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CONDITIONS FOR EXPECTED UTILITY MAXIMIZATION: THE FINITE CASE[1]

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A decision is a mapping from states of nature to consequences. Given a utility u on the set of consequences and a measure ν on the set of states, the expected utility of a decision f is $\int u(f(e))\,d\nu(e)$. By the "expected utility hypothesis" on a set of choices made by an individual we mean that there exists a utility *and a measure* such that the individual always chooses the decision of highest expected utility.

We present a set of necessary and sufficient conditions that a set of choices between two decisions be consistent with the expected utility hypothesis. We assume the set of states and the set of consequences to be finite and we do not assume the ordering, given by the choices, to be complete.

Our conditions require the individual to make new choices, between decisions which involve repetitions of states, in a consistent way. There are finitely many new choices and they do not involve utility.

1. Definitions, outline of the paper. Let X denote a finite set of outcomes, or *prizes*. (We do not assume that the elements of X are divisible or comparable. Thus if $x \in X$ we cannot require some $y \in X$ to be the "same as" $(\frac{1}{2})x$). Let E denote a finite set of states, or *events*; for simplicity we will further assume it contains two elements, say $E = \{H, T\}$. An ordered pair $(x, y) \in X \times X$ is a decision, or *ticket*. It is interpreted to yield its owner the prize x if H occurs and the prize y if T occurs.

We imagine a single *subject* who cannot influence the events H and T. We are given the information that our subject has made some *choices* between tickets. We denote the fact that he has chosen (x, y) over (z, w) by $(x, y) > (z, w)$. For example, if $X = \{x, y, z\}$, the subject may have made the choices

$$(x, z) > (y, z) \qquad (w, z) > (y, z)$$
$$(x, y) > (y, x) \qquad (y, w) > (w, y).$$

Note that the subject can make only finitely many choices (since X is finite) and that he need not make a choice between every available pair of tickets.

Given a utility u and probabilities p (of H) and q (of T), the expected utility of the ticket (x, y) is

$$E(x, y) = pu(x) + qu(y). \tag{1.1}$$

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(1.2) DEFINITION. Let $\mathbb{S}$ denote a set of choices:

$$\mathbb{S} = \{(x_{1i}, x_{2i}) > (x_{3i}, x_{4i}) : i = 1, \cdots, n\} \tag{1.3}$$

where $x_{j\,i} \in X$ for each i and j. We say $\mathbb{S}$ *is compatible with* EUH (expected utility hypothesis) if there exist p, q positive and a function u from X to the reals $\mathbb{R}$ such that

$$pu(x_{1i}) + qu(x_{2i}) > pu(x_{3i}) + qu(x_{4i}) \qquad \text{for} \quad i = 1, \cdots, n. \tag{1.4}$$

We require p and q to be positive to avoid trivial cases.

Since we assume strict preference, if a (necessarily finite) set of decisions satisfies EUH for a fixed p, q and u then any p', q', u' sufficiently close to p, q and u will also work. Thus the utilities we describe cannot be cardinal.

(1.5) Notation. For simplicity we may delete the u in formulas such as (1.4) so $x \in X$ stands interchangeably for a prize and for the utility of that prize. Thus instead of asking for a utility function $u : X \to R$ satisfying (1.4), we will ask for real numbers $x_{j\,i}$, $1 \leqslant j \leqslant 4$, $1 \leqslant i \leqslant n$, such that

$$px_{1i} + qx_{2i} > px_{3i} + qx_{4i} \qquad \text{for all} \quad i,$$

where we, of course, intend that if $x_{j\,i}$ and x_{kl} refer to the same prize, then they should be equal when we are referring to utilities. Also we denote q/p by ρ, so ρ is the "odds for T over H". Thus definition (1.2) becomes: $\mathbb{S}$ *is compatible with* EUH *if and only if there are real numbers* $x_{j\,i}$ *and a positive number* ρ *such that*:

$$x_{1i} + \rho x_{2i} > x_{3i} + \rho x_{4i} \qquad \text{for} \quad i = 1, \cdots, n. \tag{1.6}$$

Since ρ will always denote these odds, we make the *convention*: *the variable* ρ *is assumed to take on only positive values*, for the remainder of the paper.

We now outline the rest of the paper. In Section 2 we discuss other approaches to the problem of finding necessary and sufficient conditions for EUH, and describe criteria we would like such conditions to meet. None of the approaches we survey, not even our own, fully meet these criteria. Section 3 is an example illustrating the Fourier-Motzkin elimination procedure and its extension which is at the heart of our proofs of necessity and sufficiency. As mentioned above, our conditions for EUH entail repetitions of the $H - T$ experiment, and they involve more complicated tickets, which we call *compound* tickets. These and related definitions are given in Section 4, where we also extend the definition of EUH to cover choices between compound tickets, so that ordinary tickets and the EUH defined above are special cases. We will find it more convenient to give necessary and sufficient conditions for EUH in this more general setting. Since our conditions are not easily stated we devote Section 5 to carefully explaining and motivating them. Section 6 contains the detailed statements of the conditions, along with some more motivation, and Section 7 is the proof of necessity and sufficiency. In Section 8 we survey the success of our necessary and sufficient conditions in meeting the criteria discussed in Section 2. A more detailed version of this paper is available (Shapiro, 1977).

2. Other approaches. Previous work on conditions for EUH assumes structures which we do not have available. Most common is the assumption of an infinite state space (Savage, 1954) or a convex structure (Anscombe-Aumann, 1963). Other work typically assumes a complete order on the set of tickets: Pfanzagl (1968), Fishburn (1972), Luce and Krantz (1971). Fishburn (1975) does not require a total order on the set of all choices but again assumes a convex structure, through "horse lotteries." Both Fishburn (1975, page 296) and Luce and Krantz (1971, pages 257–8) (the latter assume a total order, but only on a certain set of "decisions") discuss the apparent need for additional structure beyond the initially given choices, a need we will fill by the use of compound tickets (see Section 4).

