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DISTRIBUTED LAG APPROXIMATION TO LINEAR TIME-INVARIANT SYSTEMS[1]

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An infinite distributed lag system is approximated by a truncated one, which can be consistently estimated. We investigate the rate at which the parameter space in the approximate model can be increased as sample size increases, in such a way that these estimates will provide consistent estimates of the underlying system.

1. Introduction. Let two time series, $y(n)$, $z(n)$ observed for $1 \leqslant n \leqslant N$, and one unobservable time series, $x(n)$, be related by

$$y(n) = \Sigma_{j=-\infty}^{\infty} \beta(j) z(n-j) + x(n), \qquad n = 0, \pm 1, \cdots, \tag{1.1}$$

where $Ex(n) = 0$, $Ex(m)z(n) = 0$, all m, n. Assuming the means of $y(n)$ and $z(n)$ exist and are independent of n, there is no loss of generality in taking them to be zero also. In practice (1.1) is often replaced by the finite-parameter approximate model

$$E(y(n)|z(m), -\infty < m < \infty) = \Sigma_{j=-p}^{q} b(j) z(n-j), \tag{1.2}$$

for some p, q, $0 \leqslant p, q \leqslant \infty$.

The $\beta(j)$ in (1.1) may be thought of as coefficients in the Laurent expansion of a function $\tilde{\beta}(s)$, so

$$\tilde{\beta}(s) = \Sigma_{-\infty}^{\infty} \beta(j) s^j, \qquad s \neq 0.$$

From the Weierstrass approximation theorem, it follows that if $\tilde{\beta}(s)$ is continuous on the circle $|s| = 1$, one can choose p, q and the $b(j)$ such that, for any $\varepsilon > 0$,

$$\max_{|s|=1} |\tilde{b}_{pq}(s) - \tilde{\beta}(s)| < \varepsilon, \quad \tilde{b}_{pq}(s) = \Sigma_{j=-p}^{q} b(j) s^j.$$

However, in practice, p and q are limited by sample size. For estimates based on (1.2) to be reasonably precise, $p + q$ needs to be substantially smaller than N. On the other hand, if p and q are too small, the discrepancy between (1.1) and (1.2) may be such as to produce serious bias in the estimates. In practice, therefore, the values of p and q should depend on N.

Define by $b_{Npq}(j)$ a consistent estimate of $b(j)$, for given, fixed p, q. Define also

$$\tilde{b}_{Npq}(s) = \Sigma_{j=-p}^{q} b_{Npq}(j) s^j.$$

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The preceding remarks suggest that the construction of a relevant asymptotic theory would include finding functions $p = p(N)$, $q = q(N)$, such that, as $N, p, q \to \infty$,

$$\max_{|s|=1}|\tilde{b}_{Npq}(s) - \tilde{\beta}(s)| \to 0$$

in probability or with probability one. In this paper we obtain such functions, for estimates $b_{Npq}(j)$ that are closely related to the least squares estimates. (Our q sequences apply immediately to the one-sided case $\beta(j) = 0$, all $j < 0$.) It is likely that very similar results will hold for other estimates. We do not assume $x(n)$ is white noise so for given p and q it may be possible to find more efficient estimates of the $b(j)$; for such estimates when $x(n)$ is stationary (which we do not assume) see Hannan [4, Chapter VII], Wahba [10]. However, Sims [9] has shown that the asymptotic ($N \to \infty$) covariance matrices of ordinary and generalized least squares estimates of (1.2) are identical in the limit as $p, q \to \infty$, even when $x(n)$ is not white noise. Results of this kind supply grounds for using the computationally simplest consistent estimates.

Sims [7], [8] discusses other issues in the approximation of (1.1) by (1.2). Brillinger [3] considers a frequency domain approximation to (1.1). He suggests estimates of $\tilde{\beta}(e^{i\lambda})$ based on estimates of spectra and cross-spectra, and investigates their asymptotic behavior as $N \to \infty$ and a bandwidth parameter $\to 0$. Berk [1], Shibata [6], consider autoregressive estimates of the spectral density, and obtain information on how fast one can increase the autoregressive order as $N \to \infty$.

