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ROBUST TESTS FOR SPHERICAL SYMMETRY AND THEIR APPLICATION TO LEAST SQUARES REGRESSION

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Invariance is used to show that Kariya and Eaton's test for multivariate spherical symmetry is UMP invariant against elliptically symmetric distributions. Also both the null and alternative distributions of the test statistic are found to be the same as those which occur when the sample is normally distributed. UMP and UMPU tests for serial correlation derived assuming normality are found to be even more robust against departure from this assumption than was recently demonstrated by Kariya. When applied to the linear regression model, these results give useful robustness properties for Kadiyala's $T1$ test and the Durbin-Watson test.

1. Introduction and Notation[1]. Following Kariya and Eaton (1977), let $O(n)$ denote the group of $n \times n$ orthogonal matrices in Euclidean n-space, R^n, and let $\mathscr{L}(x)$ denote the distribution law of an $n \times 1$ random vector x. A random vector x has a *spherically symmetric* distribution if $\mathscr{L}(x) = \mathscr{L}(Gx)$ for $G \in O(n)$. Let $\mathscr{F}_0^n$ denote the class of spherically symmetric pdf's, i.e., $\mathscr{F}_0^n = \{f \in \mathscr{F}^n \mid f(x) = f(Gx), x \in R^n, G \in O(n)\}$ where $\mathscr{F}^n$ is the class of all pdf's with respect to the Lebesgue measure on R^n.

In order to introduce the problems considered in this paper, set

$$Q_0 = \{q \mid q \text{ is a function on } [0, \infty)\},$$

$$Q_1 = \{q \mid q \in Q_0 \text{ and } q \text{ is nonincreasing}\},$$

$$Q_2 = \{q \mid q \in Q_1 \text{ and } q \text{ is convex}\},$$

and, for any given $n \times n$ positive definite matrix, Σ, let

$$\mathscr{F}_0^n(\Sigma) = \{f \in \mathscr{F}^n \mid f(x) = |\Sigma|^{-1/2} q(x'\Sigma^{-1}x), q \in Q_0\},$$

$$\mathscr{F}_1^n(\Sigma) = \{f \in \mathscr{F}^n \mid f(x) = |\Sigma|^{-1/2} q(x'\Sigma^{-1}x), q \in Q_1\},$$

$$\mathscr{F}_2^n(\Sigma) = \{f \in \mathscr{F}^n \mid f(x) = |\Sigma|^{-1/2} q(x'\Sigma^{-1}x), q \in Q_2\}.$$

The term *elliptically symmetric* often is used to describe a distribution with pdf belonging to $\mathscr{F}_0^n(\Sigma)$.

Suppose $x \in R^n$ is an observed random vector with pdf h. The problem of testing $H_0 : h \in \mathscr{F}_0^n$ against $h \in \mathscr{F}_1^n(\Sigma)$, $\Sigma \neq \sigma^2 I_n$, has been studied by Kariya and Eaton (1977) who found that the test which rejects H_0 for small values of $t = x'\Sigma^{-1}x/x'x$ is UMP (uniformly most powerful), and that the null distribution of t is the same as that when $\mathscr{L}(x)$ is $N(0, I_n)$. In Section 2, we show that the test is also UMP invariant against the more general alternative hypothesis, $H_a : h \in \mathscr{F}_0^n(\Sigma)$, $\Sigma \neq \sigma^2 I_n$. An interesting corollary of this result is that the distribution of t under H_a is identical to that when $\mathscr{L}(x)$ is $N(0, \Sigma)$.

When Σ is of the form

$$\Sigma(\lambda)^{-1} = I_n + \lambda A \tag{1.1}$$

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[1] Throughout we use upper-case symbols to denote matrices while vectors are represented by lower-case symbols.

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$$(1.2)\qquad \text{with } \lambda \in \Lambda \equiv \{\lambda \in R \mid \Sigma(\lambda)^{-1} \text{ positive definite}\},$$

$A \neq \sigma^2 I_n$, $A \neq 0$ and λ unknown, Kariya (1977) has shown that the test statistic

$$s = x'Ax/x'x$$

with c.r. (critical region) $s < c_1$ or $s > c_2$ provides a UMPU (UMP unbiased) test against the alternative hypothesis $h \in \mathscr{F}_2^n\ (\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda \neq 0$. We show that this test is also UMPU invariant against the more general alternative, $H_a' : h \in \mathscr{F}_0^n\ (\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda \neq 0$ and that the distributions of s under H_a' are the same as those when $\mathscr{L}(x)$ is $N(0, \Sigma(\lambda))$, for each nonzero value of λ.