Given our restatement of EUH as the solvability of the quadratic inequalities (1.6), it seems natural to use duality theory. For fixed ρ, the equations (1.6) are linear. In fact they can be written in the form $A(\rho)x > 0$ where $A(\rho)$ is a matrix for each $\rho > 0$ and x is a vector with as many components as prizes. By (Gale, 1960) $A(\rho)x > 0$ has a solution iff there is no vector $y(\rho) \gg 0$ such that $y(\rho)A(\rho) = 0$. Thus $\mathbb{S}$ is compatible with EUH iff there exists some $\rho > 0$ such that for every $y \gg 0, yA(\rho) \neq 0$.

This approach, using duality, is deficient on at least two grounds.

(2.1a). It is not constructive. Given a specific $\mathbb{S}$, it does not tell us how to go about finding whether or not $\mathbb{S}$ is compatible with EUH. Of course for a single ρ one can test whether there is a $y(\rho) \gg 0$ with $y(\rho)A(\rho) = 0$ (Kuhn, 1956), but we might have to carry out this procedure for infinitely many $\rho > 0$ before possibly finding one that worked.

(2.1b). The conditions do not intuitively justify EUH; in fact they are almost a restatement of EUH. We would like to find conditions which are reasonable in the sense that they seem to impose fewer restrictions on a subject than does EUH. A prototype of such conditions is the strong axiom of revealed preference. This requires, in the context of choice under certainty, only that a subject not express a cycle of choices (e.g., $x_1 > x_2 > x_3 > x_1$), an eminently reasonable requirement, but (under certain conditions) it is equivalent to the utility hypothesis that a subject acts to maximize the value of a utility function (Richter, 1966).

A second approach to this problem can be found in the work of Tarski. Equations such as (1.6) are polynomial equations in their unknowns which are ρ and the elements of X. The statement that they have a solution is, therefore, a theorem of the first-order theory of algebra and by Tarski's theorem on the completeness of the theory of real-closed fields (Tarski, 1951) its truth or falsity can be verified in a constructive way. This solution does satisfy our requirement (2.1a) above, of constructivity, although the methods Tarski uses are not at all simple to apply. But it does not satisfy our requirement (2.1b) as far as we know: we have been unable to interpret Tarski's work to yield what we would call reasonable conditions for a subject's behavior under uncertainty.

The comments we have made about Tarski's work also apply to the results in Richter (1975). There, another constructive method does not supply reasonable conditions.

3. Illustration of the elimination procedure. We have mentioned that a generalization of the Fourier-Motzkin elimination procedure is central to our results. That procedure is used to solve linear inequalities by elimination of variables. To illustrate its use here, we consider another example:

$$\begin{aligned} \mathbb{S} : (x,y) &> (v,x) \\ (v,y) &> (y,x) \\ (x,y) &> (y,x). \end{aligned} \tag{3.1}$$

First we rewrite the system in inequality form and collect terms to get:

$$\begin{aligned} \mathbb{S}_0 : &(1)\ x - v + \rho(y - x) > 0 \\ &(2)\ v - y + \rho(y - x) > 0 \\ &(3)\ x - y + \rho(y - x) > 0. \end{aligned}$$

Adding equations (1) and (2) cancels the v and we get

$$\begin{aligned} \mathbb{S}_1 : (1+2) \quad & x - y + 2\rho(y - x) > 0 \\ (3) \quad & x - y + \rho\,(y - x) > 0. \end{aligned}$$

Clearly $\mathbb{S}_0 \Rightarrow \mathbb{S}_1$ in the sense that if x, y, v and ρ satisfy $\mathbb{S}_0$ then the same values of x, y and ρ satisfy $\mathbb{S}_1$. Notice that also $\mathbb{S}_1 \Rightarrow \mathbb{S}_0$. For if x, y and ρ satisfy $\mathbb{S}_1$ then by $(1 + 2)$ we have

$$x + \rho(y - x) > y - \rho(y - x)$$

so we can choose v between them, i.e.,

$$x + \rho(y - x) > v > y - \rho(y - x). \tag{*}$$

The first inequality in (*) shows that x, y, v and ρ satisfy (1) and the second inequality shows (2). Thus x, y, v and ρ satisfy $\mathbb{S}_0$. The equivalence $\mathbb{S}_0 \Leftrightarrow \mathbb{S}_1$ is a special case of the Fourier-Motzkin elimination procedure, which can be applied since the variable we eliminated, v, appears linearly. This same technique is applied in Section 7 below, at formulas (7.4) and (7.5).

Now to solve the system $\mathbb{S}_1$ we introduce the variable $z = x - y$. Then $\mathbb{S}_1$ is equivalent to

$$\begin{aligned} \mathbb{S}_2 : &(4)\ z(1 - 2\rho) > 0 \\ &(5)\ z(1 - \rho) > 0. \end{aligned}$$

We could solve this system in a few simple steps, but it will be instructive to continue an "elimination procedure." We could eliminate z from $\mathbb{S}_2$ if we knew that its coefficients, $1 - 2\rho$ and $1 - \rho$, differed in sign. For example, if $1 - 2\rho < 0$ and $1 - \rho > 0$ then we could multiply (4) by $(1 - \rho) > 0$ and (5) by $-(1 - 2\rho) > 0$, add them together and get $0 > 0$. Similarly, if $1 - 2\rho > 0$ and $1 - \rho < 0$ then $0 > 0$. If either of the coefficients $1 - \rho$ or $1 - 2\rho$ equals 0, then clearly $0 > 0$. Summarizing, $\mathbb{S}_2 \Rightarrow \mathbb{S}_3$ where $\mathbb{S}_3$ is:

$$\mathbb{S}_3\colon \ (6) \ \text{if } 1 - 2\rho \leqslant 0 \text{ and } 1 - \rho \geqslant 0 \text{ then } 0 > 0$$
$$(7) \ \text{if } 1 - 2\rho \geqslant 0 \text{ and } 1 - \rho \leqslant 0 \text{ then } 0 > 0.$$

One can further show that $\mathbb{S}_3$ is equivalent to $\mathbb{S}_2$.

