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ON THE ASYMPTOTIC EFFICIENCY OF STRONGLY ASYMPTOTICALLY MEDIAN UNBIASED ESTIMATORS

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It is shown that the problem of asymptotic efficiency (in the Wolfowitz-sense) admits a satisfactory solution for the class of strongly asymptotically median unbiased estimators. The concept of strongly asymptotic median unbiasedness furthermore links up the classical approach (concerning estimator-sequences whose distribution functions converge) with Pfanzagl's concept of median unbiasedness.

Let $(X, \mathscr{A})$ be a measurable space and $P_\theta \mid \mathscr{A}$, $\theta \in \Theta$, where $\Theta \subset \mathbb{R}$ is open, a family of probability measures. Denote by $P_\theta^n \mid \mathscr{A}^n$ the independent product of n identical components $P_\theta \mid \mathscr{A}$.

One of the concepts of asymptotic efficiency of estimators (asymptotic efficiency in terms of covering probabilities) deals with the following problem: A specified class $\mathscr{E}$ of estimator-sequences $\{T_n\}$ is taken into consideration. First one concentrates on determining an upper bound $B(\theta, t_1, t_2)$, say, such that for all $\{T_n\} \in \mathscr{E}$, all $t_1, t_2 > 0$, and all $\theta \in \Theta$

$$(1) \qquad \limsup_{n\to\infty} P_\theta^n\{\mathbf{x} \in X^n : \theta - t_1 n^{-\frac{1}{2}} < T_n(\mathbf{x}) < \theta + t_2 n^{-\frac{1}{2}}\} \leqq B(\theta, t_1, t_2)$$

and then one attempts to find a sequence of estimators $\{T_n^*\} \in \mathscr{E}$ for which equality holds in (1) with $\limsup_{n\to\infty}$ replaced by $\lim_{n\to\infty}$. (This estimator-sequence $\{T_n^*\}$ then is called optimal in $\mathscr{E}$.)

The "classical" approach deals with estimator-sequences whose distribution functions converge: Let $\mathscr{E}_0$ be the class of all estimator-sequences $\{T_n\}$ with the property that the distribution functions of $a_n(T_n - \theta)$, say $F_{n,\theta}^{T_n}$, where $a_n > 0$ are chosen appropriately, converge, for each $\theta \in \Theta$, to a distribution function F_θ^T (which may depend on the sequence $\{T_n\}$) with $F_\theta^T(0) = \frac{1}{2}$, $\theta \in \Theta$. (In the classical theory the limiting distribution is assumed to be normal, but the last condition on it is the essential one.)

Consider

$$(2) \qquad \mathscr{E}_1 = \{\{T_n\} \in \mathscr{E}_0 : F_{n,\theta}^{T_n}(0) \to F_\theta^T(0) \text{ continuously in } \theta\}.$$

(For the definition of continuous convergence see, e.g., Roussas [3], Definition 2.1, page 131.) Then (under suitable regularity conditions) inequality (1) holds true with "Fisher's asymptotic bound" $B(\theta, t_1, t_2) = \Phi(t_2 I(\theta)^{\frac{1}{2}}) - \Phi(-t_1 I(\theta)^{\frac{1}{2}})$, where $I(\theta) = \int ((\partial/\partial\theta) \log p(x, \theta))^2 P_\theta(dx)$. This result can also be proved for the

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class

(3) $\mathscr{E}_2 = \{\{T_n\} \in \mathscr{E}_0 : F^{T_n}_{n,\theta}(0) \to F_\theta{}^T(0)$ locally uniformly in $\theta\}$.

It is, however, easy to see that $\mathscr{E}_2 \subset \mathscr{E}_1$: the assumption $F_\theta{}^T(0) = \frac{1}{2}$, $\theta \in \Theta$, implies that $\theta \to F_\theta{}^T(0)$ is continuous. Hence, $F^{T_n}_{n,\theta}(\theta) \to F_\theta{}^T(0)$ uniformly in θ implies that the convergence is continuous in θ (see Roussas [3], Lemma 2.2, page 132).

