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UNIQUENESS AND EVENTUAL UNIQUENESS OF OPTIMAL DESIGNS IN SOME TIME SERIES MODELS

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Using the results of Barrow, et al., and Chow on the optimal placement of knots in the approximation of functions by piecewise polynomials, we show the uniqueness or "eventual uniqueness" of optimal designs for certain time series models considered by Sacks and Ylvisaker, and Wahba. In addition, the limiting behavior (as the sample size increases) of the variance of the BLUE of the regression coefficient is characterized in terms of the density defining the design, and the density for the asymptotically optimal design is given.

1. Introduction. Consider the stochastic process

$$Y(t) = \beta f(t) + X(t), \qquad t \in [0, 1] \tag{1.1}$$

where β is an unknown parameter, f is a known regression function, and $X(\cdot)$ is a zero mean process with known covariance kernel $R(s, t)$. When an infinite observation set is observed from (1.1) an estimator for β, $\hat{\beta}$, may be constructed using the reproducing kernel Hilbert space (RKHS) techniques developed by Parzen (1961a, 1961b). For finite sampling schemes the regression design problem has been addressed by Sacks and Ylvisaker (1966, 1970) and by Wahba (1971, 1974). They consider the problem of selecting a set of n distinct design points, $T_n = \{t_1, \cdots, t_n\}$ in the interval [0, 1] so that $\hat{\beta}_{T_n}$, the best linear unbiased estimator (BLUE) of β obtained by taking observations according to T_n, would have minimum variance.

For certain functions f and covariance kernels R, they show the existence of optimal and/or asymptotically optimal designs. The difficulties they encountered constructing optimal designs however, led to the construction of design sequences, $\{T_n\}_1^\infty$, that are asymptotically optimal in the sense that

$$\lim_{n\to\infty} \frac{\mathrm{Var}(\hat{\beta}_{T_n}) - \mathrm{Var}(\hat{\beta})}{\inf_{T\in D_n} \mathrm{Var}(\hat{\beta}_T) - \mathrm{Var}(\hat{\beta})} = 1 \tag{1.2}$$

where D_n is the set of all n-point designs.

In this paper the uniqueness or eventual uniqueness of regression designs is considered for the model (1.1) for a certain class of f's. We also consider a class of covariance kernels of which Brownian bridge is a special case. This process has received increasing attention as a limiting distribution for a smoothed sample quantile process (Csörgő and Révész (1975, 1978)). Parzen (1979) has shown that a model of the form (1.1) with $X(\cdot)$ a Brownian bridge process arises in the estimation of location and scale parameters by sample quantiles. We also discuss the Brownian motion process, the importance of which is well known in the context of (1.1).

Our uniqueness results derive from the work of Barrow, et al. (1978) and extensions by Chow (1978) who shows that the best $L_2[0, 1]$ approximation of a certain class of functions by piecewise polynomials with variable knots is unique. That these results are applicable

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follows from Section 4 where it is shown that solving a certain approximation problem is equivalent to finding an optimal design. In the particular case of Brownian motion for example, where $R(s, t) = \min(s, t)$, an optimal n-point design is obtained by minimizing (with respect to $t_1, \cdots, t_n$)

$$\left\| f' - \textstyle\sum_{i=1}^{n} a_i \dfrac{\partial R(\cdot, t_i)}{\partial \cdot} \right\| \tag{1.3}$$

where $\|\cdot\|$ denotes the $L_2[0, 1]$ norm. Since $R(\cdot, t)$ is a linear spline, we are approximating f' by a spline of order 1, i.e., by piecewise constants. The optimal design points $t_1, \cdots, t_n$ correspond to the optimal knot locations. In addition to the theoretical work on unicity by Barrow, et al., (1978) and Chow (1978), Chow (1978) has developed very efficient numerical algorithms which can be used to compute optimal designs. The uniqueness of certain optimal designs and their ease of computation obviate the need in such cases for the consideration of asymptotically optimal designs.

