The Annals of Statistics
1978, Vol. 6, No. 4, 882–894

STRONG APPROXIMATIONS OF THE QUANTILE PROCESS

BY MIKLÓS CSÖRGŐ[1] AND PÁL RÉVÉSZ

Carleton University, Ottawa and Mathematical Institute, Budapest

Let $q_n(y)$, $0 < y < 1$, be a quantile process based on a sequence of i.i.d. rv with distribution function F and density function f. Given some regularity conditions on F the distance of $q_n(y)$ and the uniform quantile process $u_n(y)$, respectively defined in terms of the order statistics $X_{k:n}$ and $U_{k:n} = F(X_{k:n})$, is computed with rates. As a consequence we have an extension of Kiefer's result on the distance between the empirical and quantile processes, a law of iterated logarithm for $q_n(y)$ and, using similar results for the uniform quantile process $u_n(y)$, it is also shown that $q_n(y)$ can be approximated by a sequence of Brownian bridges as well as by a Kiefer process.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. rv with a continuous distribution function $F(\cdot)$ and let $X_{1:n} \leqq \cdots \leqq X_{n:n}$ denote the order statistics of the random sample $X_1, \cdots, X_n$. Define the empirical distribution function $F_n(x)$ and the quantile function $Q_n(y)$ as follows:

$$\begin{aligned} F_n(x) &= 0 && \text{if} \quad X_{1:n} > x \\ &= \frac{k}{n} && \text{if} \quad X_{k:n} \leqq x < X_{k+1:n}\,, \quad k = 1, 2, \cdots, n-1\,. \\ &= 1 && \text{if} \quad X_{n:n} \leqq x\,, \\ Q_n(y) &= X_{k:n} && \text{if} \quad \frac{k-1}{n} < y \leqq \frac{k}{n}\,, \quad k = 1, 2, \cdots, n\,. \end{aligned}$$

Define also the empirical process $\beta_n(x)$ and the quantile process $q_n(y)$ the following way:

$$\begin{aligned} \beta_n(x) &= n^{\frac{1}{2}}(F_n(x) - F(x))\,, && -\infty < x < +\infty \\ q_n(y) &= n^{\frac{1}{2}}(Q_n(y) - F^{-1}(y))\,, && 0 < y < 1\,. \end{aligned}$$

It is of interest to investigate how well these processes can be approximated by appropriate Gaussian processes.

In case of rv uniformly distributed over $[0, 1]$, this problem can be handled in a somewhat easier way. In order to make a distinction between this and the general case, the following notations will be used in the uniform setup:

$$\begin{aligned} &U_i \quad \text{instead of} \quad X_i\,, && i = 1, 2, \cdots\,, \\ &U_{k:n} \quad \text{instead of} \quad X_{k:n}\,, && k = 1, 2, \cdots, n\,, \end{aligned}$$

Received October 1975; revised June 1977.

[1] This research was partially supported by a Canadian Research Council Grant at Carleton University, Ottawa.

AMS 1970 *subject classifications.* Primary 62G30; Secondary 60F15.

Key words and phrases. Quantile process, strong approximations, strong invariance, Gaussian processes, convergence rates.

$$\begin{array}{llll} E_n(y) & \text{instead of} & F_n(x)\,, & 0 \leqq y \leqq 1,\ -\infty < x < +\infty\,, \\ U_n(y) & \text{instead of} & Q_n(y)\,, & 0 < y < 1\,, \\ \alpha_n(y) & \text{instead of} & \beta_n(x)\,, & 0 \leqq y \leqq 1,\ -\infty < x < +\infty\,, \\ u_n(y) & \text{instead of} & q_n(y)\,, & 0 < y < 1\,. \end{array}$$

Two Gaussian processes play an important role in our approximations: Brownian bridges and the Kiefer process, respectively defined as follows.

A Brownian bridge $\{B(y); 0 \leqq y \leqq 1\}$ is a separable Gaussian process with $EB(y) = 0$, and covariance function $EB(y_1)B(y_2) = y_1 \wedge y_2 - y_1 y_2$.