The remainder of the paper is devoted to the application of these results, through invariance, to the linear model $y = X\beta + u$, where X is an $n \times k$ fixed matrix of rank k. In Section 3, we derive a UMP invariant test of $h \in \mathscr{F}_0^n$ versus $h \in \mathscr{F}_0^n(\Sigma)$, where h is the pdf of the regression disturbance vector, u, and Σ is any fixed, positive definite matrix. This generalizes a result obtained by Kadiyala (1970) for normally distributed disturbances and is followed, in Section 4, by a generalization of Kariya's (1977) theorem on the problem of testing $h \in \mathscr{F}_0^n$ versus $h \in \mathscr{F}_0^n(\gamma\Sigma(\lambda))$, $\gamma > 0$ when either $\lambda > 0$ or $\lambda \neq 0$ and $\Sigma(\lambda)$ is given by (1.1) and (1.2). The final section deals with the application of these results to the Durbin-Watson test.

2. The Main Results. Suppose x is an $n \times 1$ random vector with pdf h. The problem of testing $H_0 : h \in \mathscr{F}_0^n$ against $H_a : h \in \mathscr{F}_0^n(\Sigma)$, $\Sigma \neq \sigma^2 I_n$ remains invariant under transformations of the form

$$g(x) = \alpha x$$

where α is a positive scalar. A maximal invariant statistic is

$$w(x) = x/(x'x)^{1/2}$$

since $w(x_1) = w(x_2)$, where x_1 and x_2 are $n \times 1$ vectors, implies $\alpha x_1 = x_2$ where $\alpha = (x_2'x_2)^{1/2}/(x_1'x_1)^{1/2}$.

THEOREM 1. *For a fixed, positive definite, $n \times n$ matrix, Σ, when testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a : h \in \mathscr{F}_0^n(\Sigma)$, $\Sigma \neq \sigma^2 I_n$, the test which rejects H_0 for small values of*

$$t = x'\Sigma^{-1}x/x'x$$

is a UMP *invariant test.*

PROOF. Under H_0, $w(x)$ is uniformly distributed on $C_n = \{x \mid x \in R^n, x'x = 1\}$. Hence, $w(x)$ has the same distribution for all $h \in \mathscr{F}_0^n$ including pdf's of the multivariate normal distribution.

The pdf of $y = w(x)$ under H_a can be shown to be

$$f(y) = \tfrac{1}{2}\Gamma(n/2)\pi^{-n/2}|\Sigma|^{-1/2}(y'\Sigma^{-1}y)^{-n/2}$$

with respect to the uniform measure on C_n. Therefore, under H_a, $w(x)$ has the same distribution for all $h \in \mathscr{F}_0^n\ (\Sigma)$ including those of the multivariate normal distribution with covariance matrix $\sigma^2\Sigma$, $\sigma \neq 0$.

Since all invariant statistics can be expressed as functions of $w(x)$, we appeal to the corresponding known result for normally distributed x (Lehmann and Stein (1948)) to complete the proof.

COROLLARY. *The null distribution of t is identical to the distribution of t when $\mathscr{L}(x)$ is $N(0, I_n)$ while the alternative distribution of t corresponds to that when $\mathscr{L}(x)$ is $N(0, \Sigma)$.*

Theorem 1 can be extended to cover the following situations when Σ is not fixed (see Kariya and Eaton (1977)): (i) $\Sigma = \sigma^2\Sigma_0$, Σ_0 known; (ii) $\Sigma = \lambda_1(I_n - M) + \lambda_2 M$, $M^2 = M$,

M known, where $\lambda_1 > \lambda_2 > 0$ (or $\lambda_2 > \lambda_1 > 0$) and (iii) $\Sigma^{-1} = \lambda_1 I_n + \lambda_2 A$, A known, $\lambda_1 > 0$, $\lambda_2 > 0$ such that Σ is positive definite.

In the latter case, using the case (i) extension, we can write Σ^{-1} in the form of (1.1) and (1.2). For a two sided test against the alternative hypothesis which allows λ in (1.1) to make any nonzero value, the arguments used to prove Theorem 1 can be applied to generalize Kariya's (1977) Theorem 4 to the following:

THEOREM 2. *The test statistic*

$$s = x'Ax/x'x$$

with c.r. $s < c_1$ *or* $s > c_2$ *provides a* UMPU *invariant test of* $H_0 : h \in \mathscr{F}_0^n$ *against* $H_a' : h \in \mathscr{F}_0^n(\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda \neq 0$, *where* $\Sigma(\lambda)$ *is given by* (1.1) *and* (1.2), $A \neq \sigma^2 I_n$, $A \neq 0$.