2. Main assumptions and definitions.

The initial assumptions are as follows.

CONDITION A.

(2.1) $$z(n) = \sum_{j=-\infty}^{\infty}\theta(j)\zeta(n-j), \quad \sum_{j=-\infty}^{\infty}|\theta(j)| < \infty;$$

(2.2) $$E(\zeta(n)|\zeta(m), m < n; x(m), -\infty < m < \infty) = 0;$$

(2.3) $$E(\zeta(n)^2|\zeta(m), m < n) = \sigma^2 < \infty;$$

(2.4) $$E|\zeta(n)|^{2\nu} \leqslant K, \qquad \text{some } \nu, \quad 1 < \nu \leqslant 2.$$

(Throughout, K represents a finite constant, not necessarily the same one, that may depend on ν but is independent of p, q, N and the $\zeta(n)$ and $x(n)$.)

CONDITION B. $E|x(n)|^{2\nu} \leqslant K$.

CONDITION C. $\sum_{j=-\infty}^{\infty}|\beta(j)| < \infty$.

It follows from (2.1)–(2.3) that $z(n)$ is wide-sense stationary, with

$$c(j) = Ez(n)z(n-j) = \sigma^2\sum_{k=-\infty}^{\infty}\theta(k)\theta(k-j), \quad -\infty < j < \infty,$$

$|c(j)| \leqslant K$. Although $x(n)$ and $y(n)$ are not necessarily stationary, Conditions A, B and C imply

$$d(j) = Ey(n)z(n-j) = \sum_{k=-\infty}^{\infty}\beta(k)c(k-j), \quad -\infty < j < \infty,$$

$|d(j)| \leqslant K$. Now define the $(p+q+1) \times (p+q+1)$ matrix A_{pq}, having (j, k)th element $c(j-k)$, and the $(p+q+1)\times 1$ vector a_{pq} having $(j+p+1)$th element $d(j)$, $-p \leqslant j \leqslant q$. Note that if (1.1) were the true model, the $(p+q+1)\times 1$ vector b_{pq}, with jth element $b(j-1-p)$, would be identified as $A_{pq}^{-1}a_{pq}$.

Now introduce

$$c_{LM}(j)=\frac{1}{M-L}\Sigma_{n=L+1}^{M}z(n)z(n-j),\quad d_{LM}(j)=\frac{1}{M-L}\Sigma_{n=L+1}^{M}y(n)z(n-j),$$

$$\bar{y}_{LM}=\frac{1}{M-L}\Sigma_{n=L+1}^{M}y(n),\quad \bar{z}_{LM}=\frac{1}{M-L}\Sigma_{n=L+1}^{M}z(n),$$

for $0\leqslant L<M$. For $N>p+q$ define the $(p+q+1)\times 1$ vector

$$b_{Npq}=A_{Npq}^{-1}a_{Npq},$$

where b_{Npq} has jth element $b_{Npq}(j-1-p)$, A_{Npq} is a $(p+q+1)\times(p+q+1)$ Toeplitz matrix with (j,k)th element $(j\geqslant k)$ $c_{j-k,N}(j-k)-\bar{z}_{0N}^2$, and a_{Npq} has $(j+p+1)$th element $d_{0,N+j}(j)-\bar{y}_{0N}\bar{z}_{0N}$ for $-p\leqslant j<0$, and $d_{jN}(j)-\bar{y}_{0N}\bar{z}_{0N}$, for $0\leqslant j\leqslant q$.

