The Annals of Statistics
1980, Vol. 8, No. 5, 1168–1170

NOTE ON THE SIGN TEST IN THE PRESENCE OF TIES

BY ALBRECHT IRLE AND KARL-HEINZ KLÖSENER

University of Münster

Correcting a result of Krauth we show that there does not exist a UMP sign test for certain nonparametric hypotheses and treat the asymptotic properties of a one-sided sign test.

Krauth (1973) treated the following problem: Let $Z_1, \cdots, Z_n$ be i.i.d. random variables,

$$n_+ = |\{Z_k : Z_k > 0\}|, n_- = |\{Z_k : k < 0\}|, n_0 = |\{Z_k : Z_k = 0\}|$$

and consider the problem of testing—based on (n_+, n_0)—the hypothesis $H : \text{Prob}(Z_k > 0) = \text{Prob}(Z_k < 0)$ against the alternative $A : \text{Prob}(Z_k > 0) > \text{Prob}(Z_k < 0)$.

Using the notations $\text{Prob}(Z_k > 0|H) = p_+$, $\text{Prob}(Z_k = 0|H) = p_0$, $\text{Prob}(Z_k < 0|H) = p_-$ and q_+, q_0, q_- for analogous probabilities under the alternative and assuming that all these probabilities are greater than zero Krauth ((1973), Theorem 1) stated that a UMP test for testing H against A with a known constant $p_0 = q_0$ is given by

$$n_+ + \tfrac{1}{2} n_0 > k_n(p_0). \tag{1}$$

From this he concluded (Krauth (1973), Theorem 2) that an asymptotic UMP test for testing H against A under the additional restriction $p_0 = q_0$ is given by

$$T_n = (2n_+ + n_0 - n)(n - n_0)^{-\frac{1}{2}} > u_\alpha, \tag{2}$$

where u_α denotes the α-fractile of the standard normal distribution $\mathfrak{N}(0, 1)$.

We will show in the following that the first statement (thus also the proof of the second statement) is not correct, and we treat the asymptotic properties of the test defined by (2). Let us remark that the arguments of Krauth are obviously only valid for the class of tests depending on $n_+ - n_-$ only.

LEMMA. *For fixed* $p_0 = q_0$ *a most powerful level* α *test* $\varphi_{n,q}$ *for testing* $\theta_0 : p_+ = p_- = (1 - p_0)/2$ *against the simple alternative* $\theta_q : q_+ = q > q_-$ *is given by*

$$n_+ + n_0/m(q) > c, \tag{3}$$

where $1 < m(q) < 2$ *and* $\lim_{q \to 2p_+} m(q) = 1$, $\lim_{q \to p_+} m(q) = 2$.

PROOF. An application of the Neyman-Pearson lemma and an obvious computation yields that $\varphi_{n,q}$ is given by (3) with $m(q) = (\log q_+ - \log q_-)/(\log p_+ - \log q_-)$. It is easily seen that $m(q)$ has the properties as stated above. ▯

Received August, 1978; revised March 1979.
AMS 1970 *subject classifications.* Primary 62G10.
Key words and phrases. Sign test, most powerful test, asymptotically uniformly most powerful test.

Now any most powerful level α test must be equal (a.s.) to $\varphi_{n,q}$ outside the set $\{n_+ + n_0/m(q) = c\}$ and so depends on q for $n > 2$. Thus it follows that there does not exist a UMP level α test for H against A with a known constant $p_0 = q_0$ if $n > 2$.

To treat the asymptotic properties of the sequence $\varphi^* = (\varphi_n^*)_{n\in\mathbb{N}}$ of tests defined by (2) we introduce the following asymptotic alternative $\tilde{A}(p_0)$:

$$\tilde{A}(p_0) = \left\{(\theta_{q(n)})_{n\in\mathbb{N}} : q(n) \in (p_+ ; 2p_+), c(n) = q(n) - p_+ = an^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}) \text{ with } a \geqq 0\right\}.$$

PROPOSITION. *φ^* is an* AUMP *level α test for θ_0 against $\tilde{A}(p_0)$*, i.e.,

$$\limsup_{n\to\infty} E_{\theta_0}\varphi_n^* \leqq \alpha \tag{4}$$

and for any sequence of tests $\tilde{\psi}$ satisfying (4)

$$\liminf_{n\to\infty} E_{\theta_{q(n)}}(\varphi_n^* - \psi_n) \geqq 0. \tag{5}$$

PROOF. Since $L(T_n|\theta_0) \to \mathfrak{N}(0, 1)$ (as stated e.g., in Putter (1955)) $E_{\theta_0}\varphi_n^* = \text{Prob}_{\theta_0}(T_n > u_\alpha) \to \alpha$, thus φ^* fulfills (4). Now consider a sequence in $\tilde{A}(p_0)$, $c(n) = an^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$.

(i) a = 0. Set

$$\tilde{S}_n = (2(n_+ + n_0/m(q(n))) - n)(n(1 - p_0))^{-\frac{1}{2}},$$
$$S_n = (2(n_+ + n_0/2) - n)(n(1 - p_0))^{-\frac{1}{2}}.$$

From $c(n) = o(n^{-\frac{1}{2}})$ we obtain $\tilde{S}_n - S_n \to 0$ in $(\theta_{q(n)})$-probability and by an application of the central limit theorem $S_n - T_n \to 0$ in $(\theta_{q(n)})$-probability. This yields

$$\tilde{S}_n - T_n \to 0 \qquad \text{in } (\theta_{q(n)})\text{-probability}. \tag{6}$$

Furthermore $L(S_n|\theta_0) \to \mathfrak{N}(0, 1)$ implies

$$L(\tilde{S}_n|\theta_0) \to \mathfrak{N}(0, 1). \tag{7}$$

Since $\varphi_{n,q(n)}$ is given by $\tilde{S}_n$, the assertion (5) follows from (6) and (7) (see Witting and Nölle (1970) page 56, page 58).

(ii) $a > 0$. Since φ_n^* is the indicator of $\{a\beta T_n > a\beta u_\alpha\}$ with $\beta = 2(1 - p_0)^{-\frac{1}{2}}$, the assertion (5) follows from the following easily verified statements

$$L(a\beta T_n|\theta_0) \to \mathfrak{N}(0, a^2\beta^2), \tag{8}$$

$$a\beta T_n - \log R_{n,q(n)} \to a^2\beta^2/2 \tag{9}$$

in θ_0-probability. ($R_{n,q(n)}$ denotes the likelihood ratio of the test $\varphi_{n,q(n)}$) (see Witting and Nölle (1970) page 68). ☐

Let us finally remark that the test proposed by Putter (1955) for this problem is shown to be UMP unbiased with respect to the class of all unbiased tests in Schlittgen (1978).

REFERENCES

[1] KRAUTH, J. (1973). An asymptotic UMP test in the presence of ties. *Ann. Statist.* **1** 166–169.
[2] PUTTER, J. (1955). The treatment of ties in some nonparametric cases. *Ann. Math. Statist.* **26** 368–386.
[3] SCHLITTGEN, R. (1978). The optimality of some tests for medians. *Statist. Hefte* **19** 15–24.
[4] WITTING, H. and G. NÖLLE (1970). *Angewandte Mathematische Statistik.* Teubner, Stuttgart.

WESTFALISCHE WILHELMS-UNIVERSITÄT MÜNSTER
INSTITUT FÜR MATHEMATISCHE STATISTIK
D-4400 MÜNSTER
ROXELER STRASSE 64
WEST GERMANY