COROLLARY *The null distribution of* s *is the same as when* $\mathscr{L}(x)$ *is* $N(0, I_n)$ *while the alternative distributions of* s *correspond to those when* $\mathscr{L}(x)$ *is* $N(0, \Sigma(\lambda))$, $\lambda \neq 0$.

3. Application to the Linear Model. In this section, Theorem 1 is applied to the linear regression model

$$y = X\beta + u \tag{3.1}$$

where X is an $n \times k$ fixed matrix of rank k, β is a $k \times 1$ vector of unknown parameters and u is an $n \times 1$ disturbance vector. This allows us to extend a result of Kadiyala (1970) obtained assuming normally distributed disturbances.

Let b denote the ordinary least squares estimate of β, i.e., $b = (X'X)^{-1} X'y$ and z the vector of associated residuals, i.e., $z = y - Xb = My = Mu$, where $M = I_n - X(X'X)^{-1}X'$. Suppose the unobservable error term u has pdf h. We shall consider the problem of testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a : h \in \mathscr{F}_0^n(\Sigma)$ where Σ is fixed.

This problem is invariant to transformations of the form

$$y^* = \gamma_0 y + X\gamma \tag{3.2}$$

where γ_0 is a positive scalar and γ is a $k \times 1$ vector.

Following Kadiyala (1970), let P be an orthogonal matrix such that

$$PMP' = \begin{bmatrix} I_{n-k} & 0 \\ 0' & 0_k \end{bmatrix} \tag{3.3}$$

and

$$PP' = P'P = I_n. \tag{3.4}$$

Let P be partitioned as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{3.5}$$

where P_1 is $(n - k) \times n$ and P_2 is $k \times n$. Note that

$$Pz = \begin{bmatrix} P_1 z \\ 0 \end{bmatrix}$$

since (3.3) and (3.4) imply

$$PM = \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$$

and $P_1 u = P_1 z$ follows from $P_1 M = P_1$. From (3.4) we have $P_1'P_1 = I_n - P_2'P_2$. Post-multiplying by M yields

$$P_1'P_1 = M. \tag{3.6}$$

$$w = P_1 z/(z'P_1'P_1 z)^{1/2} = P_1 u/(u'P_1'P_1 u)^{1/2} \tag{3.7}$$

is a maximal invariant with respect to transformations of the form (3.2).

Let g be the pdf of $v = P_1 u$. Under H_0, $g \in \mathscr{F}_0^m$ where $m = n - k$ (see Lord (1954)) while Kelker (1970) has shown that under H_a,

$$g \in \mathscr{F}_0^m(P_1 \Sigma P_1'). \tag{3.8}$$

Since the maximal invariant, w, is a function of v, the principle of invariance and Theorem 1 imply that the test with c.r.

$$t = v'(P_1 \Sigma P_1')^{-1} v/v'v < c$$

is UMP invariant where invariance is with respect to transformations of the form (3.2).

We now shall show that this test is equivalent to the likelihood ratio test derived assuming normally distributed disturbances. We require the following result from Rao (1973, page 77).

LEMMA 1. *$V^{-1} - V^{-1}U(U'V^{-1}U)^{-1}U'V^{-1} = T(T'VT)^{-1}T'$ where V is any $n \times n$ positive definite matrix and U and T are $n \times k$ and $n \times (n-k)$ matrices respectively such that if $W = (U : T)$ then $W'W = WW' = I_n$.*

LEMMA 2. *$t = v'(P_1\Sigma P_1')^{-1} v/v'v = \hat{u}'\Sigma^{-1}\hat{u}/z'z$ where P_1 is any $(n-k) \times n$ matrix for which* (3.3), (3.4) *and* (3.5) *hold and $\hat{u}$ is the vector of generalized least squares residuals,* i.e., $\hat{u} = y - X(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$.

PROOF. Applying Lemma 1 with $T = P_1'$, $U = P_2'$ and $V = \Sigma$ and using (3.6) we have

$$\begin{aligned} t &= v'(P_1\Sigma P_1')^{-1} v/v'v \\ &= u'P_1'(P_1\Sigma P_1')^{-1}P_1 u/u'P_1'P_1 u \\ &= u'(\Sigma^{-1} - \Sigma^{-1}P_2'(P_2\Sigma^{-1}P_2')^{-1}P_2\Sigma^{-1})u/u'Mu \\ &= u'(\Sigma^{-1} - \Sigma^{-1}X(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1})u/u'Mu \\ &= \hat{u}'\Sigma^{-1}\hat{u}/z'z. \end{aligned}$$

The second last equality follows because we can write $P_2' = XG$ where G is a $k \times k$ nonsingular transformation matrix.