3. Six lemmas. Lemma 1 will not be proved as it is a special case of Lemma 1 of Robinson [5].

LEMMA 1. *For $L<M$, let*

$$S_{LM}=\frac{1}{M-L}\Sigma_{L+1}^{M}u(n),\quad u(n)=\Sigma_{l,m=-\infty}^{\infty}\Sigma v_{lm}(n),\tag{3.1}$$

$$E(v_{lm}(n)|v_{lm}(n-1),v_{lm}(n-2),\cdots)=0,\quad \text{a.s.},$$

for all l, m, n. For some ν, $1<\nu\leqslant 2$, let there exist $\eta_1(l)$, $\eta_2(l)$, $-\infty<l<\infty$, such that

$$E|v_{lm}(n)|^{\nu}\leqslant|\eta_1(l)\eta_2(m)|^{\nu},\ \Sigma_{l=-\infty}^{\infty}|\eta_j(l)|<\infty,\qquad j=1,2,\tag{3.2}$$

for all l, m, n. Then

$$E|S_{LM}|^{\nu}\leqslant K(M-L)^{1-\nu}.$$

In the next two lemmas, we derive bounds on the central moments of $c_{LM}(j)$ and $d_{LM}(j)$ that are independent of j.

LEMMA 2. *Under Condition* A,

$$E|c_{LM}(j)-c(j)|^{\nu}\leqslant K(M-L)^{1-\nu}.$$

PROOF. Note that $c_{LM}(j)-c(j)$ has the form of S_{LM} in Lemma 1 if $v_{lm}(n)=\theta(l)\theta(l-j)(\zeta(n-l)^2-\sigma^2)$, $m=l-j$; $=\theta(l)\theta(m)\zeta(n-l)\zeta(n-j-m)$, $m\neq l-j$. To see this, observe that (3.1) is true because (2.2), (2.3) imply

$$E(\zeta(n)^2-\sigma^2|\zeta(n-l)^2-\sigma^2,l>0)=0,$$

$$\begin{aligned}&E(\zeta(n)\zeta(m)|\zeta(n-l)\zeta(m-l),l>0)\\&=E(E(\zeta(n)\zeta(m)|\zeta(n-l),l>0)|\zeta(n-l)\zeta(m-l),l>0)\\&=E(\zeta(m)E(\zeta(n)|\zeta(n-l),l>0)|\zeta(n-l)\zeta(m-l),l>0)\\&=0,\quad m<n.\end{aligned}$$

Also, (3.2) is true because, by (2.4),

$$E|v_{lm}(n)|^{\nu} \leqslant |\theta(l)\theta(l-j)|^{\nu}2(E|\zeta(n-l)|^{2\nu}+\sigma^{2\nu}) \leqslant K|\theta(l)\theta(l-j)|^{\nu}, \quad m = l-j,$$

$$E|v_{lm}(n)|^{\nu} \leqslant |\theta(l)\theta(m)|^{\nu}(E|\zeta(n-l)|^{2\nu}E|\zeta(n-m)|^{2\nu})^{\frac{1}{2}} \leqslant K|\theta(l)\theta(m)|^{\nu}, \quad m \neq l-j.$$

Then the result follows from Lemma 1. □

LEMMA 3. *Under Conditions* A, B *and* C,

$$E|d_{LM}(j) - d(j)|^{\nu} \leqslant K(M-L)^{1-\nu}.$$

PROOF. Write

$$d_{LM}(j) - d(j) = e_{LM}(j) + f_{LM}(j),$$

$$e_{LM}(j) = \Sigma_{k=-\infty}^{\infty}\beta(k)(c_{L-k,M-k}(j-k) - c(j-k))$$

$$f_{LM}(j) = \frac{1}{M-L}\Sigma_{n=L+1}^{M}x(n)z(n-j).$$

Now by the c_r-inequality and Jensen's inequality,

$$E|d_{LM}(j) - d(j)|^{\nu} \leqslant 2E|e_{LM}(j)|^{\nu} + 2E|f_{LM}(j)|^{\nu}. \tag{3.3}$$

By Hölder's inequality and Lemma 2,

$$E|e_{LM}(j)|^{\nu} \leqslant (\Sigma|\beta(k)|)^{\nu-1}\Sigma|\beta(k)|E|c_{L-k,M-k}(j-k) - c(j-k)|^{\nu}$$
$$\leqslant K(\Sigma|\beta(k)|)^{\nu}(M-L)^{1-\nu} = K(M-L)^{1-\nu}.$$