Thus we have reduced the problem of whether (3.1) satisfies EUH to a question about signs of certain quantities not involving utilities: see Theorem (6.0)(a) below for our use of this technique.

4. Compound lottery tickets. Ideally, a set of necessary and sufficient conditions for a set $\mathbb{S}$ of choices to satisfy EUH should involve only the set $\mathbb{S}$ itself, e.g., requiring that no cycles (appropriately defined) exist. We are unable to find such conditions which are also constructive and reasonable, as defined in (2.1). Our conditions, though constructive and (we think) reasonable, require that some "choices", other than those in $\mathbb{S}$, be made. These choices, finite in number, concern tickets involving a *repeating* of the experiment with outcomes H and T. Thus these results are not strictly in the spirit of the subjectivist school, which claims expected utility maximization even in the case of unrepeatable experiments. See Section 8 for more information on this point.

The main result of Richter-Shapiro (1978) is that the set of ρ's which appear in solutions to (1.6) can be an arbitrary polynomial set, i.e., polynomials of arbitrarily high order are needed to describe such sets. This result suggests that repetitions of events will be needed in any necessary and sufficient conditions.

In describing our necessary and sufficient conditions it will be helpful to use tickets which involve positive integer multiples of the prizes in X, as well as repetitions of the $H - T$ experiment. (We can use a fair coin to interpret these multiples, as suggested by Ramsey). It will also be convenient to use notation such as

$$(x - z, y - w) > 0, \tag{4.1}$$

instead of

$$(x, y) > (z, w). \tag{4.2}$$

For fixed $K \geqslant 1$, define

$$E_K = \{e_0, e_1, \cdots, e_K\}$$

to be a certain set of outcomes of the $H - T$ experiment repeated K times, namely e_k denotes the outcome of H occurring on the first k trials, T on the other $K - k$ trials. Notice that $e_k \in E_K$ differs from $e_k \in E_{K'}$ if $K \neq K'$, but the appropriate meaning will be clear from the context. E_1 is just the outcomes of the $H - T$

experiment: e_0 corresponds to H and e_1 to T. The set E_K is not the set of *all* outcomes; for example "T then H" is not an outcome in E_2.

We number the elements of X as $x_1, \cdots, x_L$ and we expand the set of prizes to G, the free abelian group on the original prize set X. The group G consists of expressions of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_L x_L$$

where for $1 \leqslant l \leqslant L$, α_l is an integer (possibly negative or zero). Note that the zero prize (all $\alpha_l = 0$) is in G, and one adds or subtracts elements in G in the obvious way.

A *compound lottery ticket* is a function from E_K into G, i.e., it associates with each "compound event" in E_K a "compound prize" in G. If h is a compound lottery ticket we can denote h by its values: $(h(e_0), h(e_1), \cdots, h(e_K))$. For example, $(x, 2y - z, 0)$ is a compound lottery ticket which awards x for the outcome e_0, $2y - z$ for the outcome e_1, and nothing for the outcome e_2.

A *linear utility of* G is a real-valued function u on G satisfying $u(x + y) = u(x) + u(y)$ and $u(\alpha x) = \alpha u(x)$ for $x, y \in G$ and integers α. For a fixed linear utility u on G and positive numbers p and q we define the *expected utility* function E for compound lottery tickets by

$$E(h) = \Sigma_{k=0}^{K} p^{K-k} q^k u(h(e_k)) \tag{4.3}$$

for tickets h on the event set E_K. As in Section 2, we will denote q/p by ρ. Thus

$$E(h) > 0 \qquad \text{iff} \quad \Sigma_{k=0}^{K} \rho^k u(h(e_k)) > 0. \tag{4.4}$$

We call h a *simple* lottery ticket if $h = (x, y)$ where x and y are in X. Note that if h is a simple lottery ticket then (4.3) reduces to the definition of expected utility given in (1.1). We use the phrase "h is *desirable*" to mean $h > 0$, as in (4.1).

In Section 1, EUH meant the existence of u, p, q such that a preference for (x, y) over (z, w) implied $E(x, y) > E(z, w)$, where E was defined by (1.1). Here we generalize the meaning of EUH.

(4.5) DEFINITION. Let $\mathfrak{T}$ denote a set of the form

$$\mathfrak{T} = \{h_j > 0 : j = 1, 2, \cdots, J\}$$

where each h_j is a compound lottery ticket. Then $\mathfrak{T}$ is *compatible with* EUH if there exist a linear utility u on G and $p, q > 0$ such that, for the E defined by (4.3),

$$E(h_j) > 0 \qquad \text{for} \quad j = 1, 2, \cdots, J.$$

Now our goal of finding necessary and sufficient conditions for EUH will be reached if we use Definition 4.5.

5. Motivation. Our conditions for EUH involve three steps:

S1. Show how choices reveal (potential) information about the relative likelihood of H and T.

S2. Ensure that the information is consistent.