The results of this section can be summarized as follows:

THEOREM 3. *Let h be the* pdf *of the disturbance vector, u, in the linear model* (3.1). *For testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a : h \in \mathscr{F}_0^n(\Sigma)$, where Σ is a fixed positive definite matrix, the test which rejects H_0 for small values of*

$$t = \hat{u}'\Sigma^{-1}\hat{u}/z'z$$

where $\hat{u}$ are the generalized least squares residuals assuming covariance matrix Σ and z are the ordinary least squares residuals, is UMP *invariant.*

COROLLARY. *The null distribution of t is the same as that when $\mathscr{L}(u)$ is $N(0, I_n)$ while its alternative distribution is identical to that when $\mathscr{L}(u)$ is $N(0, \Sigma)$.*

4. Testing for Serial Correlation in the Linear Model. Obviously Theorem 3 can be extended to situations where $\Sigma = \sigma^2\Sigma_0$ with σ^2 unknown and Σ_0 known. In the more interesting case where $\Sigma(\lambda)^{-1}$ is given by (1.1) and (1.2) we need the restrictive assumption that the column space of X is spanned by some k latent vectors of A. The following generalizes Kariya's Theorem 5:

THEOREM 4. *Let h be the* pdf *of the disturbance term, u, in the linear model* (3.1). *When the column space of X is spanned by some k latent vectors of A in* (1.1), *then for testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a : h \in \mathscr{F}_0^n(\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda > 0$, the test statistic*

$$s = z'Az/z'z \tag{4.1}$$

with c.r. $s < c_0$, *where z is the ordinary least squares residual vector, is* UMP *invariant unless $A = \omega^2 I_n$ or $A = 0$ and for testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a' : h \in \mathscr{F}_0^n(\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda \neq 0$, the* c.r. $s < c_1$ *or* $s > c_2$ *is* UMPU *invariant unless $A = 0$ or $A = \omega^2 I_n$.*

PROOF. The one-sided test is a straight forward application of Theorem 3 since it is well known that when the column space of X is spanned by some k latent vectors of A, $\hat{u}$ and z coincide (see, for example, Watson (1967)).

That the two-sided test is UMPU invariant can be verified by following the proof of Theorem 3 and noting that when the column space of X is spanned by some k latent vectors of A, P_1 can be chosen so that its rows correspond to the remaining $n - k$ latent vectors of $\Sigma(\lambda)$ (and hence of A). Then

$$\begin{aligned}(P_1\Sigma(\lambda)P_1')^{-1} &= P_1\Sigma(\lambda)^{-1}P_1' \\ &= I_m + \lambda P_1AP_1'\end{aligned}$$

and (3.8) becomes

$$g \in \mathscr{F}_0^m(\gamma(I_m + \lambda P_1AP_1')^{-1})$$

allowing Theorem 2 to be applied.

COROLLARY. *The null distribution of s is the same as that when $\mathscr{L}(u)$ is $N(0, I_n)$ while its alternative distributions correspond to those when $\mathscr{L}(u)$ is $N(0, \Sigma(\lambda))$, $\lambda \neq 0$.*

The properties of the s test in the neighbourhood of $\lambda = 0$ are also of interest.

THEOREM 5. *Let h be the* pdf *of the disturbance vector, u, in the linear model* (3.1). *For testing $H_0 : h \in \mathscr{F}_0^n$ versus $H_a : h \in \mathscr{F}_0^n(\gamma\Sigma(\lambda))$, $\gamma > 0$, $\lambda > 0$, where $\Sigma(\lambda)$ is given by* (1.1) *and* (1.2), *the test statistic* (4.1) *with* c.r. $s < c_0$ *is a locally best invariant test in the neighbourhood of $\lambda = 0$ unless $A = \omega^2 I_n$ or $A = 0$.*

PROOF. Let $f_0(w)$ denote the pdf of the maximal invariant w under H_0. With respect to the uniform measure on C_m, $f_0(w)$ has the form,

$$f_0(w) = \tfrac{1}{2}\Gamma(m/2)\pi^{-m/2}.$$

Let $f_\lambda(w)$ denote the pdf of w under H_a. It has the form

$$f_\lambda(w) = \tfrac{1}{2}\Gamma(m/2)\pi^{-m/2}|P_1\Sigma(\lambda)P_1'|^{-1/2}(w'(P_1\Sigma(\lambda)P_1')^{-1}w)^{-m/2}$$

with respect to the uniform measure on C_m.