A Kiefer process $\{K(y, t); 0 \leqq y \leqq 1, 0 \leqq t\}$ is a separable Gaussian process with $EK(y, t) = 0$, and covariance function $EK(y_1, t_1)K(y_2, t_2) = (t_1 \wedge t_2)(y_1 \wedge y_2 - y_1 y_2)$.

Both these processes can be represented in terms of standard Wiener processes as follows:

$$B(y) = W(y) - yW(1)\,, \qquad 0 \leqq y \leqq 1\,,$$

$$K(y, t) = W(y, t) - yW(1, t)\,, \qquad 0 \leqq y \leqq 1,\ t \geqq 0\,,$$

where $\{W(y); 0 \leqq y\}$ is a standard Wiener process with $EW(y) = 0$ $EW(y_1)W(y_2) = y_1 \wedge y_2$, and $\{W(y, t); 0 \leqq y, 0 \leqq t\}$ a standard, two dimensional Wiener process with $EW(y, t) = 0$ and

$$EW(y_1, t_1)W(y_2, t_2) = (y_1 \wedge y_2)(t_1 \wedge t_2).$$

When talking about approximation of the empirical and quantile processes by appropriate Gausian processes, we think of constructing the latter on the probability space of the former so that they should be near to each other with probability one. This can be done if this probability space is rich enough in the sense that an infinite independent sequence of Wiener processes can be defined on it, which is also independent of the originally given i.i.d. sequence.

In the sequel it will be always assumed that the underlying probability space is rich enough in the above sense.

As to the uniform empirical process the following is known:

THEOREM A (Komlós, Major and Tusnády (1975)). *One can define a Brownian bridge* $\{B_n(y); 0 \leqq y \leqq 1\}$ *for each n and a Kiefer process* $\{K(y, t); 0 \leqq y \leqq 1, 0 \leqq t\}$ *such that*

$$P\{\sup_{0 \leqq y \leqq 1} |\alpha_n(y) - B_n(y)| > n^{-\frac{1}{2}}(C \log n + z)\} < Le^{-\lambda z} \tag{1.1}$$

for all z, where C, L and λ are positive absolute constants;

$$P\{\sup_{1 \leqq k \leqq n} \sup_{0 \leqq y \leqq 1} |k^{\frac{1}{2}}\alpha_k(y) - K(y, k)| > (C \log n + z) \log n\} < Le^{-\lambda z} \tag{1.2}$$

for all z and n, where C, L and λ are positive absolute constants. These inequalities, in turn, imply

$$\sup_{0 \leqq y \leqq 1} |\alpha_n(y) - B_n(y)| =_{\text{a.s.}} O(n^{-\frac{1}{2}} \log n)\,, \tag{1.3}$$

and

(1.4) $$\sup_{0\leq y\leq 1} |n^{\frac{1}{2}}\alpha_n(y) - K(y, n)| =_{\text{a.s.}} O(\log^2 n)\,.$$

REMARK 1. It is clear from (1.1) that (1.3) can be written as

(1.3*) $$\limsup_{n\to\infty} \frac{n^{\frac{1}{2}}}{\log n} \sup_{0\leq y\leq 1} |\alpha_n(y) - B_n(y)| \leq C_0 \quad \text{a.s.},$$

where C_0 is an absolute constant. Obviously (1.4) can be also rewritten this way and we note here that, throughout this exposition, the notation $=_{\text{a.s.}} O(\cdot)$ will always have a similar meaning with an appropriate absolute constant.