To handle $f_{LM}(j)$ take

$$v_{lm}(n) = \theta(l)\zeta(n-l)x(n), \quad m = 0; \; = 0, m \neq 0$$

in Lemma 1 and note that by (2.2), (2.4) and Condition B,

$$E(\zeta(m)x(n)|\zeta(m-l)x(n-l), l > 0)$$
$$= E(x(n)E(\zeta(m)|\zeta(m-l), l>0; x(l), -\infty<l<\infty)|\zeta(m-l)x(n-l), l>0) = 0,$$

$$E|\zeta(m)x(n)|^{\nu} < (E|\zeta(m)|^{2\nu}E|x(n)|^{2\nu})^{\frac{1}{2}} < K.$$

It follows that $E|f_{LM}(j)|^{\nu} < K(M-L)^{1-\nu}$. Finally, apply (3.3). □

From Lemma 2 of [5] and Conditions A–C it may readily be shown also that

$$E|\bar{z}_{LM}|^{2\nu} \leqslant K(M-L)^{-\nu} \tag{3.4}$$

$$E|\bar{y}_{LM}|^{2\nu} \leqslant E|\Sigma\beta(k)\bar{z}_{L-k,M-k} + \bar{x}_{LM}|^{2\nu}$$
$$\leqslant 2^{\nu}\{(\Sigma|\beta(k)|)^{2\nu-1}\Sigma|\beta(k)|E|\bar{z}_{L-k,M-k}|^{2\nu} + E|\bar{x}_{LM}|^{2\nu}\}$$
$$\leqslant K(M-L)^{-\nu}, \tag{3.5}$$

where $\bar{x}_{LM} = (M-L)^{-1}(x(L+1) + \cdots + x(M))$.

Let $\|X\|$ be the square root of the greatest eigenvalue of XX', where X' is the transposed of X. Define $\Delta_{Npq} = A_{Npq} - A_{pq}$.

LEMMA 4. *Under Condition* A,

$$E\|\Delta_{Npq}\|^{\nu} \leqslant K(p+q+1)^{\nu}(N-p-q)^{1-\nu}, \qquad N > p+q.$$

PROOF. Because Δ_{Npq} is symmetric, $\|\Delta_{Npq}\|$ is its greatest eigenvalue. Then from a theorem of Perron

$$(3.6) \qquad \begin{aligned} \|\Delta_{Npq}\| &\leqslant \max_{0 \leqslant j \leqslant p+q} \Sigma_{k=0}^{p+q} |c_{|j-k|, N}(|j-k|) - \bar{z}_{0N}^2 - c(j-k)| \\ &\leqslant 2\Sigma_{j=0}^{p+q}\left(|c_{jN}(j) - c(j)| + \bar{z}_{0N}^2\right), \end{aligned}$$

the second inequality resulting because Δ_{Npq} is Toeplitz. Thus from Lemma 2, (3.4) and Minkowski's inequality

$$\begin{aligned} E\|\Delta_{Npq}\|^{\nu} &\leqslant 4(p+q+1)^{\nu-1}\Sigma_{j=0}^{p+q}\left\{E|c_{jN}(j) - c(j)|^{\nu} + E|\bar{z}_{0N}|^{2\nu}\right\} \\ &\leqslant K(p+q+1)^{\nu-1}\Sigma_{j=0}^{p+q}\left\{(N-j)^{1-\nu} + N^{-\nu}\right\} \\ &\leqslant K(p+q+1)^{\nu}(N-p-q)^{1-\nu}. \end{aligned}$$ □

Define $d_{Npq} = a_{Npq} - a_{pq}$.