We state our main results in Sections 2 and 5 with proofs in Sections 4 and 5. In Section 3, some recent results in piecewise polynomial approximation are stated from which we derive our uniqueness results. Some examples are included in Section 6.

2. Results and notation. We consider the stochastic process (1.1). In a sequence of papers Sacks and Ylvisaker (1966, 1968, 1970) study various aspects of estimating β utilizing a finite number of observations. This problem of statistical design may be stated as follows. Let $D_n := \{T_n := (t_1, \cdots, t_n) \mid 0 \le t_1 < \cdots < t_n \le 1\}$ be the set of all possible (noncoincident) designs and let

$$Y_{k,T_n} := \operatorname{span}\{Y^{(j)}(t) \mid t \in T_n \quad \text{and} \quad j = 0, 1, \cdots, k-1\} \tag{2.1}$$

where the symbol := means "is defined as." Then among all $Z \in Y_{k,T_n}$ one can find a unique Z^* satisfying

$$\begin{aligned} &E(Z^*) = \beta \\ &\operatorname{Var}(Z^*) \le \inf\{\operatorname{Var}(Z) \mid E(Z) = \beta \quad \text{and} \quad Z \in Y_{k,T_n}\}. \end{aligned} \tag{2.2}$$

We will denote Z^* by $\hat\beta_{k,T_n}$ to emphasize the dependence of Z^* on both k and T_n. The design problem is then to find $T_n^* \in D_n$ so that

$$\operatorname{Var}(\hat\beta_{k,T_n^*}) \le \inf\{\operatorname{Var}(\hat\beta_{k,T_n}) \mid T_n \in D_n\}. \tag{2.3}$$

This problem may be reformulated as a nonlinear (spline) approximation problem. For any sufficiently smooth kernel $R(s, t)$ we define

$$R^{(l,j)}(\cdot, t) = g(\cdot) \tag{2.4}$$

where

$$g(s) = D^{(l,j)}R(s, t) = \frac{\partial^l}{\partial s^l} \frac{\partial^j}{\partial t^j} R(s, t).$$

Let us set

$$S_{k,T_n} = \operatorname{span}\{R^{(0,j)}(\cdot, t) \mid t \in T_n \quad \text{and} \quad j = 0, 1, \cdots, k-1\}. \tag{2.5}$$

Finally, let $H(R)$ denote the RKHS of functions defined on [0, 1] generated by R with norm for $h \in H(R)$ denoted by $\|h\|_R$. Sacks and Ylvisaker (1966) have shown that if $f \in H(R)$ then

$$\operatorname{Var}(\hat\beta_{k,T_n}) = \|\mathscr{P}_{k,T_n} f\|_R^{-2} \tag{2.6}$$

where $\mathscr{P}_{k,T_n}$ is the orthogonal projection onto S_{k,T_n}. Thus $\|\mathscr{P}_{k,T_n} f\|_R^2 = \|f\|_R^2 - \|f - \mathscr{P}_{k,T_n} f\|_R^2$

and we conclude that minimizing the variance in (2.6) (the optimal design problem) is equivalent to finding a $T_n^* \in D_n$ so that

$$\| f - \mathscr{P}_{k,T_n^*} f \|_R \leq \inf\{ \| f - \mathscr{P}_{k,T_n} f \|_R \mid T_n \in D_n \}. \tag{2.7}$$

Recall the example of Brownian motion given in Section 1.

In general, (2.7) is a very difficult problem and very little is known about it. However, if $R(s, t)$ is the Brownian bridge kernel or a suitable generalization of it then recent results in the theory of spline approximation yield results concerning the optimal design problem. Let us now fix a positive integer k for the remainder of this section. Suppose that $R(s, t)$ is the covariance kernel for a k-fold multiple integral of Brownian motion pinned, with its derivatives, at both endpoints, i.e., $R(s, t) = (-1)^k G(s, t)$ where $G(s, t)$ is the Green's function for the boundary value problem:

$$\begin{aligned} &D^{2k} g = h \\ &0 = g(0) = \cdots = g^{(k-1)}(0) = g(1) = \cdots = g^{(k-1)}(1). \end{aligned} \tag{2.8}$$

Then we can state three theorems concerning the optimal design problem, the proofs of which are given in Section 4.