THEOREM B (Csörgő and Révész (1975)). *Concerning the uniform quantile process we have: one can define a Brownian bridge $\{B_n(y);\ 0 \leq y \leq 1\}$ for each n and a Kiefer process $\{K(y, t);\ 0 \leq y \leq 1,\ 0 \leq t\}$ such that*

(1.5) $$\sup_{0\leq y\leq 1} |u_n(y) - B_n(y)| =_{\text{a.s.}} O(n^{-\frac{1}{2}} \log n)$$

and

(1.6) $$\sup_{0\leq y\leq 1} |n^{\frac{1}{2}}u_n(y) - K(y, n)| =_{\text{a.s.}} O((n \log\log n)^{\frac{1}{4}}(\log n)^{\frac{1}{2}})\,.$$

REMARK 2. The rate of (1.3) and (1.5) is the best possible and that of (1.4) cannot be improved beyond $\log n$. As to the rate of (1.6), it seems to be far from the best possible one.

A disadvantage of the statements of (1.3) and (1.5) is that we do not know anything about the joint distribution in n of the corresponding sequences of Brownian bridges. Consequently, only "in probability" and "in distribution" limit laws can be proved for $\alpha_n(\cdot)$ and $u_n(\cdot)$ from these statements. Thus, in spite of the weaker rates of convergence in (1.4) and (1.6), an advantage of these latter ones is the possibility of producing strong laws for $\alpha_n(\cdot)$ and $u_n(\cdot)$ via establishing the same for $K(y, n)$ as $n \to \infty$. For example, knowing the law of iterated logarithm

$$\limsup_{n\to\infty} \sup_{0\leq y\leq 1} \frac{|K(y, n)|}{(2n \log\log n)^{\frac{1}{2}}} =_{\text{a.s.}} \tfrac{1}{2}$$

for the Kiefer process, one can immediately write down the same for $n^{\frac{1}{2}}u_n(y)$.

One of the aims of the present exposition is to extend the results of Theorem B to the nonuniform quantile process $q_n(y)$.

Letting $U_i = F(X_i)$, $i = 1, 2, \cdots$, we get that the U_i are independent uniform $-(0, 1)$ rv, provided $F(\cdot)$ is continuous. Also, for every $\omega \in \Omega$, $\alpha_n(F(x)) = \beta_n(x)$ and, consequently, Theorem A can be immediately generalized to the case of the empirical process $\beta_n(x)$ with an arbitrary continuous distribution function $F(x)$ by simply replacing y by $F(x)$ in the statements (1.1), (1.2), (1.3) and (1.4). As to the similar problem of strong approximations of the quantile process $q_n(y)$, however, there is no such immediate handle; that is, by simply replacing y by $F(x)$ in Theorem B we do not get the corresponding desired results for $q_n(y)$. However, the distance between $q_n(y)$ and $u_n(y)$, respectively defined in terms of

$X_{k:n}$ and $U_{k:n} = F(X_{k:n})$, can be computed accurately enough (cf. Section 3), so that Theorem B can actually be used to obtain strong approximations also for the general quantile process $q_n(y)$ (cf. Section 5). Our investigations in Section 3, when combined with a theorem of Kiefer (1970) concerning the deviations between the empirical and quantile processes, also give an extension of the latter result and a law of iterated logarithm for $q_n(y)$ (cf. Section 4). In order to sharpen our earlier result (1.5), Section 2 is devoted to the problem of strong approximations of the uniform quantile process $u_n(y)$. These latter results, in turn, are used to approximate the general quantile process $q_n(y)$ similarly. The results of this section are based on the following:

THEOREM C (Komlós, Major and Tusnády (1975)). *Let $Y_1, Y_2, \cdots$ be independent* rv *with standard exponential distribution. Then there exists a standard Wiener process $\{W(t);\, 0 \leqq t\}$ such that for all real z*

$$P\{\sup_{1 \leqq k \leqq n} |(S_k - k) - W(k)| \geqq A \log n + z\} \leqq Be^{-Cz}$$

where $S_k = Y_1 + \cdots + Y_k$ and A, B, C are positive absolute constants.

REMARK 3. Theorem C is only a special case of a more general result of Komlós, Major and Tusnády (1975).

