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TWO CONDITIONAL LIMIT THEOREMS WITH APPLICATIONS[1]

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Let $(X_1, Y_1), \cdots, (X_N, Y_N)$ be i.i.d. rv's where the X's are nonnegative integer-valued. Conditional on ΣX_k the asymptotic distributions of ΣY_k and $\Sigma a_k X_k$ are derived by general methods. Some applications are briefly discussed: sampling without replacement, the classical occupancy problem, the Wilcoxon statistic, the Poisson index of dispersion, testing geometric versus Poisson distribution.

1. Introduction. It is quite common in probability theory and statistics to come across a random variable whose distribution is the same as that of a sum of independent random variables conditioned on another such sum. A tool for obtaining asymptotic results in such a case is to study the joint asymptotic behaviour of the two sums. An example is the proof of asymptotic normality of the sample sum in sampling without replacement by Erdös and Rényi (1959). Another example is Le Cam (1958) on sums of functions of uniform spacings. The general question of when joint asymptotic normality implies asymptotic normality of a conditioned distribution was investigated by Steck (1957). In the present paper the asymptotic behaviour of such a sum is obtained by general methods when the conditional random variables are integer-valued. Also some examples are briefly discussed.

Let $(X, Y), (X_1, Y_1), \cdots, (X_N, Y_N)$ be i.i.d. rv's (independent identically distributed random variables) with X nonnegative integer-valued. In Section 2 a partial inversion formula by Bartlett (1938) for the joint characteristic function of (X, Y) is given and some applications of the formula are discussed. In Section 3 the limit behaviour of ΣY_k and the linear combination $\Sigma a_k X_k$, conditional on ΣX_k, is investigated under general conditions. To study $\mathfrak{L}(\Sigma Y_k | \Sigma X_k)$ a general method by Le Cam (1958) is used. A technique similar to that of Erdös and Rényi (1959) is used for $\mathfrak{L}(\Sigma a_k X_k | \Sigma X_k)$. The methods are general and can be applied in other cases. Some applications of the limit theorems are considered in Section 4: the classical occupancy problem, the Wilcoxon statistic, sampling without replacement, the Poisson index of dispersion and testing a geometric versus a Poisson distribution.

2. A partial inversion formula. The following formula for a conditional distribution was given by Bartlett (1938).

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THEOREM 1. *Let (X, Y) be a two-dimensional random vector with X integer-valued. Then, for n such that $P(X = n) > 0$,*

(2.1) $E(e^{itY}|X = n) = (2\pi P(X = n))^{-1} \cdot \int_{-\pi}^{\pi} E(\exp(is(X - n) + itY))\, ds.$

PROOF. Using conditional expectation it follows that

(2.2) $$E(\exp(isX + itY)) = \Sigma_{j=-\infty}^{\infty} P(X = j) E(e^{itY}|X = j) \cdot e^{isj}.$$

By the well-known formula for Fourier-series the coefficient of e^{isn} is obtained from

(2.3) $P(X = n)E(e^{itY}|X = n) = (2\pi)^{-1} \cdot \int_{-\pi}^{\pi} E(\exp(isX + itY)) \cdot e^{-isn}\, ds,$

which proves the assertion. The formula (2.1) is quite useful as the following examples may indicate.

EXAMPLE 1. (The classical occupancy problem.) Consider N different urns and throw n balls randomly into the urns. If U_k is the number of balls hitting the kth urn then the vector $U = (U_1, \cdots, U_N)$ is distributed as multinomial $(n, 1/N, \cdots, 1/N)$. The number of empty urns can be written $V = \Sigma_k I(U_k = 0)$. It is well known that if $X_1, \cdots, X_N$ are i.i.d. rv's with a Poisson distribution then $X = (X_1, \cdots, X_N)$ conditioned on $\Sigma_k X_k = n$ has the same distribution as U. Thus, with $Y = \Sigma_k I(X_k = 0)$ it follows that $\mathcal{L}(V) = \mathcal{L}(Y|\Sigma_k X_k = n)$. From (2.1) the characteristic function of V is obtained:

(2.4) $$E(e^{itV}) = (2\pi P(\Sigma X_k = n))^{-1} \cdot \int_{-\pi}^{\pi} \left(E\left(\exp\left(itI(X_1 = 0) + is\left(X_1 - \frac{n}{N}\right)\right)\right)\right)^N ds.$$

The mean EX_1 is arbitrary. An explicit expression is easily obtained for the integrand in (2.4). With $EX_1 = n/N$ the formula (2.4) was given by Rényi (1962) with a quite different derivation. Rényi used it to prove limit theorems for V when $n, N \to \infty$. Rényi's results have been generalized to more general "multinomial situations" with different statistical applications. A recent paper on this is Medvedev (1977); see also Holst (1979).