S3. Show how this information can be used to derive choices and, possibly, a contradiction.

We need a few definitions in order to clarify these steps. Let h be a compound lottery ticket on E_K. Then h is given by

$$h(e_k) = \Sigma_{l=1}^{L} \alpha_l^k x_l \qquad \text{for} \quad 1 \leqslant k \leqslant K,$$

for some integers α_l^k. We define the *restriction* of h to a prize x_l, denoted $h|x_l$, to be the ticket defined by

$$(h|x_l)(e_k) = \alpha_l^k x_l \text{ for } 1 \leqslant k \leqslant K.$$

Thus $h|x_l$ is the game h with all prizes ignored except x_l. Given positive probabilities p and q, and a linear utility u, the expected value of $h|x_l$ is clearly

$$E(h|x_l) = \Sigma_{k=0}^{K} p^{K-k} q^k \alpha_l^k u(x_l).$$

Setting $\rho = q/p$, we obtain

(5.1) $$E(h|x_l) > 0 \quad \text{iff} \quad u(x_l)\Sigma_{k=0}^{K} \alpha_l^k \rho^k > 0.$$

The polynomial $\Sigma_{k=0}^{K} \alpha_l^k \rho^k$ is called the *characteristic polynomial* of h, and it is denoted $P[h|x_l]$. Note that $P[h|x_l]$ is just the coefficient of $u(x_l)$ in the formula for $E(h)$. The intuitive meaning of $P[h|x_l]$ is this: it is positive iff x_l is more likely to be gained than lost, from using the ticket h.

Given the previous definitions, we can explain S1: each ticket-choice $h > 0$ potentially reveals, for each $x \in X$, that $P[h|x]$ is positive, zero or negative.

Next we explain the term "consistent" in S2.

(5.2) DEFINITION. A set of choices of signs of characteristic polynomials $P[h_j|x_l]$ (i.e., choices of their being $>$, $<$, or $=$, 0) for various j and l is *consistent* if there is a positive value of ρ, say $\bar{\rho}$, which gives all those polynomials their chosen sign, e.g., if we have chosen $P[h_j|x_l]$ to be positive for some j and l then we must have $P[h_j|x_l](\bar{\rho}) > 0$ for that j and l.

Step S3, deriving new choices, is complicated and we postpone it for now. But a *contradiction* is as hinted above—it is the choice (as desirable) of a ticket h having all prizes equal to zero, i.e., $h(e_k) = 0$ for all k, for example $(0, 0, 0) > 0$.

In general the steps S1–S3 get quite involved—there are potential revelations for each ticket and each prize in X, and these lead to new tickets, then new potential revelations, and so on ad nauseam (but not ad infinitum—everything is finite). Meanwhile we must keep track of which "potential" revelations lead to contradictions, so they can be discarded (the remaining revelations are "actual").

Because it will simplify matters greatly, we state our conditions contrapositively —that is, we require a set of "actual" revealed information (revelations which do not lead to a contradiction) to exist. We emphasize three points:

(1) This revealed information involves whether $P[h|x]$ is positive, negative, or zero, where $x \in X$ and h is either a ticket from the original set or a derived new ticket (see S3).

(2) The number of "new" tickets created is finite, so in deciding whether a set of revelations is actual—i.e., does not lead to a contradiction—one needs to go through only a finite number of steps.
(3) It is possible to express the decision as to whether $P[h|x]$ is $>$, $<$, or $= 0$ in terms of choices between tickets; we explain this at the end of Section 6.

6. Statement of necessary and sufficient conditions—the main theorem.

(6.0) THEOREM. *Let $\mathfrak{S}$ be a set of simple ticket-choices,*

$$\mathfrak{S} = \{(x_j, y_j) > (z_j, w_j) : \quad j = 1, 2, \cdots, J\}.$$

Define $h_j = (x_j - z_j, y_j - w_j)$ for $1 \leqslant j \leqslant J$ and define

$$\mathfrak{T}_1 = \{h_j > 0 : \quad j = 1, \cdots, J\}.$$

Then $\mathfrak{S}$ is compatible with EUH iff *$\mathfrak{T}_1$ is compatible with* EUH, iff *for each $l = 1, 2, \cdots, L$ it is possible to choose signs of characteristic polynomials $P[h|x_l]$, for $h \in \mathfrak{T}_l$, such that*

(a) *The choices are consistent,* i.e., *there exists $\bar{\rho} > 0$ such that each number $P[h|x_l](\bar{\rho})$, for $1 \leqslant l \leqslant L$ and $h \in \mathfrak{T}_l$, has a sign equal to that chosen for $P[h|x_l]$.*
(b) *The sets $\mathfrak{T}_l$ defined (inductively—see below) by these choices do not contain a contradiction,* i.e., *a member $h > 0$ where $h(e_k) = 0$ for all k.*

REMARKS. (i) The sets $\mathfrak{T}_l$, $l = 1, 2, \cdots, L + 1$, will be defined in the remainder of this section. For $1 \leqslant l \leqslant L$, $\mathfrak{T}_{l+1}$ will be constructed from the tickets in $\mathfrak{T}_l$ and the choices of signs of the restricted polynomials $P[h|x_l]$ as h ranges over $\mathfrak{T}_l$.

(ii) A similar theorem could be stated where $\mathfrak{S}$ contains arbitrary compound tickets, of arbitrary degree, and the proof would differ only in notation.

(iii) That $\mathfrak{S}$ is compatible with EUH iff $\mathfrak{T}_1$ is, was shown in the comments following (4.5); the rest of the proof of 6.0 is in Section 7.