2. Approximation of the uniform quantile process by Brownian bridges. First we prove

THEOREM 1. *If the uniform* $-(0, 1)$ rv $U_1, U_2, \cdots$ *are defined on a rich enough probability space, then one can define, for each n, a Brownian bridge $\{B_n(y);\, 0 \leqq y \leqq 1\}$ on the same probability space such that, for all z, we have*

$$P\{\sup_{0 \leqq y \leqq 1} |u_n(y) - B_n(y)| > n^{-\frac{1}{2}}(A \log n + z)\} \leqq Be^{-Cz}\,,$$

where A, B, C are positive absolute constants. Whence we also have (1.5).

PROOF. Put $Y_k = \log(1/U_k)$, $k = 1, 2, \cdots$, $S_0 = 0$, $S_k = \sum_{j=1}^{k} Y_j$, $k = 1, 2, \cdots$, and

$$\tilde{U}_n(y) = S_k/S_{n+1} \quad \text{if} \quad \frac{k-1}{n} < y \leqq \frac{k}{n}\,, \qquad k = 1, 2, \cdots, n\,, \tag{2.1}$$

$$\tilde{u}_n(y) = n^{\frac{1}{2}}(\tilde{U}_n(y) - y)\,, \qquad 0 \leqq y \leqq 1\,. \tag{2.2}$$

Then the Y_k are independent exponential rv with mean value one and, for each n,

$$\{\tilde{U}_n(y);\, 0 \leqq y \leqq 1\} =_D \{U_n(y);\, 0 \leqq y \leqq 1\}\,; \tag{2.3}$$

hence

$$\{\tilde{u}_n(y);\, 0 \leqq y \leqq 1\} =_D \{u_n(y);\, 0 \leqq y \leqq 1\}\,. \tag{2.4}$$

A simple calculation yields

$$\tilde{u}_n\left(\frac{k}{n}\right) = n^{\frac{1}{2}}\left(\frac{S_k}{S_{n+1}} - \frac{k}{n}\right) = n^{-\frac{1}{2}}\frac{n}{S_{n+1}}\left[(S_k - k) - \frac{k}{n}(S_{n+1} - n)\right]. \tag{2.5}$$

Let $W(k)$ be as in Theorem C. Define $B_n(y) = n^{-\frac{1}{2}}(W(ny) - yW(n)), 0 \leqq y \leqq 1$, and put $1 + \varepsilon_n = n/S_{n+1}$. Now consider

$$\text{(2.6)} \qquad \tilde{u}_n\left(\frac{k}{n}\right) - B_n\left(\frac{k}{n}\right) = n^{-\frac{1}{2}}\left[((S_k - k) - W(k)) - \frac{k}{n}((S_n - n) - W(n)) - \frac{k}{n} Y_{n+1} + \varepsilon_n((S_j - k) - \frac{k}{n}(S_{n+1} - n))\right].$$

We have for all z

$$P\left\{\sup_{1\leqq k\leqq n} n^{-\frac{1}{2}}|(S_k - k) - W(k)| \geqq \left(\frac{A\log n + z}{5}\right)n^{-\frac{1}{2}}\right\} \leqq Be^{-Cz},$$

$$P\left\{n^{-\frac{1}{2}}|(S_n - n) - W(n)| \geqq \left(\frac{A\log n + z}{5}\right)n^{-\frac{1}{2}}\right\} \leqq Be^{-Cz},$$

$$P\left\{n^{-\frac{1}{2}}Y_{n+1} \geqq \left(\frac{A\log n + z}{5}\right)n^{-\frac{1}{2}}\right\} \leqq Be^{-Cz},$$

$$P\left\{\sup_{1\leqq k\leqq n} n^{-\frac{1}{2}}|S_k - k| \geqq \frac{(A\log n + z)}{5^{\frac{1}{2}}}\right\} \leqq Be^{-Cz},$$

$$P\left\{|\varepsilon_n| \geqq \frac{(A\log n + z)}{5^{\frac{1}{2}}n^{\frac{1}{2}}}\right\} \leqq Be^{-Cz},$$

on choosing A, B, C appropriately. Consequently, (2.6) and the above inequalities imply

$$P\left\{\sup_{1\leqq k\leqq n}\left|\tilde{u}_n\left(\frac{k}{n}\right) - B_n\left(\frac{k}{n}\right)\right| > n^{-\frac{1}{2}}(A\log n + z)\right\} \leqq 6Be^{-Cz},$$

which gives

$$\text{(2.7)} \qquad P\{\sup_{0\leqq y\leqq 1}|\tilde{u}_n(y) - B_n(y)| > n^{-\frac{1}{2}}(A\log n + z)\} \leqq 6Be^{-Cz}.$$

