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A MONOTONE UNIMODAL DISTRIBUTION WHICH IS NOT CENTRAL CONVEX UNIMODAL

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For symmetric univariate distributions the usual definition of unimodality due to Khintchine has several equivalent formulations. When these concepts are generalized to higher dimensions in an attempt to define multivariate unimodality questions concerning their equivalence naturally arise. Of particular interest in this area is the relationship between two concepts first studied by Sherman and more recently by Dharmadhikari and Jogdeo. They asked if requiring that a distribution belong to the closed convex hull of all uniform distributions on symmetric convex bodies was the same as requiring that the probability it assigns to a symmetric convex set decrease as the set is translated away from the origin in a fixed direction. Sherman conjectured that the two concepts were the same while Dharmadhikari and Jogdeo felt that this was not so and they suggested a possible counterexample to Sherman's conjecture. In this paper it is shown that their example is indeed a counterexample.

1. Notation and terminology. A subset A of $\mathbb{R}^n$ will be called symmetric if $x \in A$ implies $(-x) \in A$. If $A \subset \mathbb{R}^n$ denote by $\bar{A}$ the symmetric set $A \cup (-A)$, let $\mathrm{Cl}(A)$ denote its closure in the usual topology; $V(A)$ will mean the Lebesgue measure of A when defined. If $A, B \subset \mathbb{R}^n$ then by $A \backslash B$ we mean A intersect the complement of B. If $x_1, \cdots, x_k \in \mathbb{R}^n$ then $[x_1, \cdots, x_k]$ is the closed convex hull generated in $\mathbb{R}^n$ by these points. A collection of probability distributions on $\mathbb{R}^n$ will be called convex if it is closed under finite mixtures and will be called closed if it is closed in the topology of weak convergence.

2. Central convex unimodality and monotone unimodality. The following definitions are based on the results of Sherman (1955) and were formulated by Dharmadhikari and Jogdeo (1976).

DEFINITION 2.1. A distribution P on $\mathbb{R}^n$ is called *central convex* unimodal if it belongs to the closed convex hull of all uniform distributions on symmetric convex bodies in $\mathbb{R}^n$.

DEFINITION 2.2. A distribution P on $\mathbb{R}^n$ is called *monotone* unimodal if for every symmetric convex subset K of $\mathbb{R}^n$ and every $x \in \mathbb{R}^n$ the quantity $P(K + tx)$ is nonincreasing in $t \geqq 0$.

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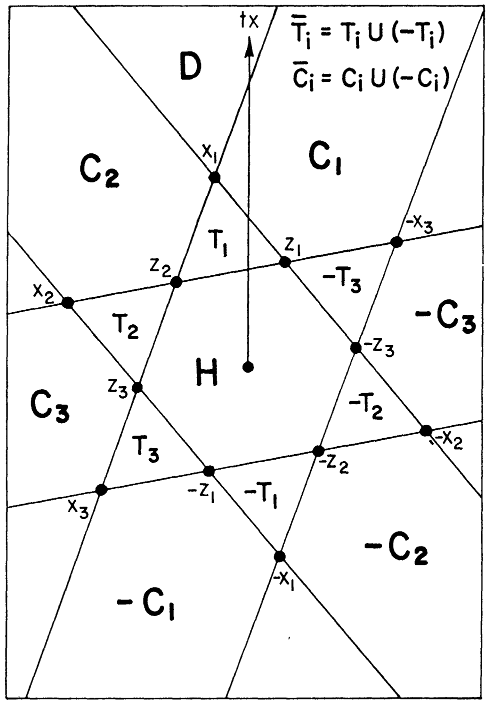

FIG. 1.

Sherman (1955) conjectured the equivalence of these definitions after having proved

THEOREM 2.1. *Every central convex unimodal distribution is monotone unimodal.*

Dharmadhikari and Jogdeo (1976) suggest a possible counterexample to the converse of Theorem 2.1. In the following section this example is given along with a proof that their example is indeed a counterexample.