The first theorem deals with uniqueness of optimal design.

THEOREM 2.1. *Let k be a fixed positive integer and let $f \in H(R)$, $f \in C^{2k}[0, 1]$ with $f^{(2k)} > 0$ on $[0, 1]$ and $\log f^{(2k)}$ concave on $(0, 1)$. Then for each positive integer n there is a unique optimal n-point design for the problem* (2.3).

That is, there is a unique $T_n^* \in D_n$ so that $\mathrm{Var}(\hat{\beta}_{k,T_n^*}) < \mathrm{Var}(\hat{\beta}_{k,T_n})$ if $T_n \in D_n$ and $T_n \neq T_n^*$. The second theorem concerns a concept introduced by Barrow, et al., (1978), called "eventual uniqueness."

THEOREM 2.2. *Considering the design problem* (2.3), *if we assume that $f \in H(R)$, $f^{(2k)} > 0$ and $f^{(2k+3)}$ is continuous on $[0, 1]$, then there is a positive integer n_0 such that for all $n > n_0$ the regression problem has a unique optimal n-point design.*

Finally, we state a theorem that Sacks and Ylvisaker (1970) anticipated in an asymptotic sense.

THEOREM 2.3. *If k is even, $f \in H(R)$, $f^{(2k)} > 0$, and T_n^* is an optimal n-point design as in* (2.3), *then* $\mathrm{Var}(\hat{\beta}_{k,T_n^*}) = \mathrm{Var}(\hat{\beta}_{k-1,T_n^*})$.

We remark that this last theorem shows one of the unexpected advantages of optimal designs: namely, the use of fewer observations can yield equivalent resolution. For example, the BLUE from the optimal n-point design with $k = 1$ has the same variance as the BLUE from the optimal n-point design with $k = 2$. This means that derivative information (in the form of $Y^{(1)}(t)$) is not useful in improving the precision of the best estimator. A special case of Theorem 2.3 has been given by Wahba (1971).

3. Approximation by piecewise polynomials and splines. This section is devoted to the mathematical and approximation theoretic preliminaries which will be necessary for later sections. Let D_n be as in Section 2. The set of polynomials of order k (degree less than k) will be denoted by P^k and the set of piecewise polynomials of order k with breakpoints at a particular $T \in D_n$ will be represented by $P^k(T)$. Further, $P_n^k := \cup_{T \in D_n} P^k(T)$ will denote the set of all piecewise polynomials of order k with n breakpoints in $[0, 1]$. Since most of our results involve approximation in the L_2 norm we will set $\| f \| := (\int_0^1 | f(t) |^2 \, dt)^{1/2}$.

With this notation we can now present several recent theorems concerning approximation by splines and piecewise polynomials.

THEOREM 3.1. *Let $f \in C^k[0, 1]$ with $f^{(k)} > 0$ on $[0, 1]$. Suppose that $\log f^{(k)}$ is concave on $(0, 1)$. Then for each positive integer n, f has a unique best $L_2[0, 1]$ approximation from P_n^k.*

This theorem and the one following were originally proved by Barrow, et al., (1978), for the case $k = 2$ and Chow (1978) proved the general result. If we instead ask whether one eventually gets uniqueness as n goes to infinity then the following theorem is pertinent.

THEOREM 3.2. *Let $f \in C^{k+3}[0, 1]$ with $f^{(k)} > 0$ on $[0, 1]$. Then there is a particular integer n_0 such that for every $n > n_0$, f has a unique best $L_2[0, 1]$ approximation from P_n^k.*

It should be noted that this is not an asymptotic result in the usual sense because unicity is obtained for some *finite* $n_0 + 1$ and for every n thereafter. In addition to these results, Barrow and Smith (1978) and Chow (1978) have also observed the following.

PROPOSITION. *If $s^* \in P_n^k$ is a best $L_2[0, 1]$ approximate to f from P_n^k and if $f^{(k)} > 0$ then*

$$(f - s^*)(\tau-) = (-1)^k(f - s^*)(\tau+) \tag{3.1}$$

for all $\tau \in [0, 1]$.