Since (2.4) holds and the above defined $B_n(y)$ is a Brownian bridge for each n, (2.7) also completes the proof of Theorem 1.

3. A strong theorem for the uniform quantile process and the distance of the latter from the general one. Csáki (1977) investigated the limit superior of the sequence

$$\sup_{\varepsilon_n\leqq y\leqq 1-\varepsilon_n}(y(1-y)\log\log n)^{-\frac{1}{2}}|\alpha_n(y)|,$$

and succeeded in evaluating this lim sup for a wide class of sequences $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$. Here we mention only one special case of his many results for later use.

THEOREM D (Csáki (1977)). *With* $\varepsilon_n = dn^{-1}\log\log n$ *and* $d = 0.236\cdots$ *we have*

$$\limsup_{n\to\infty}\sup_{\varepsilon_n\leqq y\leqq 1-\varepsilon_n}(y(1-y)\log\log n)^{-\frac{1}{2}}|\alpha_n(y)| =_{\text{a.s.}} 2.$$

One of the aims of this section is to prove an analogue to Theorem D for the uniform quantile process. The next result is weaker than the corresponding result of Csáki. Applying his method, however, it does not seem to be too difficult to get complete analogues. The one presented herewith suffices for our purposes in the sequel.

THEOREM 2. *With* $\delta_n = 25n^{-1}\log\log n$ *we have*

$$\limsup_{n\to\infty} \sup_{\delta_n \leq y \leq 1-\delta_n} (y(1-y)\log\log n)^{-\frac{1}{2}}|u_n(y)| \leq 4 \quad \text{a.s.}$$

PROOF. Let

$$x_1 = x_1(y) = y - 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}},$$
$$x_2 = x_2(y) = y + 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}.$$

Then for $n \geq 3$

$$(3.1) \qquad \varepsilon_n \leq x_1 < x_2 \leq 1 - \varepsilon_n, \qquad \text{provided} \quad \delta_n \leq y \leq 1 - \delta_n,$$

where again $\varepsilon_n = dn^{-1}\log\log n$ ($d = 0.236\cdots$). In order to see (3.1) holds for $n \geq 3$ we note that

$$x_1 - \varepsilon_n \geq \left(\frac{y}{25} - \varepsilon_n\right) + (\tfrac{24}{25}y^{\frac{1}{2}} - 4(n^{-1}\log\log n)^{\frac{1}{2}})y^{\frac{1}{2}} \geq 0,$$

and similarly, $1 - x_2 - \varepsilon_n \geq 0$. Hence for x_1 as defined at the beginning of this proof and n large, Theorem D gives

$$(3.1^*) \qquad \begin{aligned} nF_n(x_1) &\leq nx_1 + 2(x_1(1-x_1)n\log\log n)^{\frac{1}{2}} \\ &= ny - 4(y(1-y)n\log\log n)^{\frac{1}{2}} + 2(x_1(1-x_1)n\log\log n)^{\frac{1}{2}} \\ &\leq ny, \end{aligned}$$

where the last inequality follows from the inequality $x_1(1-x_1) \leq 4y(1-y)$. The latter, in turn, is true since $x_1 < y$ and $(1-x_1)/(1-y) = 1 + 4(y/(1-y)n^{-1}\log\log n)^{\frac{1}{2}} \leq 1 + (\frac{1}{25})^{\frac{1}{2}} < 4$. Similarly we have $nF_n(x_2) \geq ny$; that is to say we now have $nF_n(x_1) \leq ny \leq nF_n(x_2)$. Hence, for n large, $x_1 \leq U_n(y) \leq x_2$ whenever $\delta_n \leq y \leq 1 - \delta_n$, since $F_n(\cdot)$ is monotone nondecreasing. This, in turn, is the statement of Theorem 2.