We will now define the sets $\mathfrak{T}_l$, $l = 2, \cdots, L + 1$, mentioned in the theorem. Each $\mathfrak{T}_l$ will consist of tickets on E_K, $K = 2^{l-1}$. Intuitively, $\mathfrak{T}_{l+1}$ will be built from $\mathfrak{T}_l$ by "eliminating" the prize x_l; it consists of the "new" ticket-choices mentioned in S3.

Fix l, $1 \leqslant l \leqslant L$. Assume $\mathfrak{T}_l$ has been defined so that it consists of tickets on E_K, $K = 2^{l-1}$, and each ticket in $\mathfrak{T}_l$ involves only the prizes $x_l, x_{l+1}, \cdots, x_L$. (Clearly these hypotheses are satisfied for $l = 1$ by the theorem's $\mathfrak{T}_1$.) As in the theorem, assume we are given signs of the characteristic polynomials $P[h|x_l]$ for all $h \in \mathfrak{T}_l$; these signs will be used to define $\mathfrak{T}_{l+1}$. The set $\mathfrak{T}_{l+1}$ is the smallest set of tickets satisfying N1, N2 and N3 below:

(N1) Suppose $h \in \mathfrak{T}_l$ and $h|x_l(e_k) = 0$ for all k. Then $h^0 \in \mathfrak{T}_{l+1}$ where $h^0(e_k) = h(e_k)$ for $1 \leqslant k \leqslant K$ and $h^0(e_k) = 0$ for $K < k \leqslant 2K$.

(N2) Suppose $h \in \mathfrak{T}_l$ and $h|x_l(e_k) \neq 0$ for some k. If $P[h|x_l] = 0$, then $h^{\#} \in \mathfrak{T}_{l+1}$ where $h^{\#}$ is defined as follows: if

$$h(e_k) = \Sigma_{m=l}^{L} \alpha_{km} x_l \qquad \text{for} \quad 0 \leqslant k \leqslant K,$$

then

$$h^{\#}(e_k) = \Sigma_{m=l+1}^{L} \alpha_{km} x_l \qquad \text{for} \quad 0 \leqslant k \leqslant K$$

and

$$h^{\#}(e_k) = 0 \qquad \text{for} \quad K < k \leqslant 2K.$$

Conditions (N1) and (N2) say essentially "if $h \in \mathfrak{T}_l$ and x_l is irrelevant in h, then put h, properly extended, in $\mathfrak{T}_{l+1}$." The meaning of "properly extended" is that, regardless of the value of p, q and the utility u, we do not change the sign of the expected utility of h. The following lemma, whose proof is a simple computation, is a mathematical version of the relevant part of this paragraph.

(6.1) LEMMA. *Fix p, q and a utility u, and let E be the expected utility defined by them.*

(a) *If $h|x_l(e_k) = 0$ for all k then $E(h) > 0$* iff $E(h^0) > 0$.

(b) *If $E(h|x_l) = 0$ then $E(h) > 0$* iff $E(h^{\#}) > 0$.

The third method of elimination (N3) will assume there are $h, h' \in \mathfrak{T}_l$ satisfying $P[h|x_l] > 0$, $P[h'|x_l] < 0$ and will put a new ticket $h *_l h'$ in $\mathfrak{T}_{l+1}$. It is analogous to the following elimination method for inequalities: suppose we are to eliminate x from the inequalities

(6.2a) $$R_1 x + R_2 y + \cdots > 0$$

(6.2b) $$Q_1 x + Q_2 y + \cdots > 0$$

where the R_1 and Q_1 are real numbers. If $R_1 > 0$ and $Q_1 < 0$ we can multiply (6.2a) by $-Q_1$ and (6.2b) by R_1 to get

(6.3a) $$-Q_1 R_1 x - Q_1 R_2 y - \cdots > 0$$

(6.3b) $$R_1 Q_1 x + R_1 Q_2 y + \cdots > 0.$$

Adding these two inequalities, the x terms cancel:

(6.4) $$-Q_1 R_2 y + R_1 Q_2 y - \cdots > 0.$$

Note that if the coefficients R_i, Q_i in (6.2) are polynomials of degree K then those in (6.3) are of degree $2K$. (That's why if $\mathfrak{T}_l$ contains tickets on E_K, $\mathfrak{T}_{l+1}$ contains tickets on E_{2K}). The conditions $R_1 > 0$ and $Q_1 < 0$ will be analogues of $P[h|x_l] > 0$ and $P[h'|x_l] < 0$.

We now define the analogues of (6.3a) and (6.3b), $h*$ and $h'*$: If

(6.5) $$h|x_l(e_k) = \alpha_k x_l \qquad \text{and} \qquad h'|x_l(e_k) = \beta_k x_l$$

then

(6.6)
$$h^*(e_k) = -\Sigma_{m+n=k} \beta_m h(e_n), \qquad 0 \leqslant k \leqslant 2K$$
$$h'^*(e_k) = \Sigma_{m+n=k} \alpha_m h'(e_n), \qquad 0 \leqslant k \leqslant 2K.$$

Then, as (6.4) is the sum of (6.3a) and (6.3b), we define $h{*}_l h'$ to be the sum of h^* and h'^*:

(6.7) $$h{*}_l h'(e_k) = h^*(e_k) + h'^*(e_k) \quad \text{for} \quad 1 \leqslant k \leqslant 2K.$$

(N3) If h and h' are in $\mathfrak{T}_l$ and $P[h|x_l] > 0$ and $P[h'|x_l] < 0$ then $h{*}_l h' \in \mathfrak{T}_{l+1}$.