LEMMA 5. *Under Conditions* A, B *and* C,

$$E\|\Delta_{Npq}|^{\nu} \leqslant K(p+q+1)^{\nu}(N-p-q)^{1-\nu}, \qquad N > p+q.$$

PROOF. In this case

$$\begin{aligned} E\|\delta_{Npq}\|^{\nu} &\leqslant E\left\{\Sigma_{j=-p}^{-1}|d_{0, N+j}(j) - \bar{y}_{0N}\bar{z}_{0N} - d(j)|\right. \\ &\quad \left. + \Sigma_{j=0}^{q}|d_{jN}(j) - \bar{y}_{0N}\bar{z}_{0N} - d(j)|\right\}^{\nu} \\ &\leqslant 4p^{\nu-1}\Sigma_{-p}^{-1}E|d_{0, N+j}(j) - d(j)|^{\nu} + 4(q+1)^{\nu-1}\Sigma_{0}^{q}E|d_{jN}(j) - d(j)|^{\nu} \\ &\quad + 2(p+q+1)^{\nu}\left\{E|\bar{y}_{0N}|^{2\nu}E|\bar{z}_{0N}|^{2\nu}\right\}^{\frac{1}{2}} \\ &\leqslant K\left\{p^{\nu}(N-p)^{1-\nu} + (q+1)^{\nu}(N-q)^{1-\nu} + (p+q+1)^{\nu}N^{-\nu}\right\} \\ &\leqslant K(p+q+1)^{\nu}(N-p-q)^{1-\nu}, \end{aligned}$$

using Lemma 3 and (3.4), (3.5). □

For the final lemma we introduce

CONDITION D. $\Sigma_{j=-\infty}^{\infty}\theta(j)s^{j}$ is bounded away from zero on $|s| = 1$.

This is equivalent to the power spectrum of $z(n)$ (which exists under A) being bounded away from zero.

LEMMA 6. *Under Condition* D,

$$\|A_{pq}^{-1}\| \leqslant K.$$

PROOF. If $\|A_{pq}^{-1}\|$ exists it is the reciprocal of the smallest eigenvalue, ρ, of A_{pq}.

Thus for some real numbers w_j, $0 \leqslant j \leqslant p + q$, $w_0^2 + \cdots + w_{p+q}^2 = 1$,

$$\rho = \Sigma_{j,k=0}^{p+q}\Sigma w_j c(j-k) w_k = \sigma^2 \Sigma_{j,k=0}^{p+q}\Sigma\Sigma_{l=-\infty}^{\infty} w_j \theta(l)\theta(l+j-k) w_k$$
$$= \sigma^2 \int_{|s|=1} |\Sigma_{j=0}^{p+q} w_j s^j|^2 |\Sigma_{j=-\infty}^{\infty} \theta(j) s^j|^2 \, ds > K \int_{|s|=1} |\Sigma_{j=0}^{p+q} w_j s^j|^2 \, ds,$$

by Condition D. The last expression is $2\pi K(w_0^2 + \cdots + w_{p+q}^2) = 2\pi K > 0$, so $\rho^{-1} \leqslant K$. □

4. Weak convergence. We introduce nonnegative integers u, v and real values ξ, χ, $0 \leqslant \xi, \chi < 1$, which describe the smoothness of $\tilde{\beta}(s)$ as follows:

CONDITION E. $\Sigma_{j=-\infty}^{0} \beta(j) s^j$ is u-times differentiable on $|s| = 1$, its uth derivative satisfying a Lipschitz condition of order ξ; $\Sigma_{j=1}^{\infty} \beta(j) s^j$ is v-times differentiable on $|s| = 1$, its vth derivative satisfying a Lipschitz condition of order χ.

This condition implies C when $\xi > \frac{1}{2}$ and $\chi > \frac{1}{2}$ (Zygmund [11, page 64]).

Let β_{pq} be the $(p + q + 1) \times 1$ vector with jth element $\beta(j - 1 - p)$.