3. A monotone unimodal distribution which is not central convex. Let $\pm x_i$, $i = 1, 2, 3$, be points in the plane which are the vertices of a regular hexagon centered at the origin. Let A be the triangle $[x_1, x_3, -x_2]$ as in Figure 1 and let $\bar{A} = A \cup (-A)$. The star $\bar{A}$ consists of 6 smaller triangles surrounding an inner hexagon. Let P be the distribution supported by $\bar{A}$ which has density α on each of the outer triangles and density 2α on the inner hexagon.

It is not difficult to show that P is not central convex unimodal; in the remainder of this section a proof that P is in fact monotone unimodal will be developed gradually.

Let the points $\pm x_i$, $\pm z_i$, $i = 1, 2, 3$, the outer triangles $\pm T_i$, $i = 1, 2, 3$, the inner hexagon H and the infinite regions D and $\pm C_i$, $i = 1, 2, 3$, be as indicated in Figure 1. All of the above regions will be taken to be closed. Let $x \in \mathbb{R}^2$ and let K be a symmetric convex region of the plane which is *open* but which has compact closure. By symmetry and a limit argument, it suffices to show that the quantity $P(K + tx)$ is nonincreasing in nonnegative t when the ray $\{tx: t \geqq 0\}$ passes through the interior of the line segment $[x_1, z_1]$.

For notational convenience let $K(t) = K + tx$ and for each $S \subset \mathbb{R}^2$ let $\bar{S} = S \cup (-S)$. Assume also that $\alpha = 1$ so that

$$(3.1) \qquad P(K + tx) = P(K(t)) = 2V(H \cap K(t)) + \textstyle\sum_{i=1}^{3} V(\bar{T}_i \cap K(t)) .$$

Since $P(K(t))$ depends continuously on t it suffices to show that for every $s \geqq 0$ there exists $s' > s$ such that $P(K(t)) \leqq P(K(s))$ if $s < t < s'$. Throughout the rest of this section the following lemma will be used to show that for each $s \geqq 0$ such an s' can be found.

LEMMA 3.1. *Let $s \geqq 0$ be given. If there exists $s' > s$ and $i = 1, 2$ or 3 such that whenever $s < t < s'$ either*

$$(3.2) \qquad V(\bar{C}_i \cap K(t)) \geqq V(\bar{C}_i \cap K(s))$$

or

$$(3.3) \qquad V(\bar{T}_i \cap K(t)) \leqq V(\bar{T}_i \cap K(s))$$

then $P(K(t)) \leqq P(K(s))$ for $s < t < s'$.

PROOF. For definiteness take $i = 1$. Let $s < t < s'$ and let $S_1 = \bar{C}_1 \cup H \cup \bar{T}_1 \cup \bar{T}_3$, $S_2 = \bar{T}_2 \cup H$ and $S_3 = \bar{T}_3 \cup H$. Each of these regions is symmetric convex and so by Theorem 2.1

$$(3.4) \qquad V(S_1 \cap K(t)) + V(S_2 \cap K(t)) \leqq V(S_1 \cap K(s)) + V(S_2 \cap K(s))$$

and

$$(3.5) \qquad V(S_2 \cap K(t)) + V(S_3 \cap K(t)) \leqq V(S_2 \cap K(s)) + V(S_3 \cap K(s)) .$$

Comparing (3.4) with (3.1) we see that (3.4) is the same as $V(\bar{C}_1 \cap K(t)) + P(K(t)) \leqq V(\bar{C}_1 \cap K(s)) + P(K(s))$. Thus if (3.2) holds so does the inequality $P(K(t)) \leqq P(K(s))$. Inequality (3.5) is the same as $P(K(t)) - V(\bar{T}_1 \cap K(t)) \leqq P(K(s)) - V(\bar{T}_1 \cap K(s))$, so (3.3) will also imply $P(K(t)) \leqq P(K(s))$ and the lemma is proved.

At this point we pause to give a brief sketch of the arguments to be used in the remainder of the section.