Since the restriction to estimator-sequences in $\mathscr{E}_0$ lacks any operational justification Pfanzagl [1] considers the intuitively appealing class of *median unbiased estimators*

(4) $$\mathscr{E}^* = \{\{T_n\} : P_\theta{}^n\{\mathbf{x} \in X^n : T_n(\mathbf{x}) \geqq \theta\} \geqq \tfrac{1}{2} \text{ and } P_\theta{}^n\{\mathbf{x} \in X^n : T_n(\mathbf{x}) \leqq \theta\} \geqq \tfrac{1}{2} \text{ for all } n \in \mathbb{N}, \theta \in \Theta\}$$

and proves the above cited result in Theorem 1, page 1502, of his paper for the class $\mathscr{E}^*$.

Since inequality (1) is an asymptotic result, it is not surprising that the assumption of median unbiasedness, which refers to finite sample sizes, may be weakened. As a matter of fact, we show in our theorem that Pfanzagl's result can be extended to the larger class $\mathscr{E}^{**}$ of strongly asymptotically median unbiased estimators, where we define an estimator-sequence $\{T_n\}$ to be *strongly asymptotically median unbiased* iff for all $\theta \in \Theta$ and all $t > 0$,

$$\liminf_{n\to\infty} P^n_{\theta+tn^{-\frac{1}{2}}}\{\mathbf{x} \in X^n : T_n(\mathbf{x}) \geqq \theta + tn^{-\frac{1}{2}}\} \geqq \tfrac{1}{2}$$

and

$$\liminf_{n\to\infty} P^n_{\theta-tn^{-\frac{1}{2}}}\{\mathbf{x} \in X^n : T_n(\mathbf{x}) \leqq \theta - tn^{-\frac{1}{2}}\} \geqq \tfrac{1}{2} .$$

Let us remark at this stage in which way our concept links up the classical approach with Pfanzagl's approach: Since Θ is open we obviously have $\mathscr{E}^* \subset \mathscr{E}^{**}$. On the other hand, it is straightforward to prove that $\mathscr{E}_1 \subset \mathscr{E}^{**}$ (recall, furthermore, that $\mathscr{E}_2 \subset \mathscr{E}_1$).

To see the advantages of the enlargement of the class $\mathscr{E}^*$ to the class $\mathscr{E}^{**}$ concerning the existence of optimal estimator-sequences one has to compare our theorem with Theorem 4.3 of Pfanzagl ([2], page 162), where it is shown that a given (asymptotically efficient) estimator-sequence can be adjusted (by a suitable randomization procedure) in such a way that (i) each estimator of the sequence becomes median unbiased, (ii) the asymptotic behavior of the sequence remains unchanged. Besides the stronger regularity conditions which are needed to prove Pfanzagl's theorem he only obtains a theoretical result on the existence of optimal median unbiased estimators, whereas in our class the sequence of maximum likelihood estimators is optimal.

The assumptions of our theorem are the same as Pfanzagl's in [1], Theorem 1, page 1502. (Needed are results on the convergence of the distributions of the log-likelihood functions under $P_\theta{}^n$ and $P^n_{\theta+tn^{-\frac{1}{2}}}$, i.e., formulas (5.1) and (5.2) in Roussas [3], page 148.)

THEOREM. *Under the above cited regularity conditions we have*:

(i) *For every* $\{T_n\} \in \mathscr{E}^{**}$, *the class of strongly asymptotically median unbiased estimators, and for all* $t_1, t_2 > 0$, $\theta \in \Theta$,

$$\limsup_{n\to\infty} P_\theta^n\{\mathbf{x} \in X^n : \theta - t_1 n^{-\frac{1}{2}} < T_n(\mathbf{x}) < \theta + t_2 n^{-\frac{1}{2}}\} \leqq \Phi(t_2 I(\theta)^{\frac{1}{2}}) - \Phi(-t_1 I(\theta)^{\frac{1}{2}}) .$$

(ii) *Assume that the sequence of maximum likelihood estimators* $\{T_n{}^*\}$ *belongs to the class* $\mathscr{E}_1$ *such that* $F_\theta^{T^*}(t) = \Phi(tI(\theta)^{\frac{1}{2}})$. *Then* $\{T_n{}^*\}$ *is optimal in* $\mathscr{E}^{**}$.

PROOF. Part (i) follows by straightforward modification of Pfanzagl's proof. Since $\mathscr{E}_1 \subset \mathscr{E}^{**}$, part (ii) is obvious.

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