THEOREM 1. *Let Conditions* A, B, D *and* E *hold, with* $u + \xi > \frac{1}{2}$, $v + \chi > \frac{1}{2}$. *Then*

$$p \lim_{N,p,q\to\infty} \|b_{Npq} - \beta_{pq}\| = 0, \tag{4.1}$$

if

$$q \leqslant Kp^{2(u+\xi)-\varepsilon}, \quad p \leqslant Kq^{2(v+\chi)-\varepsilon}, \qquad \textit{any} \quad \varepsilon > 0, \tag{4.2}$$

$$(p + q)/N^{(\nu-1)/\nu} \to 0. \tag{4.3}$$

PROOF. Write

$$b_{Npq} - \beta_{pq} = A_{Npq}^{-1}(a_{Npq} - A_{Npq}\beta_{pq})$$
$$= A_{Npq}^{-1}(\delta_{Npq} - \Delta_{Npq}\beta_{pq} + a_{pq} - A_{pq}\beta_{pq})$$

so that

$$\|b_{Npq} - \beta_{pq}\| \leqslant \|A_{Npq}^{-1}\|(\|\delta_{Npq}\| + \|\Delta_{Npq}\| \, \|\beta_{pq}\| + \|a_{pq} - A_{pq}\beta_{pq}\|). \tag{4.4}$$

Now

$$\|A_{Npq}^{-1}\| \leqslant \|A_{Npq}^{-1} - A_{pq}^{-1}\| + \|A_{pq}^{-1}\| \leqslant (\|A_{Npq}^{-1}\| \, \|\Delta_{Npq}\| + 1)\|A_{pq}^{-1}\|,$$

so

$$\|A_{Npq}^{-1}\|(1 - \|A_{pq}^{-1}\| . \|\Delta_{pq}^{N}\|) \leqslant \|A_{pq}^{-1}\|.$$

By Markov's inequality it follows from Lemma 4 that $\|\Delta_{Npq}\| \to 0$, in probability, if

$$(p + q + 1)^\nu (N - p - q)^{1-\nu} \to 0, \qquad \text{i.e.,} \quad (p + q)^\nu N^{1-\nu}\left(1 - \frac{p+q}{N}\right)^{1-\nu} \to 0.$$

Thus from Lemma 6

$$p \lim_{N,p,q\to\infty} \|A_{Npq}^{-1}\| \leqslant \lim_{p,q\to\infty} \|A_{pq}^{-1}\| < \infty,$$

for sequences (4.3). We are left with the factor in parentheses in (4.4). By Lemma 5 and Markov's inequality, $\|\delta_{Npq}\| \to 0$ i.p. for sequences (4.3). The second term $\to 0$ i.p. since $\|\Delta_{Npq}\| \to 0$ i.p. and

$$\|\beta_{pq}\| = \left(\Sigma_{j=-p}^{q}\beta(j)^2\right)^{\frac{1}{2}} \leqslant \Sigma_{j=-p}^{q}|\beta(j)| \leqslant \Sigma_{j=-\infty}^{\infty}|\beta(j)| < \infty,$$

by E and $u + \xi$, $v + \chi > \frac{1}{2}$. Finally, for p and q that are not too small,

$$\begin{aligned}\|a_{pq} - A_{pq}\beta_{pq}\| &= \left\{\Sigma_{j=-p}^{q}\left(d(j) - \Sigma_{k=-p}^{q}\beta(k)c(j+k)\right)^2\right\}^{\frac{1}{2}}\\ &= \left\{\Sigma_{j=-p}^{q}\left(\Sigma_{k=-\infty}^{-p-1}\beta(k)c(j+k) + \Sigma_{k=q+1}^{\infty}\beta(k)c(j+k)\right)^2\right\}^{\frac{1}{2}}\\ &= \frac{\sigma^2}{2\pi}\left\{\Sigma_{-p}^{q}\left[\int_{|s|=1}\left(\Sigma_{-\infty}^{-p-1}\beta(k)s^k + \Sigma_{q+1}^{\infty}\beta(k)s^k\right)s^j|\Sigma_{-\infty}^{\infty}\theta(l)s^l|^2\,ds\right]^2\right\}^{\frac{1}{2}}\\ &\leqslant c(0)(p+q+1)^{\frac{1}{2}}\left\{\max_{|s|=1}|\Sigma_{-\infty}^{-p-1}\beta(k)s^k| + \max_{|s|=1}|\Sigma_{q+1}^{\infty}\beta(k)s^k|\right\}\\ &\leqslant K\left(p^{\frac{1}{2}} + q^{\frac{1}{2}}\right)\left(\frac{\ln p}{p^{u+\xi}} + \frac{\ln q}{q^{v+\chi}}\right)\end{aligned}$$