EXAMPLE 2. (Bose-Einstein statistics and nonparametrics.) Consider again N urns. Put n indistinguishable balls in the urns in such a way that each distinguishable outcome has the same probability, i.e.,

$$1/\binom{n + N - 1}{n},$$

cf. Feller (1968, page 40). Let U_k be the number of balls in the kth urn and let $X_1, \cdots, X_N$ be i.i.d. rv's with a geometric distribution. It is easily verified that $\mathcal{L}((U_1, \cdots, U_N)) = \mathcal{L}((X_1, \cdots, X_N)|\Sigma X_k = n)$. Therefore, by (2.1) the characteristic function of a random variable of the form $V = \Sigma h_k(U_k)$, where the h_k's are given functions, can be expressed using the characteristic function of the sum of the independent random vectors $(X_1, h_1(X_1)), \cdots, (X_N, h_N(X_N))$.

The above urn model occurs e.g. in connection with the nonparametric two-sample problem. The numbers n and $N - 1$ correspond to the sizes of the two samples. The ranks of the "second" sample can be written $R_j = j + \Sigma_{k=1}^{j} U_k$ so $\Sigma_{k=1}^{N} kU_k$ is a linear transformation of the Wilcoxon statistic.

EXAMPLE 3. (Sampling without replacement.) In a finite population the real numbers $a_1, \cdots, a_N$ are associated with the N elements. Consider a simple random sample of size n drawn without replacement. Let V be the sample sum of the corresponding a's. Set $U_k = 1$ (0) if (not) element k is drawn and let $X_1, \cdots, X_N$ be i.i.d. Bernoulli rv's. It is easily seen that $\mathcal{L}((U_1, \cdots, U_N)) = \mathcal{L}((X_1, \cdots, X_N)|\Sigma X_k = n)$ and therefore $\mathcal{L}(V) = \mathcal{L}(\Sigma a_k U_k) = \mathcal{L}(\Sigma a_k X_k | \Sigma X_k = n)$. Using (2.1) an expression for the characteristic function of V can be obtained. The same as that of Erdös and Rényi (1959), who used a direct derivation.

3. Two conditional limit theorems

THEOREM 2. *For* $j = 1, 2, \cdots$ *let* $(X_j, Y_j), (X_{1j}, Y_{1j}), \cdots, (X_{N_j j}, Y_{N_j j})$ *be* i.i.d. *two-dimensional random vectors, where* X_j *is nonnegative integer-valued with* $Ex_j = \theta_j$ *and* Var $X_j = \sigma_j^2(\theta_j)$.

Suppose that

$$(3.1) \qquad \Sigma_{k=1}^{N_j} X_{kj} \quad \text{is sufficient for} \quad \theta_j,$$

$$(3.2) \qquad N_j \to \infty \quad \text{when} \quad j \to \infty,$$

for every $\varepsilon > 0$ *there exists* $K_\varepsilon < 1$ *such that for* $\varepsilon \leqslant |t| \leqslant \pi$,

$$(3.3) \qquad |E(\exp(it(X_j - \theta_j)))| \leqslant K_\varepsilon < 1 \quad \text{for all} \quad j,$$

$$(3.4) \qquad \sigma_j(\theta_j) \to \sigma(\theta) > 0 \quad \text{when} \quad j \to \infty \quad \text{and} \quad \theta_j \to \theta > 0$$

and that there exists $A_j(\theta_j)$, $B_j(\theta_j)$ *such that*

$$(3.5) \qquad \Sigma_{k=1}^{N_j}\big(N_j^{-\frac{1}{2}}(X_{kj} - \theta_j)/\sigma_j(\theta_j), (Y_{kj} - A_j(\theta_j))/B_j(\theta_j)\big) \to (Z_1, Z_2), \quad \text{in law when} \quad j \to \infty \quad \text{and} \quad \theta_j \to \theta > 0$$

where the (infinitely divisible) characteristic function of (Z_1, Z_2) *can be written*

$$(3.6) \qquad E(\exp(isZ_1 + itZ_2)) = g_\theta(t) \cdot \exp\Big(-\frac{1}{2}(s^2 + 2C(\theta)st + D(\theta)t^2)\Big)$$

for some $C(\theta)$, $D(\theta)$ *and where the characteristic function* $g_\theta(t)$ *has no normal component.*