More motivation for the definition of $h{*}_l h'$ can be found in the next lemma, especially if one sees $E(h)$, $E(h')$, $P[h|x_l]$ and $P[h'|x_l]$ as analogues of (6.2a), (6.2b), R_1 and Q_1 respectively.

(6.8) LEMMA. *Fix positive probabilities p and q, and a linear utility u on G, and set* $\rho = q/p$.

(a) *Using the definitions of* h^*, h'^* *in* (6.6), *we have*

$$p^K E(h^*) = -P[h'|x_l](\rho)E(h)$$

$$p^K E(h'^*) = P[h|x_l](\rho)E(h')$$

(b) *If* $E(h)$, $E(h')$, $P[h|x_l](\rho)$ *and* $-P[h'|x_l](\rho)$ *are positive, so is* $E(h{*}_l h')$.

(c) *The ticket* $h{*}_l h'$ *does not include any nonzero multiple of* x_l *in any of its prizes.*

PROOF. (a) From the various definitions involved we get

$$P[h'|x_l](\rho) = \Sigma_{m=0}^{K}\beta_m\rho^m$$

$$E(h) = p^K\Sigma_{n=0}^{K}u(h(e_n))\rho^n.$$

The formula for multiplication of polynomials gives

$$P[h'|x_l](\rho)E(h) = p^K\Sigma_{k=0}^{2K}\rho^k\Sigma_{m+n=k}\beta_m u(h(e_n)).$$

Now use the fact that u is linear:

$$P[h'|x_l](\rho)E(h) = p^K\Sigma_{k=0}^{2K}\rho^k u(\Sigma_{m+n=k}\beta_m h(e_n)).$$

The right-hand side is clearly $-p^K E(h*)$. The proof of the other half of (a) is similar.

(b) Since $p > 0$, the hypotheses and (a) imply that $E(h*)$ and $E(h'*)$ are positive, so $E(h{*}_l h') = E(h*) + E(h'*)$ is positive.

(c) The terms involving x_l in $h*$ and h'^* are $\Sigma_{m+n=k}\ \beta_m\alpha_n x_l$ and $-\Sigma_{m+n=k}\alpha_m\beta_n x_l$, respectively, and these clearly add up to zero.

Lemma 6.8 also provides the key to an intuitive explanation of (N3): let us say you are presented with h and h', both desirable tickets, both involving a certain prize x, and in h you feel you are likely to get x (i.e., $P[h|x] > 0$), in h' you are likely to lose x (i.e., $P[h'|x] < 0$). Now suppose that for some reason the prize x is unavailable. You should be willing to exchange h and h' for some new ticket which is a combination of h and h', but which does not involve x. The ticket $h{*}h'$ is just such a ticket, if $x = x_l$.

Finally, we explain how the choice "$P[h|x]$ is $>$, $<$, or $= 0$" is equivalent to a choice between tickets. The key to this is (5.1). Suppose for the moment that $x \in X$

and $u(x)$ is positive. Then (5.1) becomes

$$(6.10) \qquad E(h|x) > 0 \quad \text{iff} \quad P[h|x](\rho) > 0.$$

Thus, instead of asking whether $P[h|x]$ were positive, we could ask a subject whether the restricted ticket $h|x$ were desirable. By (6.10), desirability of $h|x$ corresponds to positivity of $P[h|x]$. Similarly, desirability of $-h|x$ corresponds to $P[h|x] < 0$. As in (4.1), (4.2) desirability of tickets can be stated in terms of choices between tickets not involving "negative" prizes. Our assumption $u(x) > 0$ is not justifiable, but one can get around it by creating a new prize **a**, which the subject agrees is itself desirable, then replacing x by **a** in $h|x$, and asking whether the resulting ticket is desirable.

7. Proofs of necessity and sufficiency in the main theorem.

PROOF OF NECESSITY IN (6.0). Assume that $\mathfrak{T}_1$ is compatible with EUH and let $\bar{\rho}, u(x_1), \cdots, u(x_l)$ denote the values of ρ and u which define the expected utility function E such that

$$h > 0 \in \mathfrak{T}_1 \quad \text{iff} \quad E(h) > 0.$$

For this proof, E is to be computed using those values. The choices of signs of $P[h|x]$ can be made consistently if one chooses $P[h|x] > 0$ iff $E(h|x) > 0$, etc., proving the condition (6.9)(a). We claim that if $g \in \mathfrak{T}_l$ for any l then $E(g) > 0$. This will imply that some outcome of g is nonzero, proving that (6.9)(b) holds. By induction, assume $E(h) > 0$ for every h in some $\mathfrak{T}_l$, $l \leqslant L$, and let $g \in \mathfrak{T}_{l+1}$. If g arises from (N1) then $g = h^0$ for some $h \in \mathfrak{T}_l$, and $E(h) > 0$. Thus by Lemma (6.1)(a), $E(h^0) > 0$, and $E(g) > 0$. A similar argument is used in cases (N2) and (N3), appealing to Lemmas (6.1)(b) and (6.8)(b), respectively.

PROOF OF SUFFICIENCY IN (6.0). We assume the choices made of signs of $P[h|x_l]$ are consistent and we let $\bar{\rho}$ be a value of ρ which shows that they are, as in (5.2). We will isolate the main step of the proof in a definition and a lemma.

DEFINITION. Let $\mathfrak{T}$ be a set of compound lottery tickets. Then $\rho(\mathfrak{T})$ is the set of q/p such that for some real numbers $u(x_i)$, $i = 1, \cdots, L$, the expected utility function E defined by p, q and $\{u(x_i)\}$ satisfies $E(h) > 0$ if $h \in \mathfrak{T}$.

Clearly, $\mathfrak{T}$ is compatible with EUH iff $\rho(\mathfrak{T}) \neq \varnothing$.