(Zygmund [11, page 120]). From $u+\xi$, $v+\chi > \frac{1}{2}$ and (4.2), this $\to 0$ as $p, q \to \infty$. □

When $\beta(j) = 0$, all $j < J$, some $J > -\infty$, the conditions for (4.1) are that A, B, D, and E hold, with $v + \chi > \frac{1}{2}$, and $qN^{(\nu-1)/\nu} \to 0$ as $N \to \infty$.

THEOREM 2. *Let Conditions* A, B, D *and* E *hold, with* $u + \xi > 1$, $v + \chi > 1$. *Then*

$$p\lim_{N,p,q\to\infty}\max_{|s|=1}|\tilde{b}_{Npq}(s) - \tilde{\beta}(s)| = 0,$$

if

(4.5) $$q \leqslant Kp^{u+\xi-\varepsilon},\ p \leqslant Kq^{v+\chi-\varepsilon}, \quad \text{any} \quad \varepsilon > 0,$$

(4.6) $$(p+q)/N^{(2(\nu-1)/3\nu)} \to 0.$$

PROOF. By the triangle inequality

$$\max_{|s|=1}|\tilde{b}_{Npq}(s) - \tilde{\beta}(s)| \leqslant \Sigma_{j=-p}^{q}|b_{Npq}(j) - \beta(j)| + \Sigma_{j=-\infty}^{-p-1}|\beta(j)| + \Sigma_{j=q+1}^{\infty}|\beta(j)|.$$

The last two terms on the right $\to 0$ as $p, q \to \infty$. The first term is bounded by $(p+q+1)^{\frac{1}{2}}\|b_{Npq} - \beta_{pq}\|$, from the Schwarz inequality. Thus the theorem is proved if

$$\begin{aligned}p\lim_{N,p,q\to\infty}(p+q)^{\frac{1}{2}}\|\Delta_{Npq}\| &= p\lim_{N,p,q\to\infty}(p+q)^{\frac{1}{2}}\|\delta_{Npq}\|\\ &= \lim_{p,q\to\infty}(p+q)^{\frac{1}{2}}\|a_{pq} - A_{pq}\beta_{pq}\| = 0.\end{aligned}$$

The first two limits are zero under (4.5), from Lemmas 4 and 5. Finally, as in the proof of Theorem 1,

$$(p+q)^{\frac{1}{2}}\|a_{pq} - A_{pq}\beta_{pq}\| \leqslant K(p+q)\left(\frac{\ln p}{p^{u+\xi}} + \frac{\ln q}{q^{v+\chi}}\right) \to 0,$$

from Zygmund [11, page 120], $u + \xi$, $v + \chi > 1$, and (4.5).

Conditions (4.2), (4.3), (4.5) and (4.6) limit the ultimate rate of increase of p and q relative to one another, and to N.

5. Strong convergence. The mode of convergence can be strengthened if the growth of p and q is further restricted, to a small degree.

THEOREM 3. *Let Conditions* A, B, D *and* E *hold, with* $u + \xi > \frac{1}{2}$, $v + \chi > \frac{1}{2}$. *Then*

$$\lim_{N,p,q\to\infty}\|b_{Npq} - \beta_{pq}\| = 0, \quad \text{a.s.},$$

under (4.2), *and the conditions*

$$p + q \leqslant C2^l, \qquad \textit{some} \quad C < 1, \tag{5.1}$$

where l is an integer such that $2^l \leqslant N$, *and*

$$p + q \leqslant KN^{(\nu-1)/\nu} / \left\{(\ln N)^{(\nu+1)/\nu} \ln \ln N\right\}. \tag{5.2}$$