Then for $n_j \to \infty$ *such that* $n_j/N_j \to \theta > 0$

$$(3.7) \qquad \mathcal{L}\big(\Sigma_{k=1}^{N_j}(Y_{kj} - A_j(n_j/N_j))/B_j(n_j/N_j)|\Sigma_{k=1}^{N_j} X_{kj} = n_j\big) \to \mathcal{L}(Z), \quad j \to \infty,$$

where Z *has the characteristic function*

$$(3.8) \qquad E(e^{itZ}) = g_\theta(t) \cdot \exp\Big(-\frac{1}{2}(D(\theta) - C(\theta)^2)t^2\Big).$$

REMARK. As pointed out by Le Cam (1958, page 8), (3.5) implies the representation (3.6).

PROOF. As $\Sigma_1^{N_j} X_{kj}$ is sufficient for θ_j the conditional distribution in (3.7) is independent of θ_j. Therefore, without any lack of generality, it can be assumed that $\theta_j = n_j/N_j$. This choice is used in the following. To facilitate notation set $\sigma_j = \sigma_j(n_j/N_j) = \sigma_j(\theta_j)$, $\sigma_{Nj} = N_j^{\frac{1}{2}} \cdot \sigma_j$, $A_j = A_j(\theta_j)$ and $B_j = B_j(\theta_j)$.

Consider a sequence $\{M_j\}$ such that $M_j/N_j \to a$, $0 < a < 1$. From Theorem 1 it follows that

$$(3.9)\quad \varphi_{M_j}(t) = E\big(\exp\big(it\Sigma_1^{M_j}(Y_{kj} - A_j)/B_j \big| \Sigma_1^{N_j} X_{kj} = n_j\big)\big)$$
$$= \big[(2\pi)^{\frac{1}{2}}\sigma_{Nj} P\big(\Sigma_1^{N_j}(X_{kj} - \theta_j) = 0\big)\big]^{-1}$$
$$\times (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\pi\sigma_{Nj}}^{\pi\sigma_{Nj}} E\big(\exp\big(it\Sigma_1^{M_j}(Y_{kj} - A_j)/B_j + iu\Sigma_1^{N_j}(X_{kj} - \theta_j)/\sigma_{Nj}\big)\big)\, du.$$

Using the assumptions (3.3) and (3.4) the proof of the local limit theorem for lattice distributions in, e.g. Gnedenko (1962, page 297) can be used for proving

$$(3.10)\qquad (2\pi)^{\frac{1}{2}}\sigma_{Nj} P\big(\Sigma_1^{N_j}(X_{kj} - \theta_j) = 0\big) \to 1, \qquad j \to \infty.$$

By the independence of the $(X_{k,j}, Y_{k,j})$ the integrand in (3.9) is a product of one factor in "$\Sigma_1^{M_j}$" and one in "$\Sigma_{M_j+1}^{N_j}$". The second factor is dominated by

$$(3.11)\quad f_j(u) = \big|E\big(\exp\big(iu\Sigma_{M_j+1}^{N_j}(X_{kj} - \theta_j)/\sigma_{Nj}\big)\big)\big| \to \exp(-(1-a)u^2/2), \quad j \to \infty.$$

Without loss of generality we can suppose that $N_j - M_j$ is an even integer. Let X_{kj}^s denote X_{kj} symmetrized. Again, as in the local limit theorem it follows that

$$(3.12)\quad \textstyle\int_{-\pi\sigma_{Nj}}^{\pi\sigma_{Nj}} f_j(u)\, du = \sigma_{Nj} \cdot \int_{-\pi}^{\pi} E\big(\exp\big(it\Sigma_1^{(N_j-M_j)/2} X_{kj}^s\big)\big)\, dt$$
$$= \big(N_j/(N_j - M_j)\big)^{\frac{1}{2}} \cdot (N_j - M_j)^{\frac{1}{2}} \cdot \sigma_j(\theta_j) \cdot 2\pi \cdot P\big(\Sigma_1^{(N_j-M_j)/2} X_{kj}^s = 0\big)$$
$$\to (2\pi/(1-a))^{\frac{1}{2}} = \textstyle\int_{-\infty}^{\infty} \exp(-(1-a)u^2/2)\, du.$$