Let $s \geqq 0$ be given. Various cases will be considered which arise according to whether or not the intersection of $K(s)$ with various regions labelled in Figure 1 is void. The particular cases to be considered are those in which either $\bar{C}_3 \cap K(s) = \emptyset$, $D \cap \mathrm{Cl}[K(s)] \neq \emptyset$ or $(\pm C_i) \cap K(s) = \emptyset$, $i = 1$ or 2. In each case Lemma 3.1 will be applied to show for some $s' > s$ the inequality $P(K(t)) \leqq P(K(s))$ holds for $s < t < s'$. Given that none of these cases occurs, it will be shown that there exists $s' > s$ such that $V(\bar{T}_1 \cap K(t)) \leqq V(\bar{T}_1 \cap K(s))$ if $s < t < s'$. An application of Lemma 3.1 will then complete the proof that P is monotone unimodal.

Throughout the rest of this section s will be a nonnegative real number.

Suppose that $\bar{C}_3 \cap K(s) = \emptyset$. Then for every $t > s$ inequality (3.2) will hold for $i = 3$. Applying Lemma 3.1 we have

THEOREM 3.1. *If* $\bar{C}_3 \cap K(s) = \emptyset$, *then* $P(K(t)) \leqq P(K(s))$ *for every* $t > s$.

To deal with the next case requires two additional lemmas.

LEMMA 3.2. *If* $\bar{C}_3 \cap K(s) \neq \emptyset$, *then for* $t > s$ $(-T_1) \cap K(t) \subset K(s)$.

PROOF. Let $t > s$ and let $x_0 \in (-T_1) \cap K(t)$. Let $z \in \bar{C}_3 \cap K(s)$ and let $z' = -z + 2sx$. By symmetry $z' \in K(s)$ and by convexity $[z, z'] \subset K(s)$. On considering how z and z' must be situated relative to $-T_1$ from Figure 1 it is clear that for some $r \geqq 0$ we have $x_0 + rx \in [z, z'] \subset K(s)$. Since $x_0 \in K(t)$ we also have $x_0 - (t - s)x \in K(s)$, so $x_0 \in K(s)$ since $x_0 \in [x_0 - (t - s)x, x_0 + rx] \subset K(s)$.

LEMMA 3.3. *Suppose* $\bar{C}_3 \cap K(s) \neq \emptyset$ *and that* $D \cap K(s) \neq \emptyset$. *Then* $T_1 \cap K(t) \subset K(s)$ *if* $t > s$.

PROOF. Let $t > s$ and let $x_0 \in T_1 \cap K(t)$. Let $z \in \bar{C}_3 \cap K(s)$ and let $y \in D \cap K(s)$. Again we have $z' \in K(s)$ where $z' = -z + 2sx$. The possible positions of the points y, z and z' relative to T_1 are all such that $x_0 + rx \in [y, z, z']$ for some $r \geqq 0$. By convexity $[y, z, z'] \subset K(s)$ and since $x_0 - (t - s)x \in K(s)$ it follows that $x_0 \in K(s)$ which establishes the lemma.

Combining all of the previous results we obtain

THEOREM 3.2. *If* $D \cap \mathrm{Cl}[K(s)] \neq \emptyset$, *then* $P(K(t)) \leqq P(K(s))$ *if* $t > s$.

PROOF. Using continuity we can replace the hypothesis $D \cap \mathrm{Cl}[K(s)] \neq \emptyset$ by $D \cap K(s) \neq \emptyset$. Let $t > s$. By Theorem 3.1 we may assume $\bar{C}_3 \cap K(s) \neq \emptyset$. Together Lemmas 3.2 and 3.3 then imply that $\bar{T}_1 \cap K(t) \subset K(s)$ and hence that $\bar{T}_1 \cap K(t) \subset \bar{T}_1 \cap K(s)$. The latter implies $V(\bar{T}_1 \cap K(t)) \leqq V(\bar{T}_1 \cap K(s))$ and by applying Lemma 3.1 the proof is completed.