PROOF. The proof follows that of Theorem 1 as soon as we prove

$$\lim_{N,p,q\to\infty}\|\Delta_{Npq}\| = \lim_{N,p,q\to\infty}\|\delta_{Npq}\| = 0, \quad \text{a.s.}$$

To establish the first limit, consider (3.6). The term $2(p + q + 1)\bar{z}_{0N}^2$ in this has νth moment

$$K\left(\frac{p+q+1}{N}\right)^{\nu} \leqslant \frac{K}{N(\ln N)^{\nu+1}(\ln \ln N)^{\nu}},$$

under (5.2), and thus $\to 0$ a.s. by Markov's inequality and the Borel-Cantelli lemma. The other term in (3.6) is twice

$$\Sigma_{j=0}^{p+q}|c_{jN}(j) - c(j)| \leqslant \Sigma_{j=0}^{p+q}\left\{\left(\frac{L-j}{N-j}\right)|c_{jL}(j) - c(j)| + \left(\frac{N-L}{N-j}\right)|c_{LN}(j) - c(j)|\right\}$$

$$\leqslant g_L + h_L,$$

$$g_L = \frac{L}{L-p-q}\Sigma_{j=0}^{p+q}|c_{jL}(j) - c(j)|,$$

$$h_L = \frac{L}{L-p-q}\max_{L<N<2L}\Sigma_{j=0}^{p+q}|c_{LN}(j) - c(j)|,$$

for $p + q < L < N < 2L$. Consider g_L. For N not too small,

$$\begin{aligned} Eg_L^{\nu} &\leqslant (p + q + 1)^{\nu-1}L(L - p - q)^{-\nu}\Sigma_{j=0}^{p+q}E|c_{jL}(j) - c(j)|^{\nu} \\ &\leqslant K(p + q + 1)^{\nu}L^{\nu}(L - p - q)^{1-2\nu} \\ &\leqslant K\{L/(L - p - q)\}^{2\nu-1}\{(\ln L)^{\nu+1}(\ln \ln L)^{\nu}\}^{-1}, \end{aligned}$$

from Lemma 2 and (5.2). Now if we choose $L = 2^l$, l integer, and use (5.1), the last expression is bounded by $Kl^{-\nu}$. Thus by Markov's inequality and the Borel-

Cantelli lemma, $g_L \to 0$ a.s. For h_L,

$$Eh_L^\nu \leqslant (p+q+1)^{\nu-1}L^\nu(L-p-q)^{-\nu}\Sigma_{j=0}^{p+q}E\{\max_{L<N<2L}|c_{LN}(j)-c(j)|\}^\nu$$
$$\leqslant K(p+q+1)^\nu L^\nu(L-p-q)^{-\nu}(\log_2 4L)^\nu L^{1-\nu}$$
$$\leqslant K(\log_2 4L)^\nu\{(\ln L)^{\nu+1}(\ln\ln L)^\nu\}^{-1} \leqslant Kl^{-1}(\ln l)^{-\nu},$$

using Billingsley [2, page 102], Lemma 2, (5.1) and (5.2). Thus $h_L \to 0$ a.s. by Markov's inequality and the Borel-Cantelli lemma, completing the proof that $\|\Delta_{Npq}\| \to 0$ a.s. The proof that $\|\delta_{Npq}\| \to 0$ a.s. is very similar so we omit it. □

THEOREM 4. *Let Conditions* A, B, D *and* E *hold with* $u+\xi>1$, $v+\chi>1$. *Then*

$$\lim_{N,p,q\to\infty}\max_{|s|=1}|\tilde{b}_{Npq}(s)-\tilde{\beta}(s)| = 0, \quad \text{a.s.},$$

under (4.5), (5.1) *and*

$$p+q \leqslant KN^{(2(\nu-1)/3\nu)}/\left\{(\ln N)^{(2(\nu+1)/3\nu)}(\ln\ln N)^{\frac{2}{3}}\right\}.$$

The proof is omitted because it uses Theorem 3 in precisely the way the proof of Theorem 2 used Theorem 1.

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