The factor in "$\Sigma_1^{M_j}$" of the integrand in (3.9) is dominated by 1, and by (3.5) we have

$$(3.13)\quad E\big(\exp\big(it\Sigma_1^{M_j}(Y_{kj} - A_j)/B_j + iu\Sigma_1^{M_j}(X_{kj} - \theta_j)/\sigma_{Nj}\big)\big)$$
$$\to \big(g_\theta(t) \cdot \exp(-(u^2 + 2C(\theta)ut + D(\theta)t^2)/2)\big)^a.$$

By the extended form of Lebesgue's dominated convergence theorem (see, e.g., Rao (1973, page 136)) (3.10), (3.12) and (3.13) imply that

$$(3.14)\quad \varphi_{M_j}(t) \to g_\theta(t)^a \cdot \exp\Big(-\frac{1}{2}\big(aD(\theta) - (aC(\theta))^2\big)t^2\Big) = \varphi_a(t), \qquad j \to \infty.$$

The function φ_a satisfies

$$(3.15)\qquad \varphi_a(t) \to 1,\ a \to 0,$$

$$(3.16)\qquad \varphi_a(t) \to g_\theta(t)\exp\Big(-\frac{1}{2}\big(D(\theta) - C(\theta)^2\big)t^2\Big), \qquad a \to 1.$$

The assertion of the theorem follows from (3.15) and (3.16) by an argument in Le Cam (1958, page 13).

THEOREM 3. *Let* $\{N_j, a_{1j}, \cdots, a_{N_j j}\}$ *be real numbers satisfying*

$$N_j \to \infty, j \to \infty, \tag{3.17}$$

$$\Sigma_1^{N_j} a_{kj} = 0, \tag{3.18}$$

$$\Sigma_1^{N_j} a_{kj}^2 / N_j \to 1, \qquad j \to \infty, \tag{3.19}$$

$$\max_{1 \leqslant k \leqslant N_j} a_{kj}^2 / N_j \to 0, \qquad j \to \infty. \tag{3.20}$$

Let $X_j, X_{1j}, \cdots, X_{N_j j}$ *be* i.i.d. *nonnegative integer-valued* rv's *with* $EX_j = \theta_j$, Var $X_j = \sigma_j^2(\theta_j)$, *satisfying* (3.1), (3.3) *and* (3.4). *Then for* $n_j \to \infty$ *such that* $n_j / N_j \to \theta > 0$

$$\mathcal{L}\big(N_j^{-1/2} \Sigma_1^{N_j} a_{kj} X_{kj} / \sigma_j(n_j/N_j) | \Sigma_1^{N_j} X_{kj} = n_j\big) \to N(0, 1), \qquad j \to \infty. \tag{3.21}$$

PROOF. Set $g_j(t) = E\big(\exp(it(X_j - \theta_j))\big)$. From the assumptions (3.3) and (3.4) the following estimates hold uniformly in j

$$g_j(t) = \exp\Big(-\frac{1}{2}\big(\sigma_j^2(\theta_j) + o(1)\big)t^2\Big), \qquad t \to 0, \tag{3.22}$$

$$|g_j(t)| \leqslant \exp(-K_1 t^2), \qquad |t| \leqslant \varepsilon, \tag{3.23}$$

$$|g_j(t)| \leqslant \exp(-K_2), \qquad \varepsilon \leqslant |t| \leqslant \pi, \tag{3.24}$$

for some $\varepsilon, K_1, K_2 > 0$.

As in the proof of Theorem 2 it can be assumed that $\theta_j = n_j / N_j$. With the same notation as before Theorem 1 gives

$$\begin{aligned} &E\big(\exp\big(it\Sigma_1^{N_j} a_{kj} X_{kj} / \sigma_{Nj}\big) | \Sigma_1^{N_j} X_{kj} = n_j\big) \\ &= \big[(2\pi)^{1/2} \sigma_{Nj} P\big(\Sigma_1^{N_j} X_{kj} = n_j\big)\big]^{-1} \cdot (2\pi)^{-1/2} \textstyle\int_{-\pi\sigma_{Nj}}^{\pi\sigma_{Nj}} \Pi_1^{N_j} g_j\big((ta_{kj} + u)/\sigma_{Nj}\big) du. \end{aligned} \tag{3.25}$$

That the first factor on the right hand side of (3.25) converges to 1 is proved in (3.10). The convergence of the integral can be studied in almost the same way as in the proof of the local limit theorem.