In view of Theorems 3.1 and 3.2, to show that there exists $s' > s$ such that $P(K(t)) \leqq P(K(s))$ if $s < t < s'$ one may assume

$$\bar{C}_3 \cap K(s) \neq \emptyset \tag{3.6}$$

and

$$D \cap \mathrm{Cl}[K(s)] = \emptyset\ . \tag{3.7}$$

Next we show that one may also assume

$$(-C_i) \cap K(s) \neq \emptyset\ , \qquad i = 1, 2 \tag{3.8}$$

and

$$C_i \cap K(s) \neq \emptyset\ , \qquad i = 1, 2\ . \tag{3.9}$$

Observe that for $t > s$ and $i = 1, 2$ $(C_i \cap K(s)) + (t - s)x \subset (C_i \cup D) \cap K(t)$. Since by (3.7) $D \cap \mathrm{Cl}[K(s)] = \emptyset$, there exists $s' > s$ such that $s < t < s'$ implies $D \cap K(t) = \emptyset$. For such t and $i = 1, 2$ we have $(C_i \cap K(s)) + (t - s)x \subset C_i \cap K(t)$, whence $V(C_i \cap K(s)) \leqq V(C_i \cap K(t))$. This implies that if for $i = 1$

or 2 $(-C_i) \cap K(s) = \emptyset$, then for $s < t < s'$ we will have

$$V(\bar{C}_i \cap K(s)) = V(C_i \cap K(s)) \leqq V(C_i \cap K(t)) \leqq V(\bar{C}_i \cap K(t))$$

which together with Lemma 3.1 implies that $P(K(t)) \leqq P(K(s))$. This justifies assumptions (3.8). Assume then that for $i = 1$ or 2 $x_0 \in (-C_i) \cap K(s)$. Then $-x_0 \in C_i$ and $-x_0 + 2sx \in K(s)$. This last point will lie in $C_i \cup D$ but cannot lie in D by (3.7). Therefore $-x_0 + 2sx \in C_i \cap K(s)$ and (3.9) is justified.

Next it will be shown that with (3.6)—(3.9) assumed there exists $s' > s$ such that $V(\bar{T}_1 \cap K(t)) \leqq V(\bar{T}_1 \cap K(s))$ for $s < t < s'$. This will be done by constructing a symmetric convex set $R \cup \bar{T}$ with $V(R \cap \bar{T}) = 0$ and $\bar{T} \subset \bar{T}_1$. Theorem 2.1 will then be used to show that for each t in some open interval (s, s') the inequality $V(\bar{T} \cap K(t)) \leqq V(\bar{T} \cap K(s))$ holds. That for $s < t < s'$ $V[(\bar{T}_1 \backslash \bar{T}) \cap K(t)] \leqq V[(\bar{T}_1 \backslash \bar{T}) \cap K(s)]$ will also be established, hence $V(\bar{T}_1 \cap K(t)) \leqq V(\bar{T}_1 \cap K(s))$ for such t. Since the construction of the region $R \cup \bar{T}$ differs only slightly in the two cases $C_3 \cap K(s) \neq \emptyset$ and $(-C_3) \cap K(s) \neq \emptyset$, in place of (3.6) and in addition to (3.7)—(3.9) it will be assumed that

(3.10) $$C_3 \cap K(s) \neq \emptyset \,.$$

As a first step in constructing the region $R \cup \bar{T}$ we prove that (3.7)—(3.10) imply

