ sits midway between these two extreme surfaces.

The fundamental results in the theory of copulas are expressed by the following three theorems. The first is due to A. Sklar, 1959, the second to M. Fréchet (1951, 1957, 1958).

THEOREM 1. *Let X, Y be random variables with individual distributions F, G and joint distribution H. Then there exists a copula* C_{XY} *such that*

$$H(x, y) = C_{XY}(F(x), G(y)), \tag{16}$$

for all real x, y. If F and G are continuous then C_{XY} *is unique; otherwise* C_{XY} *is uniquely determined on* (Range F) × (Range G).

Note. In this paper, in order to keep the main ideas in focus and for the sake of brevity, we restrict our attention to continuous distributions F, G. Also, when there is no danger of confusion, we write C instead of C_{XY}.

THEOREM 2. *Let X, Y be random variables with continuous distributions F, G, joint distribution H and (unique) copula C. Then*

(i) *X and Y are independent if and only if* $C = C^0$.

(ii) $Y = f(X)$ a.s., *where f is strictly increasing* (resp, *decreasing*) a.s. *on the range of X if and only if* $C = C^+$ (resp, $C = C^-$).

Note that if the distribution function of X is continuous and f is strictly monotone a.s. on the range of X, then the distribution function of $f(X)$ is also continuous.

THEOREM 3. *Let X, Y, F, G, H and C be as in Theorem 2. Then*

(i) *If f and g are strictly increasing* a.s. *on* Range X *and* Range Y, *respectively, then* $C_{f(X)g(Y)} = C$.

(ii) *If f and g are strictly decreasing* a.s. *on* Range X *and* Range Y, *respectively, then the copulas* C_1, C_2 *and* C_3 *of the pairs* $(f(X), Y)$, $(X, g(Y))$ *and* $(f(X), g(Y))$, *respectively, are independent of the particular choices of f and g and are given by*

$$\begin{aligned} C_1(u, v) &= v - C(1 - u, v), \\ C_2(u, v) &= u - C(u, 1 - v), \\ C_3(u, v) &= u + v - 1 + C(1 - u, 1 - v). \end{aligned} \tag{17}$$

(iii) *C is the restriction to the unit square of the joint distribution function of the probability transforms* $F(X)$ *and* $G(Y)$.

(iv) *If* F_1 *and* G_1 *are any given continuous distributions then the random variables* $F_1^{-1}F(X)$ *and* $G_1^{-1}G(Y)$ *have distributions* F_1 *and* G_1, *respectively, and copula C.*

A proof of Theorem 1 is given in Schweizer and Sklar (1974) (see also Moore and Spruill (1975) and Whitt (1976)). The proof of Theorem 2, modulo some readily supplied details, is given in Frechét. The proofs of (i) and (ii) of Theorem 3 are a series of straightforward verifications. To prove (iii), let $H_{F(X)G(Y)}$ be the joint distribution function of $F(X)$ and $G(Y)$. Then using (16), the fact that $F(X)$ and $G(Y)$ are both uniformly distributed on $[0, 1]$, and (i) with $f = F$ and $g = G$, we have

$$H_{F(X)G(Y)}(s, t) = C_{F(X)G(Y)}(s, t) = C(s, t). \tag{18}$$

Finally, (iv) follows from (i) and a simple verification.

Theorem 1 shows that a copula is a function which links a bivariate distribution to its one-dimensional margins. Thus, since a copula is itself a continuous bivariate distribution on the unit square, with uniform margins, (7) and (16) show that much of the study of joint distributions can be reduced to the study of copulas. But for us the true importance of copulas lies in a combination of Theorems 1 and 3. For, from the structure of (16) and the fact that, under a.s. strictly increasing transformations of X and Y, the copula is invariant while the margins may be changed at will, it follows that it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strictly increasing transformations. Hence the study of rank statistics—insofar as it is the study of properties invariant under such transformations—may be characterized as the study of copulas and copula-invariant properties.

3. Rényi's axioms. The following conditions form a reasonable set of disiderata for a symmetric, nonparametric measure of dependence $R(X, Y)$ for two continuously distributed random variables X and Y.

(A) $R(X, Y)$ is defined for any X and Y.

(B) $R(X, Y) = R(Y, X)$.