Here, $\tau-$ and $\tau+$ denote limits from the left and right respectively. For a simple proof the reader is urged to differentiate the error functional with respect to the breakpoints of s^* which yields the equation $|f - s^*|(\tau-) = |f - s^*|(\tau+)$. One then removes the absolute value signs by recalling the oscillation of f with its best approximant from P^k.

Instead of approximating by piecewise polynomials it is many times more natural to approximate by a smooth subspace of $P^k(T)$, namely $S^k(T)$, which we define by $S^k(T) := P^k(T) \cap C^{k-2}[0, 1]$. S^k is called the subspace of splines or order k with knot sequence T. We can define S_n^k similarly to P_n^k by setting $S_n^k := \cup_{T \in D_n} S^k(T)$. Sacks and Ylvisaker (1968) call a sequence of knots $\{T_n = (0 = T_n^0, T_n^1, \cdots, T_n^{n+1} = 1)\}$ a *regular sequence* provided there is a continuous density h so that $\int_{T_n^i}^{T_n^{i+1}} h(x)\, dx = 1/(n + 1)$, $i = 0, 1, \cdots, n$. This relationship is abbreviated: $\{T_n\}$ is $\mathrm{RS}(h)$.

Two recent results in the theory of spline approximation proved by Barrow and Smith (1978) will be used in Section 5. Here the notation $\mathrm{dist}(f, V)$ means $\inf\{\| f - v \| \mid v \in V\}$.

THEOREM 3.3. *Let $f \in C^k[0, 1]$ and suppose that $\{T_n\}$ is $\mathrm{RS}(h)$. Then*

$$\lim_{n\to\infty} n^k \mathrm{dist}(f, S^k(T_n)) = C_k\left(\int_0^1 \frac{(f^{(k)}(x))^2}{h^{2k}(x)}\, dx\right)^{1/2} \tag{3.2}$$

where $C_k = (|B_{2k}|/(2k)!)^{1/2}$ and B_{2k} is the $2k$th Bernoulli number.

Finally, the last theorem deals with choosing the best knot sequence in (3.2).

THEOREM 3.4. *Let k be a positive integer and let $\gamma = (k + 1/2)^{-1}$. Suppose that $h(x) = |f^{(k)}(x)|^\gamma / \int_0^1 |f^{(k)}(\tau)|^\gamma\, d\tau$. Then if $\{T_n\}$ is $\mathrm{RS}(h)$ we have*

$$\lim_{n\to\infty} n^k \mathrm{dist}(f, S^k(T_n)) = \lim_{n\to\infty} n^k \mathrm{dist}(f, S_n^k) = C_k\left(\int_0^1 |f^{(k)}(x)|^\gamma\, dx\right)^{1/\gamma}$$

where C_k and f are as in Theorem 3.3.

4. Proofs of theorems and applications. In this section we discuss the application of Theorems 3.1 and 3.2 of the preceeding section to the case of the Brownian bridge covariance kernel and certain of its generalizations. In particular we will prove Theorems 2.1, 2.2 and 2.3.

Throughout this section k will be a fixed positive integer and $R(s, t)$ will be the covariance kernel derived from the Green's function for the problem (2.8). (The Brownian bridge covariance kernel corresponds to $k = 1$.) Thus it follows that $g(s) = R(s, t)$ is a spline of order $2k$ and continuity class C^{2k-2} with a knot at t. This means that S_{k,T_n} defined in (2.5) is a linear subspace of splines of continuity class C^{k-1} with knots at T_n.