(7.1) LEMMA. *Suppose $\bar{\rho}$ is a value of ρ which satisfies all the choices of signs of $P[h|x_l]$ for all $h \in \mathfrak{T}_l$ and all l, $1 \leqslant l \leqslant L$, and $\bar{\rho} \in \rho(\mathfrak{T}_{l+1})$ for some l, $1 \leqslant l \leqslant L$. Then $\bar{\rho} \in \rho(\mathfrak{T}_l)$.*

PROOF. Let

$$\mathfrak{T}_l = \{h_j > 0 : \quad j = 1, 2, \cdots, J\}$$

$$\mathfrak{T}_{l+1} = \{g_i > 0 : \quad i = 1, 2, \cdots, I\}.$$

By construction, the tickets in $\mathfrak{T}_l$ involve prizes $x_l, x_{l+1}, \cdots, x_L$ and those in $\mathfrak{T}_{l+1}$ involve $x_{l+1}, \cdots, x_L$. (If $l = L$ then $\mathfrak{T}_{l+1}$ is the empty set or has one ticket, all of whose outcomes are zero).

That $\bar{\rho} \in \rho(\mathfrak{T}_{l+1})$ means there exist real numbers $u(x_{l+1}), \cdots, u(x_L)$ such that the resulting expected utility function E satisfies $E(g) > 0$ for all $g \in \mathfrak{T}_{l+1}$. Our task is to find a value for $u(x_l)$ so that the E defined by $\bar{\rho}, u(x_l), u(x_{l+1}), \cdots, u(x_L)$, which we call E', satisfies $E'(h) > 0$ for all $h \in \mathfrak{T}_l$. Each of the J conditions $E'(h_j) > 0$ will impose a condition on $u(x_l)$ and we must show they all can be satisfied. We denote h_j by just h and see what conditions $E'(h) > 0$ imposes. As in (4.4), $E'(h) > 0$ iff

$$\Sigma_{k=l}^{K}(\bar{\rho})^k u(h(e_k)) > 0.$$

Now use the linearity of u to rearrange the left side:

$$E'(h) = \Sigma_{m=l}^{L} P_m(\bar{\rho}) u(x_m) > 0, \tag{7.2}$$

where each $P_m(\rho)$ is an integer polynomial in ρ. Notice that $E'(h|x_l) = P_l(\bar{\rho})u(x_l)$ and $P[h|x_l] = P_l$.

Case 1. $P_l(\rho)$ is the zero polynomial. Then the ticket h does not involve the prize x_l, so by (N1) the ticket h^0 is in $\mathfrak{T}_{l+1}$. Thus $E(h^0) > 0$, and Lemma (6.1)(a) implies $E'(h) > 0$, regardless of the value of $u(x_l)$. So in this case $E'(h) > 0$ imposes no restriction on $u(x_l)$.

Case 2. P_l not identically zero, but $P_l(\bar{\rho}) = 0$. Since $E'(h|x_l) = P_l(\bar{\rho})u(x_l)$, in this case $E'(h|x_l) = 0$, regardless of the value of $u(x_l)$. Since $\bar{\rho}$ is a value of ρ which makes all the choices of signs of $P[h|x_l]$ consistent, we cannot have $P[h|x_l] > 0$ or $P[h|x_l] < 0$. Thus this case corresponds to (N2), and $h^{\#} \in \mathfrak{T}_{l+1}$. Again $E(h^{\#}) > 0$, so $E'(h) > 0$, regardless of the value of $u(x_l)$, and no restriction is imposed.

Case 3. $P_l(\bar{\rho}) \neq 0$. Since we have eliminated all other cases, we can, without loss of generality, assume that all the h_j's in $\mathfrak{T}_l$ fall into this case. For each h_j we rewrite $E'(h_j) > 0$, as in (7.2):

$$E'(h_j) = \Sigma_{m=l}^{L} P_m^j(\bar{\rho}) u(x_m) > 0. \tag{7.3}$$

By renumbering, we can assume that $P_l^j(\bar{\rho}) > 0$ for $1 \leqslant j \leqslant J'$ and $P_l^j(\bar{\rho}) < 0$ for $J' < j' \leqslant J$. We can also assume $1 < j' < J$ since if, for example, $P_l^j(\bar{\rho}) > 0$ for all j, $1 \leqslant j \leqslant J$, then we can achieve (7.3) by just making $u(x_l)$ sufficiently positive. We can rearrange (7.3) as:

$$u(x_l) > -(P_l^j(\bar{\rho}))^{-1} \Sigma_{m=l+1}^{L} P_m^j(\bar{\rho}) u(x_m), \qquad 1 \leqslant j \leqslant J'$$

(7.4)

$$-(P_l^{j'}(\bar{\rho}))^{-1} \Sigma_{m=l+1}^{L} P_m^{j'}(\bar{\rho}) u(x_m) > u(x_l), \qquad J' < j' \leqslant J.$$

It is clear that, given $u(x_m)$ for $m \geqslant l+1$ and $\bar{\rho}$, there exists a real number $u(x_l)$ satisfying (7.4) iff for every j and j' with $1 \leqslant j \leqslant J'$ and $J' < j' \leqslant J$,

$$(7.5) \quad -\left(P_l^{j'}(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m^{j'}(\bar{\rho})u(x_m) > -\left(P_l^{j}(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m^{j}(\bar{\rho})u(x_m).$$

Thus the lemma will be proved if we can show: for every h_j and $h_{j'}$ in $\mathfrak{T}_l$ with $P_l^j(\bar{\rho}) > 0$, $P_l^{j'}(\bar{\rho}) < 0$,

$$(7.6) \quad \left(P_l^{j'}(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m^{j'}(\bar{\rho})u(x_m) < \left(P_l^{j}(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m^{j}(\bar{\rho})u(x_m).$$