(3.11) $$z_1, z_2, -z_1 \in K(s) \,.$$

Indeed, by (3.8)—(3.10) we may choose points $y_1, \cdots, y_5 \in K(s)$ lying, respectively, in the regions $C_1, C_2, C_3, -C_1$ and $-C_2$. Obviously the pentagon $[y_1, \cdots, y_5]$ will contain both z_2 and $-z_1$ as well as the point z_3. Thus, $-z_1, z_2, z_3 \in K(s)$. To see that $z_1 \in K(s)$, let $z_1' = z_1 + 2sx$ and $z_3' = -z_3 + 2sx$. By symmetry $z_1', z_3' \in K(s)$ and so $Q \subset K(s)$ where Q is the parallelogram $[-z_1, z_3, z_1', z_3']$. Because of (3.7) z_1' cannot lie in the region D and so the side $[z_1', z_3]$ of Q will intersect $[x_1, z_1]$. Since $[x_1, z_1]$, $[-z_1, z_3]$ and $[z_1', z_3']$ are all parallel and of equal length, it must be that $z_1 \in Q$. Thus (3.11) is established.

Next we note that since $-z_1 \in K(s)$ by (3.11) and since $(-C_2) \cap K(s) \neq \emptyset$ by (3.8), there is a point $-y \in K(s)$ lying on the segment $[-x_1, -z_2]$. Hence $y + 2sx \in K(s)$ and $y \in [x_1, z_2]$. By (3.11), $z_1, z_2 \in K(s)$ so that $[z_1, z_2, y + 2sx]$ is contained in $K(s)$. Since y lies in the interior of this triangle, $y \in K(s)$. This shows that there is a y such that

$$y, -y \in K(s) \qquad \text{and} \qquad y \in [x_1, z_1] \,.$$

Let R be the parallelogram $[y, z_1, -y, -z_1]$. Then $R \subset K(s)$ and since K was assumed to be open there exists $s' > s$ such that if $s < t < s'$ then $R \subset K(t)$. Let t lie in this interval and let $T = [x_1, y, z_1] \subset K(s)$. The region $R \cup \bar{T}$ is symmetric convex so $V[(R \cup \bar{T}) \cap K(t)] \leqq V[(R \cup \bar{T}) \cap K(s)]$ as a consequence of Theorem 2.1. By choice of t, $R \subset K(s) \cap K(t)$ therefore

(3.12) $$V(\bar{T} \cap K(t)) \leqq V(\bar{T} \cap K(s)) \,.$$

Now $T_1 \backslash \bar{T} = [y, z_1, z_2] \subset K(s)$ by (3.11) and choice of y, whence $(T_1 \backslash \bar{T}) \cap K(t) \subset T_1 \backslash \bar{T} = (T_1 \backslash \bar{T}) \cap K(s)$. It follows that

$$V[(T_1 \backslash \bar{T}) \cap K(t)] \leqq V[(T_1 \backslash \bar{T}) \cap K(s)] . \tag{3.13}$$

By Lemma 3.2 we also have

$$[(-T_1) \backslash \bar{T}] \cap K(t) \subset (-T_1) \cap K(t) \subset K(s)$$

so that

$$[(-T_1) \backslash \bar{T}] \cap K(t) \subset [(-T_1) \backslash \bar{T}] \cap K(s) .$$

Consequently,

$$V([-T_1) \backslash \bar{T}] \cap K(t)) \leqq V([(-T_1) \backslash \bar{T}] \cap K(s)) . \tag{3.14}$$

Combining (3.12), (3.13), (3.14) and the fact that $\bar{T} \subset \bar{T}_1$ we obtain $V(\bar{T}_1 \cap K(t)) \leqq V(\bar{T}_1 \cap K(s))$. Applying Lemma 3.1 we have $P(K(t)) \leqq P(K(s))$ if $s < t < s'$. Thus it has been shown that for every $s \geqq 0$ there exists $s' > s$ such that $P(K(t)) \leqq P(K(s))$ if $s < t < s'$ and the proof that P is monotone unimodal is complete.

4. An open question and a reference for related problems. Sherman (1955) proved that the class of central convex unimodal distributions was closed under convolutions. Since monotone unimodality does not imply central convex unimodality one wonders if the convolution of monotone unimodal distributions is again monotone unimodal.

A summary of the work done on multivariate unimodality along with a list of unsolved problems in this area is given in Dharmadhikari and Jogdeo (1976).

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