The Generalized Neyman-Pearson Lemma implies that a locally best invariant test is given by the c.r.

$$\frac{\partial}{\partial\lambda}[f_\lambda(w)/f_0(w)]|_{\lambda=0} > c_0'. \tag{4.2}$$

$$\begin{aligned}&\frac{\partial}{\partial\lambda}f_\lambda(w)|_{\lambda=0} \\ &= -\tfrac{1}{2}\left[\frac{\partial}{\partial\lambda}\{|P_1\Sigma(\lambda)P_1'|\}|P_1\Sigma(\lambda)P_1'|^{-1}f_\lambda(w)\right]\Bigg|_{\lambda=0}\end{aligned}$$

$$-\frac{m}{2}\left[\frac{\partial}{\partial\lambda}\{w'(P_1\Sigma(\lambda)P_1')^{-1}w\}\{w'(P_1\Sigma(\lambda)P_1')^{-1}w\}^{-1}f_\lambda(w)\right]\Bigg|_{\lambda=0}$$

$$= -\tfrac{1}{2}[c_1' + mw'P_1AP_1'w]f_0(w)$$

where $\Sigma(\lambda) = (I_n + \lambda A)^{-1}$ and c_1' a scalar constant. Clearly (4.2) is equivalent to the c.r. $s < c_0$.

5. Application to the Durbin-Watson Test. Suppose the components of the disturbance vector of the linear model (3.1) are generated by the first-order autoregressive scheme

$$u_t = \rho u_{t-1} + e_t \qquad t = 2, \cdots, n$$

where

$$u_1 = e_1/(1-\rho^2)^{1/2}, \; |\rho| < 1.$$

Let $e = (e_1, \cdots, e_n)'$, then $Tu = e$ and $u = T^{-1}e$ where

$$T = \begin{bmatrix} (1-\rho^2)^{1/2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & & 0 & 0 \\ 0 & -\rho & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}.$$

Durbin and Watson (1950) have shown that their statistic $d = z'A_1z/z'z$ with c.r. $d < d_0$ where

$$A_1 = \begin{bmatrix} 1 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & & & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ & & & & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

is an approximately UMP test of $H_0: \rho = 0$ against $H_a: \rho < 0$ when the column space of X is spanned by k latent vectors of A_1 and $\mathscr{L}(e)$ is $N(0, \sigma^2 I_n)$. Kariya (1977) generalized this result to $g \in \mathscr{F}_1^n$, where g is the pdf of e, and also found that, with the same restriction on the X matrix, the two-sided Durbin-Watson test is an approximately UMPU test of $H_0: \rho = 0$ against $H_a': \rho \neq 0$ when $g \in \mathscr{F}_2^n$. For general X, Durbin and Watson (1971) have demonstrated that their one-sided test is locally UMP invariant in the neighbourhood of $\rho = 0$.

Suppose $g \in \mathscr{F}_0^n$. Then we can write (see Kelker (1970)),

$$(5.1) \qquad g(e) = q(e'e), \qquad q \in Q_0$$

and the pdf of u is given by

$$h(u) = |T| q(u'T'Tu)$$
$$= |\Sigma|^{-1/2} q(u'\Sigma^{-1}u)$$

where $\Sigma^{-1} = (T'T)$

$$= \begin{bmatrix} 1 & -\rho & 0 & \cdots & & 0 \\ -\rho & 1+\rho^2 & -\rho & & & 0 \\ 0 & -\rho & 1+\rho^2 & & & 0 \\ \vdots & & & \ddots & & \vdots \\ & & & & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$$

$$= (1-\rho)^2 I_n + \rho A_1 + \rho(1-\rho)C$$

and

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Following Durbin and Watson (1950), approximate Σ^{-1} by $\Sigma^{-1} \approx (1 - \rho)^2 I_n + \rho A_1$. If we assume $g \in \mathscr{F}_0^n$, Theorem 4 implies that when the column space of X is spanned by k latent roots of A_1, the one-sided Durbin-Watson test is an approximately UMP invariant test of H_0 against H_a while the two-sided test is an approximately UMPU invariant test of H_0 versus H_a'. For general X, that the one-sided Durbin-Watson test is approximately locally best invariant in the neighbourhood of $\rho = 0$ follows from Theorem 5. Note that for a given value of ρ, the distribution of d is independent of the form of q in (5.1) and hence can be calculated assuming $\mathscr{L}(e)$ is $N(0, \sigma^2 I)$.

Application of Theorem 4 and Theorem 5 will yield similar results for the higher-order Durbin-Watson type statistics studied by Wallis (1972), Vinod (1973) and King and Giles (1977).

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