(C) $0 \leqq R(X, Y) \leqq 1$.

(D) $R(X, Y) = 0$ if and only if X and Y are independent.

(E) $R(X, Y) = 1$ if and only if each of X, Y is a.s. a strictly monotone function of the other.

(F) If f and g are strictly monotone a.s. on Range X and Range Y, respectively, then $R(f(X), g(Y)) = R(X, Y)$.

(G) If the joint distribution of X and Y is bivariate normal, with correlation coefficient r, then $R(X, Y)$ is a strictly increasing function ϕ of $|r|$.

(H) If (X, Y) and (X_n, Y_n), $n = 1, 2, \cdots$, are pairs of random variables with joint distributions H and H_n, respectively, and if the sequence $\{H_n\}$ converges weakly to H, then $\lim_{n\to\infty} R(X_n, Y_n) = R(X, Y)$.

Rényi's original axioms, which were introduced in Rényi (1959) (see also Rényi (1970)), differed from the above in that: (1) They were not restricted to continuously distributed random variables; (2) (E) was "$R(X, Y) = 1$ if either $X = f(Y)$ or $Y = g(X)$ for some Borel-measurable functions f and g"; (3) (F) was "If f and g are Borel-measurable, one-one mappings of the real line into itself then $R(f(X), g(Y)) = R(X, Y)$"; (4) In (G), $R(X, Y)$ was required to be equal to $|r|$; (5) (H) was not included. However, from the very outset this original set of axioms left something to be desired. For Rényi himself showed in 1959 that, among various well-known measures of dependence, the only one which satisfies all of his axioms is the maximal correlation coefficient

$$S(X, Y) = \sup_{f,g} r(f(X), g(Y)), \tag{19}$$

where the supremum is taken over all Borel functions f, g for which $r(f(X), g(Y))$ is defined. And as Hall pointed out in 1969, S has a number of major drawbacks, e.g., it equals 1 too often and is generally not effectively computable. In addition, in Schweizer and Wolff (1976) and Wolff (1977) we have given several examples which indicate that, at least for nonparametric measures, Rényi's conditions are too strong.

4. The measures σ, γ and κ.

THEOREM 4. *Let X and Y be continuously distributed random variables with copula C. Then the quantity $\sigma(X, Y)$ given by* (8) *satisfies the conditions* (A)–(H), *with the function ϕ in* (G) *given by*

$$\phi(|r|) = \frac{6}{\pi} \arcsin(|r|/2). \tag{20}$$

PROOF. It is clear from (8) that $\sigma(S, Y)$ is well-defined. Next, it follows from (7) that $C_{XY}(u, v) = C_{YX}(v, u)$ which yields (B). (C) follows from (8) and the readily established fact that, for any copula C,

$$\int_0^1 \int_0^1 |C(u, v) - uv| \, du \, dv \leqq 1/12. \tag{21}$$

(D) follows from (i) of Theorem 2 and the continuity of C and C^0. (E) follows from (ii) of Theorem 2 and the fact that equality holds in (21) if and only if $C = C^+$ or $C = C^-$. As regards (F), if both f and g are strictly increasing a.s., this is an immediate consequence of (i) of Theorem 3; if f is strictly increasing a.s., and g strictly decreasing a.s. then (F) follows from (i) and (ii) of Theorem 3, together with the observation that $(C_{f(X)g(Y)} - C^0)(u, v) = C^0 - C_{XY})(1 - u, v)$; similar arguments establish (F) in the remaining cases. Turning to (G), (20) can be established by a series of routine calculations using, among other things, Schläfli's differential recursion relation for the bivariate normal density (see Slepian (1962), page 482). However, it is more instructive to exploit the relationship of σ to Spearman's ρ as evidenced by (5) and (8). To this end we first note that when the joint distribution of X and Y is bivariate normal then $C > C^0$, $C = C^0$ or $C < C^0$ according as $r > 0$, $r = 0$ or $r < 0$. Thus, in this case, $\sigma(X, Y) = |\rho(X, Y)|$. But it is well-known (Kruskal (1958), page 827) that for the bivariate normal distribution

$$\rho(X, Y) = \frac{6}{\pi} \arcsin(r/2), \tag{22}$$

whence (20) follows. Lastly, if $H_n \to_W H$ then it follows from (7) that the corresponding copulas C_n converge pointwise to C. By the Lipschitz condition (14), any family of copulas is equicontinuous, whence the convergence is uniform. This establishes (H) and completes the proof.