Let H denote the Hilbert space of functions which are k-fold integrals of $L_2[0, 1]$ functions which additionally satisfy the homogeneous boundary conditions in (2.8). The inner product on H is given by

$$\langle h, g\rangle = \int_0^1 h^{(k)}(x)g^{(k)}(x)\,dx. \tag{4.1}$$

It is easy to verify that H is a RKHS with reproducing kernel $R(s, t)$. This means that $H = H(R)$, the RKHS mentioned in Section 2 when R is defined via the Green's function for (2.8). Let us now define $S^k_{k,T_n} = \{s^{(k)} \mid s \in S_{k,T_n}\}$ and the mapping $\mathscr{Q}_{T_n}$ which will be the $L_2[0, 1]$ orthogonal projector onto S^k_{k,T_n}. Then we note that

$$\| f - \mathscr{P}_{k,T_n} f \|_R = \| f^{(k)} - (\mathscr{P}_{k,T_n} f)^{(k)} \| = \| f^{(k)} - \mathscr{Q}_{T_n} f^{(k)} \|. \tag{4.2}$$

Recalling (2.7) one sees that finding an optimal design T_n^* is equivalent to finding the best knot locations for functions in S^k_{k,T_n}.

We could read off Theorems 2.1–2.3 directly from Theorems 3.1, 3.2, and the Proposition except that the latter apply to approximation from $P^k(T_n)$, not $S^k_k(T_n)$. That these two spaces are not the same is proved in the following lemma which also provides the missing link in applying the results of Section 3 to prove those of Section 2.

LEMMA. *If $f \in H$ then $\mathscr{Q}_{T_n} f^{(k)}$ is the best $L_2[0, 1]$ approximation to $f^{(k)}$ from $P^k(T_n)$.*

PROOF. First observe that S^k_{k,T_n} is a subspace of $P^k(T_n)$. In addition, $\dim S^k_{k,T_n} = nk$ (assuming t_1 and t_n are not endpoints) while $\dim P^k(T_n) = (n + 1)k$. Thus, S^k_{k,T_n} is properly contained in $P^k(T_n)$. The orthogonal complement of S^k_{k,T_n} in $P^k(T_n)$ is seen to be P^k since P^k is also a subspace of $P^k(T_n)$ and for any $f \in H$ and $p \in P^k$, we have

$$\int_0^1 f^{(k)}(x)p(x)\,dx = 0 \tag{4.3}$$

as may be easily verified by repeated integration by parts and application of the boundary conditions (2.8). Now $f^{(k)} - \mathscr{Q}_{T_n} f^{(k)}$ is the error from the orthogonal projection of $f^{(k)}$ onto S^k_{k,T_n}. It is thus orthogonal to S^k_{k,T_n}. In addition, from (4.3) it is orthogonal to elements of P^k. Since

$$P^k(T_n) = P^k \oplus S^k_{k,T_n} \tag{4.4}$$

we have $\int_0^1 (f^{(k)}(x) - \mathscr{Q}_{T_n} f^{(k)}(x))g(x)\,dx = 0$ for all $g \in P^k(T_n)$. That is, the L_2 error from the projection of $f^{(k)}$ onto S^k_{k,T_n} is orthogonal to $P^k(T_n)$. This completes the proof.

Hence, Theorem 2.1 follows from Theorem 3.1 with the f in the latter theorem replaced by $f^{(k)}$. Similarly, Theorems 2.2 and 2.3 follow from Theorem 3.2 and the Proposition upon noting that when k is even the Proposition implies that $\mathscr{Q}_{T_n^*} f^{(k)} \in P^k_n \cap C[0, 1]$.

Situations with boundary conditions other than those given in (2.8) can be treated analogously. We illustrate this point by examining the case of Brownian motion whose covariance kernel is $R(s, t) = \min(s, t)$, $0 \le s, t \le 1$, with corresponding RKHS $= \{f \mid f(0) = 0$ and $f' \in L_2[0, 1]\}$. Following the notation in this section and in Section 2 we see that $k = 1$, and the usual design problem reduces again to minimizing the norm in (4.2) over all admissible designs. Unfortunately, (4.3) and thus (4.4) are no longer valid since we do not have the boundary condition $f(1) = 0$. The situation is still amenable to analysis, however, once we realize that S^1_{1,T_n} in this case corresponds to the subspace of piecewise constants which are constant on the subintervals (t_i, t_{i+1}) and zero on $(t_n, 1]$. Thus we see that (3.1) still must hold at all breakpoints which means here that

$$(4.5) \qquad |f' - \mathscr{Q}_{T_n} f'|(\tau+) = |f' - \mathscr{Q}_{T_n} f'|(\tau-)$$

for all $\tau \in T_n$ (except of course for $\tau = 1$ if t_n happens to be 1).