Choose an $A > 0$. By (3.22)

$$(2\pi)^{-1/2} \textstyle\int_{|u| < A} \Pi_1^{N_j} g_j\big((ta_{kj} + u)/\sigma_{Nj}\big) du \to (2\pi)^{-1/2} \int_{|u| < A} e^{-\frac{1}{2}(t^2 + u^2)} du \tag{3.26}$$

where the right hand side converges to $e^{-t^2/2}$ when $A \to \infty$. From (3.23) it follows that

$$\Big|\textstyle\int_{A \leqslant |u| \leqslant \varepsilon\sigma_{Nj}} \Pi_1^{N_j} g_j\big((ta_{kj} + u)/\sigma_{Nj}\big) du\Big| \leqslant \int_{A \leqslant |u|} e^{-K_3(t^2 + u^2)} du, \tag{3.27}$$

which converges to 0 when $A \to \infty$. By (3.24) the remaining part of the integral in (3.25) is estimated as

$$(3.28) \qquad \left|\int_{\epsilon\sigma_{N_j} \leqslant |u| \leqslant \pi\sigma_{N_j}} \Pi_1^{N_j} g_j\big((ta_{kj} + u)/\sigma_{Nj}\big)du\right| \leqslant K_4 \cdot N_j^{1/2} \cdot e^{-K_2 N_j},$$

converging to 0 when $j \to \infty$. Combining the estimates proves that the characteristic function in (3.25) converges to $e^{-t^2/2}$, from which the assertion (3.21) follows.

4. Applications of the limit theorems.

EXAMPLE 1 (ctd). Consider a triangular array of urn schemes. Let N_j be the number of urns in the jth scheme and n_j the number of balls, with $N_j, n_j \to \infty$, $j \to \infty$ such that $n_j/N_j \to \theta > 0$. Set $Y_j = I(X_j = 0)$ where X_j is distributed as Poisson with mean n_j/N_j. The assumptions of Theorem 2 are easily verified and the limit distribution for $(V_j - N_j e^{-n_j/N_j})/N_j^{1/2}$ is $N(0, e^{-\theta}(1 - e^{-\theta}) - \theta e^{-2\theta})$. Rényi (1962) proved the same result with different methods.

EXAMPLE 2 (ctd). As in the example above a triangular array is considered here. To study the Wilcoxon statistic set $a_{kj} = \big(k - b(n_j, N_j)\big)/c(n_j, N_j)$ for suitable chosen $b(\cdot)$ and $c(\cdot)$. From Theorem 3 the well-known limit distribution for the Wilcoxon test is found.

EXAMPLE 3 (ctd). In a similar way as in the previous examples, asymptotic normality is proved, cf. Erdös and Rényi (1959) and Rényi (1966, page 379). In this case Hájek (1960) proved that the conditions on the a's are also necessary for convergence to the normal distribution.

The examples above are one-dimensional involving dependent random variables. The dependence can be relaxed by using independent random vectors. This type of dependence occurs in a natural way in connection with conditioning by sufficient statistics. For a discussion on the use of sufficient statistics see e.g. Cox and Hinkley (1974, page 73).

EXAMPLE 4 (The Poisson index of dispersion). Suppose that under the hypothesis H_0 the rv's $X_1, \cdots, X_N$ are i.i.d. in a Poisson distribution of unknown mean θ. The sample sum $\Sigma X_k = N \cdot \bar{X}$ is sufficient for θ under H_0. The conditional distribution $\mathcal{L}\big((X_1, \cdots, X_N)|\bar{X}\big)$ is multinomial with equal probabilities. A possible conditional test of H_0 is the usual chi-square suggested by Fisher (see Gart (1975) for further discussion). The test statistic is the index of dispersion $\Sigma(X_k - \bar{X})^2/\bar{X} = T_N$. By Theorem 2 T_N is As $N(N, 2N)$ or As $\chi^2(N - 1)$. The usual "chi-square" argument is not rigorous here because N is not fixed.

EXAMPLE 5 (Geometric versus Poisson distribution). Consider the same hypothesis H_0 as in the previous example against the alternative H_A, $X_1, \cdots, X_N$ and geometric with mean λ. The sample sum is sufficient for θ in H_0 and also for λ in H_A. Simple calculations give that conditional on $\Sigma X_k = n$ the likelihood ratio critical region is of the form $\Sigma \log X_k! \geqslant C$. Theorem 2 gives conditional large sample distributions of the test statistic both under H_0 and H_A. By different methods these limits are also derived in Cox and Hinkley (1974, pages 149, 336).

The methods used above can also be used for the case of nonhomogeneous rv's and for random vectors. For the continuous case a similar approach is also useful. Such generalizations and further applications will be discussed elsewhere; e.g. in Holst (1979) urn models are considered in more detail.

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