Using completely analogous arguments, it is easy to show that the quantity γ given by (9), as well as all the other normalized L_p-distances, $1 < p < \infty$, satisfy the conditions (A)-(H). However, the explicit form of the function ϕ in (G) remains to be determined.

The situation changes slightly when one considers the L_∞-distance κ given by (10). The above arguments show that κ satisfies all of the conditions (A)-(H) with the sole exception of (E). Furthermore, the function ϕ in (G) is given by

$$\phi(|r|) = \frac{2}{\pi} \arcsin(|r|). \tag{23}$$

As regards (E), if either X or Y is a.s. a strictly monotone function of the other then $\kappa(X, Y) = 1$. However, since there exist copulas C, distinct from C^+ and C^-, for which sup $|C - C^0| = 1/4$, the converse is false. On the other hand, if $\kappa(X, Y) = 1$ then $\sigma(X, Y) \geqq 3(\ln 2) - 3/2 \approx 0.58$, and this inequality is best-possible (Wolff (1977)).

5. Comparisons and examples. It follows immediately from (5) and (8) that $|\rho(X, Y)| \leqq \sigma(X, Y)$ for any X, Y. The difference between σ and $|\rho|$ can be large—exactly how large remains to be determined. For example, if X is the identity map on [0, 1] and Y is defined by

$$Y(w) = \begin{cases} w, & 0 \leqq w \leqq 1/2, \\ 3/2 - w, & 1/2 < w \leqq 1, \end{cases}$$

then $\sigma(X, Y) - |\rho(X, Y)| = 3(\ln 2) - 3/2 \approx 0.58$ (Wolff (1977)).

It is easy to show that $|\rho|$ satisfies the conditions (A)-(H), with the important exception of (D). The same is true for $|\tau|$, where τ is given by (6). And $|r|$ always satisfies (A), (B), (C) and (G) but, as is well-known, satisfies (E) and (F) if and only if the functions f and g are linear. It is also well-known that $|r|$ fails to satisfy both (D) and (H).

In the case of the bivariate normal distribution, σ is a strictly increasing function of $|r|$. In general, however, there is no functional relationship whatever between these quantities. Indeed, one can construct a sequence $\{(X_n, Y_n)\}$ for which $\sigma(X_n, Y_n) = 1$, for all n, whereas $\lim_{n\to\infty} r(X_n, Y_n) = 0$; but the limiting values, cannot be attained, for we have:

THEOREM 5. *Let X, Y, F, G, H and C be as in Theorem 2. Suppose that $\sigma(X, Y) = 1$ and that $r(X, Y)$ exists. Then $|r(X, Y)| > 0$.*

PROOF. Since $\sigma(X, Y)$ satisfies (E) it follows from (ii) of Theorem 2 that $C = C^+$ or $C = C^-$. Assume $C = C^+$. Then $H(s, t) = C^+(F(s), G(t))$ whence, using (1), we have

$$E(XY) - E(X)E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [C^+(F(s)G(t)) - F(s)G(t)]\, ds\, dt. \tag{24}$$

Now $C^+(F(s), G(t)) \geqq F(s)G(t)$ for all s, t, with strict inequality whenever $0 < F(s)G(t) < 1$. Therefore, since F and G are continuous, the integral in (24) is positive, whence $r(X, Y) > 0$. By the same argument, $r(X, Y) < 0$ when $C = C^-$.

In a recent paper (1978) G. Kimeldorf and A. R. Sampson introduced a measure of dependence, which we denote by ν, and which is obtained from (19) by restricting the supremum to monotone functions f and g. As they show, ν satisfies (A)–(D), (F) and (G), but neither (E) nor (H). As regards (E), if either X or Y is a.s. a strictly monotone function of the other then $\nu(X, Y) = 1$, but not conversely. Indeed, there are simple examples in which $\nu(X, Y) = 1$ while all of the measures $\sigma(X, Y)$, $\rho(X, Y)$, $\kappa(X, Y)$, $\gamma(X, Y)$ and $r(X, Y)$ are, simultaneously, arbitrarily small. In addition, ν is generally difficult to compute. Thus it appears that ν suffers from many of the same defects as the maximal correlation coefficient itself.