Thus, if $f \in C^2[0, 1]$ and f'' is positive with $\log f''$ concave we see by arguments similar to those that led to Theorem 3.1 and the Proposition that (4.5) is a necessary condition for a minimum to occur. For example, if f'' and f' are both positive on $[0, 1]$ then the only way (4.5) can occur is for $t_n = 1$. Then we see that the approximation problem (4.2) is equivalent to piecewise constant approximation with $(n - 1)$ breakpoints on $[0, 1]$. Hence, theorems similar to Theorems 2.1 and 2.2 may be obtained in this case as well.

5. Asymptotic properties of the BLUE. We have already seen how piecewise polynomial approximation naturally arises from the expression for the variance of the linear unbiased estimators of β from the observations $Y_{k,T}$. In general, information involving derivatives is difficult to obtain so that one would prefer to use the set $Y_{1,T}$ instead of $Y_{k,T}$ for $k > 1$. If we consider the covariance kernels $R(s, t)$ occurring in Section 4 arising from the Green's function for (2.8), we see that for a fixed positive integer k using the observations Y_{1,T_n} instead of Y_{k,T_n} yields the approximation problem (see (4.2))

$$\| f - \mathscr{P}_{1,T_n} f \|_R = \| f^{(k)} - (\mathscr{P}_{1,T_n} f)^{(k)} \| = \| f^{(k)} - \mathscr{Q}_{1,T_n} f^{(k)} \|,$$

where $\mathscr{Q}_{1,T_n}$ is the $L_2[0, 1]$ orthogonal projector onto $S^k_{1,T_n} = \{s^{(k)} \mid s \in S_{1,T_n}\}$. Just as in Section 4 one notices that $S^k(T_n) = P^k \oplus S^k_{1,T_n}$ which means that $\mathscr{Q}_{1,T_n} f^{(k)}$ is the best $L_2[0, 1]$ approximation to $f^{(k)}$ from $S^k(T_n)$.

Thus, Theorems 3.3 and 3.4 are applicable to the design problem, whence we obtain

THEOREM 5.1. *Let $\{T_n\}$ be* RS$\{h\}$ *with $f \in H(R)$ and $f \in C^{2k}[0, 1]$ then*

$$\lim_{n\to\infty} n^{2k}\{\operatorname{Var}(\hat\beta_{1,T_n}) - \operatorname{Var}(\hat\beta)\} = C_k^2\left(\int_0^1 \frac{(f^{(2k)}(x))^2}{h^{2k}(x)}\,dx\right)\operatorname{Var}^2(\hat\beta),$$

where C_k is the constant in Theorem 3.3.

Recall that $\hat\beta_{1,T_n}$ is the BLUE of β using the design T_n and no derivative information. This theorem characterizes the asymptotic behavior of the variance of the BLUE in terms of the density defining a design. If one wants asymptotically *optimal* designs then the following theorem is of interest.

THEOREM 5.2. *Let $\gamma = (k + ½)^{-1}$, $f \in H(R)$, $f \in C^{2k}[0, 1]$, and suppose that $h(x) = |f^{(2k)}(x)|^\gamma / \int_0^1 |f^{(2k)}(\tau)|^\gamma\, d\tau$. Then if $\{T_n\}$ is* RS$\{h\}$ *we have*

$$\lim_{n\to\infty} n^{2k}\{\operatorname{Var}(\hat\beta_{1,T_n}) - \operatorname{Var}(\hat\beta)\} = C_k^2\left(\int_0^1 |f^{(2k)}(x)|^\gamma\,dx\right)^{2/\gamma}\operatorname{Var}^2(\hat\beta).$$

Furthermore, if U_n is RS(p) *with $p \neq h$ we have for large n* $\mathrm{Var}(\hat{\beta}_{1,T_n}) < \mathrm{Var}(\hat{\beta}_{1,U_n})$.