Recall that $P_l^j(\bar{\rho}) = P[h_j|x_l](\bar{\rho})$. Thus $P_l^j(\bar{\rho}) > 0$, $P_l^{j'}(\bar{\rho}) < 0$ imply $P[h_j|x_l] > 0$, $P[h_{j'}|x_l] < 0$ and the pair h_j, $h_{j'}$ falls into case (N3). From now on we leave off the "j"s in our formulas, and denote $h_{j'}$ by h'. By (N3) the ticket $g = h_l^* h'$ is in $\mathfrak{T}_{l+1}$, so $E(g) > 0$. Recall $g = h^* + h'^*$, as defined in (6.7), so $E(g) > 0$ implies

$$E'(h'^*) > -E'(h^*),$$

regardless of the value of $u(x_l)$. By Lemma (6.8)(a)

$$P[h|x_l](\bar{\rho})E(h') > P[h'|x_l](\bar{\rho})E(h).$$

Since $P[h|x_l] = P_l$ and similarly for h' and P_l', this implies

$$P_l(\bar{\rho})E(h') > P_l'(\bar{\rho})E(h).$$

Now expanding $E(h')$ and $E(h)$, we get

$$P_l(\bar{\rho})\left[P_l'(\bar{\rho})u(x_l) + \Sigma_{m=l+1}^{L} P_m'(\bar{\rho})u(x_m)\right] > P_l'(\bar{\rho})\left[P_l(\bar{\rho})u(x_l) + \Sigma_{m=l+1}^{L} P_m(\bar{\rho})u(x_m)\right].$$

If we multiply out, the first terms $P_l(\bar{\rho})P_l'(\bar{\rho})u(x_l)$ cancel. Then recall $P_l(\bar{\rho}) > 0$, $P_l'(\bar{\rho}) < 0$ so we can divide through by $(P_l(\bar{\rho})P_l'(\bar{\rho})) < 0$ to get

$$\left(P_l'(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m'(\bar{\rho})u(x_m) < \left(P_l(\bar{\rho})\right)^{-1}\Sigma_{m=l+1}^{L} P_m(\bar{\rho})u(x_m),$$

which is exactly (7.6).

Now for the proof of sufficiency: assume conditions (6.9)(a) and (6.9)(b). Consider $\mathfrak{T}_{L+1}$. Since all prizes have been eliminated at that stage (see (6.8)(c)) the only possible ticket in $\mathfrak{T}_{L+1}$ is the zero ticket. This is excluded by assumption (6.9)(b). Therefore $\mathfrak{T}_{L+1} \neq \varnothing$ and, by default, $\rho(\mathfrak{T}_{L+1}) = \{\rho : \rho > 0\}$. Now let $\bar{\rho}$ be a positive value of ρ satisfying all the sign-choices—such a $\bar{\rho}$ exists by (6.9)(a). Since $\bar{\rho} \in \rho(\mathfrak{T}_{L-1})$, by the previous lemma $\bar{\rho} \in \rho(\mathfrak{T}_L)$. For similar reasons $\bar{\rho} \in \rho(\mathfrak{T}_{L-1})$, etc., until we get $\bar{\rho} \in \rho(\mathfrak{T}_1)$. Since $\rho(\mathfrak{T}_1) \neq \varnothing$, $\mathfrak{T}_1$ is compatible with EUH.

8. Criticisms. Let us review the extent to which our conditions satisfy the criteria mentioned in Section 2.

Partial order on decisions. We meet this criteria unless one interprets the statement "there exist choices of signs · · · " in (6.9) as requiring the subject to make other choices than those in the original partial order (cf. (6.10)). Even then, these other choices do not require an extension of the partial order to a complete one, or even an extension to choices between constant tickets. Luce and Krantz (1971, page 258) make this comment about requiring such other choices:

"The measurer must be prepared to present for serious consideration by the decision maker some rather artificial alternatives, and the decision maker must be induced to make realistic decisions among them."

Finite sets of events and consequences. We have assumed that the set of events contains only two elements, but this was merely for simplicity; our methods could equally well apply to any finite set. They do not, however, apply directly to an infinite set of events or to an infinite set of consequences.

Constructive. Our method seems on the surface to be constructive, since everything is finite. There are finitely many choices of signs of characteristic polynomials $P[h|x_l]$ for each $\mathfrak{T}_l$, thus finitely many possible $\mathfrak{T}_{l+1}$'s given $\mathfrak{T}_l$. Each $\mathfrak{T}_l$ is finite so it is easy to check if it contains a zero ticket. But checking whether a choice of signs is consistent is not so simple. In general it involves verifying that a certain set of polynomials in one variable (the characteristic polynomials) have a common point of positive value, i.e., $P(\bar{\rho}) > 0$ for all P in that set. This problem is decidable in the mathematical sense, since it is a special case of the solvability of polynomial inequalities discussed in Section 2, but so is the original problem of solving the inequalities (1.6). So by some simple steps we have reduced the problem of deciding solvability of the polynomial inequalities (1.6) to the solvability of some polynomial inequalities in *one* unknown ρ. The main result of Richter-Shapiro (1978) is that given any set $\mathcal{G}$ of polynomial inequalities in one unknown, there is a set $\mathcal{G}$ of simple (i.e., as in Section 1) ticket-choices such that $\mathcal{G}$ has a solution with $p/q = \rho$ iff ρ is a solution to $\mathcal{G}$.

Reasonability is a purely subjective judgement and our method does require *repetition of events.*

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