Recently, S. Kotz and N. L. Johnson (1977) compared σ to Q, the mean square contingency, by evaluating both quantities when X and Y have Iterated Generalized Farlie Gumbel Morgenstern Distributions. They found the expressions for $\sigma(X, Y)$ to be uniformly simpler than the ones for $Q(X, Y)$ whence, as they remark, comparisons of the dependence properties of these IGFGM distributions are more clearcut when σ is used.

We conclude this section with several examples.

EXAMPLE 1. Let the pair (X, Y) be uniformly distributed on the circumference of the unit circle. As is well-known, X and Y are dependent, yet $r(X, Y) = \rho(X, Y) = \tau(X, Y) = 0$. In this case the measures $\sigma(X, Y)$, $\kappa(X, Y)$ and $\gamma(X, Y)$ are easily evaluated by using (iii) of Theorem 3 and the fact that the pair of probability transforms $(F(X), G(Y))$ is uniformly distributed on the diamond with vertices $(\frac{1}{2}, 0)$, $(1, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(0, \frac{1}{2})$. We find that $\sigma(X, Y) = \kappa(X, Y) = \gamma(X, Y) = \frac{1}{4}$.

EXAMPLE 2. Let X be the identity function on $[0, 1]$; let $0 \leqq z \leqq 1$; let g be defined on $[0, 1]$ by $g(0) = g(1) = 0$, $g(u) = u/z$ for $0 < u \leqq z$ and $g(u) = (1 - u)/(1 - z)$ for $z < u < 1$; and let $Y = g(X)$. Then X and Y are uniformly distributed on $[0, 1]$, whence $C = H$; and routine calculations yield $\sigma(X, Y) = 2(z - \frac{1}{2})^2 + \frac{1}{2}$, $\kappa(X, Y) = |z - \frac{1}{2}| + \frac{1}{2}$, and $\gamma(X, Y) = [3(z - \frac{1}{2})^2 + \frac{1}{4}]^{1/2}$. In this example $|r(X, Y)| = |\rho(X, Y)| = 2|z - \frac{1}{2}|$ while $S(X, Y)$ is identically 1.

EXAMPLE 3. Let X be the identity function on $[0, 1]$ and let $y = h(X)$, where $h : [0, 1] \to [0, 1]$ is the piecewise linear function determined by $h(j/2k) = j (\text{mod } 2)$, for $j = 0, 1, \cdots, 2k$. In this case $\sigma(X, Y) = \kappa(X, Y) = \gamma(X, Y) = \frac{1}{2}k$. This result is appealing in that, for a given value of Y, there are $2k$ equiprobable possible values of X. Here $r(X, Y) = \rho(X, Y) = \tau(X, Y) = 0$ and $S(X, Y) = 1$.

6. Concluding remarks. Throughout this paper we have restricted our discussion to random variables with continuous distribution functions. The measures σ, κ, etc., can also be defined for random variables whose distribution functions are not continuous. In this case, one can work with one of the nonunique copulas whose existence is guaranteed by Theorem 1, with the "subcopula" which is uniquely determined on (Range F) × (Range G) and which is given by (7), or with the representations (11), (13), etc. In each instance natural measures of dependence can be defined.

Theorem 1 is also valid for n-dimensional distributions, $n > 2$ (Moore and Spruill (1975), Sklar (1959, 1973) and Whitt (1976)). That is, for every such distribution there exists an n-dimensional copula joining the distribution to its margins. Thus the definitions of our various measures extend easily to any collection of n random variables. Here one can use either a single number to measure the collective dependence or, on taking all the marginal distributions into account, a set of 2^{n-1} numbers. Finally, upon letting $n \to \infty$, the limiting measure could serve to measure the collective dependence of a sequence of random variables. Details are given in Wolff (1977) and will be presented in a subsequent paper.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
AMHERST, MASSACHUSETTS 01003

DEPARTMENT OF MATHEMATICS
BEAVER COLLEGE
GLENSIDE, PENNSYLVANIA 19038