The interest in the above theorems is two-fold. First, the fact that the limit exists was not previously known although Sacks and Ylvisaker (1970) knew that $\lim_{n\to\infty} n^{2k}\{\mathrm{Var}(\hat{\beta}_{1,T_n}) - \mathrm{Var}(\hat{\beta})\}$ was bounded by two numbers. Second, it is of interest that the constant C_k in the limit involves the Bernoulli numbers.

6. Examples.

EXAMPLE 1. The location and scale parameter model assumes that a random sample $X_1, \cdots, X_n$ is taken from a distribution of the form

$$F(x) = F_0\left(\frac{x-\mu}{\sigma}\right)$$

where F_0 is a known distributional form and μ and σ are respectively location and scale parameters (either or both of which may be assumed to be unknown). F_0 is assumed to be absolutely continuous with associated density f_0. Let $Q_0(u) := F_0^{-1}(u)$ and denote by f_0Q_0 the composition of f_0 and Q_0, i.e., $f_0Q_0(u) := f_0(Q_0(u))$. The sample quantile function is defined by $\tilde{Q}(u) := X_{(j)}$, $(j-1)/n < u \leq j/n$ where $X_{(j)}$ is the jth order statistic from the random sample $X_1, \cdots, X_n$.

Parzen (1979) has shown that, for large samples, location and/or scale parameter estimation can be considered as a problem in continuous parameter time series regression by writing

$$f_0Q_0(u)\tilde{Q}(u) = \mu f_0Q_0(u) + \sigma Q_0(u)f_0Q_0(u) + \sigma_B B(u) \tag{6.1}$$

where $B(\cdot)$ is a Brownian bridge process and $\sigma_B = \sigma/\sqrt{n}$. Using this framework Eubank (1979) has noted the equivalence of optimal design selection for the model (6.1) and the selection of optimal spacings for the sample quantiles utilized in constructing (asymptotically) best linear unbiased estimators of μ and σ.

Since the error process for the model (6.1) is Brownian bridge, Theorem 2.1 can be applied to the case of estimating $\mu(\sigma)$ when $\sigma(\mu)$ is known. By noting that

$$f_0Q_0(u) = u(1-u) \qquad \text{for the logistic distribution}$$

$$Q_0(u)f_0Q_0(u) = \beta(1-u) - \beta(1-u)^{1+(1/\beta)} \qquad \text{for the Pareto distribution}$$

$$Q_0(u)f_0Q_0(u) = m(1-u)\log\frac{1}{1-u} \qquad \text{for the Weibull distribution}$$

where β and m are both positive shape parameters, it is seen that Theorem 2.1 applies to all three cases provided $\beta < 1$. Thus, Theorem 2.1 provides an independent proof of results previously obtained by Gupta and Gnanadesikan (1966), Kulldorf and Vannman (1973) and Ogawa (see Sarhan and Greenberg, 1962; pages 371–380) regarding the unicity of optimal spacings for the logistic, Pareto, and exponential distributions respectively.

Since the classical approach to optimal spacing selection has dealt with finding spacings that satisfy certain necessary conditions for maximum estimator efficiency (see Sarhan and Greenberg, 1962), one of the principal problems has been that the number of solutions is not known. Theorem 2.1, when applicable, alleviates such difficulty.

EXAMPLE 2. Consider the design problem connected with the Brownian bridge process $B(\cdot)$ of the type $Y(t) = \beta\phi(t) + B(t)$, $0 \leq t \leq 1$. If $\phi(t) = t^k$ with $k \geq 2$ then Theorem 2.1 guarantees unique optimal designs. If, on the other hand, $\phi(t) = e^{t^2}$ then Theorem 2.1 no longer applies since $\log(\phi'') = t^2 + \log(4t^2+2)$ is not concave. However, Theorem 2.2 is applicable and we conclude that for some positive integer n_0 unique optimal designs exist for all $n > n_0$.

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