In connection with the proof of Theorem 2 we note that when applying Theorem D above, we have treated its "$\limsup_{n\to\infty}$" as if it were a "$\sup_n$," and this, of course, is incorrect. Indeed, in both lines of (3.1*) above the constant 2 should be replaced by the random $2 + o(1)$. In order to avoid writing lots of $o(1)$'s later on, a similar convention will be used in the proof of Theorem 3 whenever using Theorem 2 there the way we have applied Theorem D above.

Now we are in the position to estimate the distance between $q_n(y)$ and $u_n(y)$ when they are respectively defined in terms of $X_{k:n}$ and $U_{k:n} = F(X_{k:n})$. We have

THEOREM 3. *Let* $X_1, X_2, \cdots$ *be* i.i.d. rv *with a continuous distribution function* F *which is also twice differentiable on* (a, b), *where* $-\infty \leq a = \sup\{x : F(x) = 0\}$, $+\infty \geq b = \inf\{x : F(x) = 1\}$ *and* $F' = f \neq 0$ *on* (a, b). *Let the quantile process* $q_n(y)$ resp. $u_n(y)$ *be defined in terms of* $X_{k:n}$ resp. $U_{k:n} = F(X_{k:n})$. *Assume that for some* $\gamma > 0$,

$$(3.2) \qquad \sup_{a<x<b} F(x)(1-F(x))\left|\frac{f'(x)}{f^2(x)}\right| \leq \gamma.$$

Then, with δ_n as in Theorem 2,

$$\limsup_{n\to\infty} \frac{n^{\frac{1}{2}}}{\log\log n} \sup_{\delta_n \leqq y \leqq 1-\delta_n} |f(F^{-1}(y))q_n(y) - u_n(y)| \leqq K \quad \text{a.s.}, \tag{3.3}$$

where $K = 40\gamma 10^\gamma$.

If, in addition to (3.2), *we also assume that f is*

(3.4) *nondecreasing* (*nonincreasing*) *on an interval to the right of* a (*to the left of* b),

then

$$\begin{aligned} &\sup_{0<y<1} |f(F^{-1}(y))q_n(y) - u_n(y)| \\ &\quad =_{\text{a.s.}} O(n^{-\frac{1}{2}} \log\log n) && \text{if } \gamma < 1 \\ &\quad =_{\text{a.s.}} O(n^{-\frac{1}{2}}(\log\log n)^2) && \text{if } \gamma = 1 \\ &\quad =_{\text{a.s.}} O(n^{-\frac{1}{2}}(\log\log n)^\gamma(\log n)^{(1+\varepsilon)(\gamma-1)}) && \text{if } \gamma > 1, \end{aligned} \tag{3.5}$$

where $\varepsilon > 0$ is arbitrary. The respective constants of the $=_{\text{a.s.}} O(\cdot)$ of (3.5) *may be taken to be*: $(45 \vee 25(2^\gamma/(1-\gamma)))2 + K$ *if* $\gamma < 1$, 102 *if* $\gamma = 1$ *and* $(45 \vee (2^\gamma/(\gamma - 1))25^\gamma)2$ *if* $\gamma > 1$.

The following lemma is going to be useful in the sequel.

LEMMA 1. *Under condition* (3.2) *of Theorem* 3 *we have*

$$\frac{f(F^{-1}(y_1))}{f(F^{-1}(y_2))} \leqq \left\{\frac{y_1 \vee y_2}{y_1 \wedge y_2} \cdot \frac{1-(y_1 \wedge y_2)}{1-(y_1 \vee y_2)}\right\}^\gamma. \tag{3.6}$$

for every pair $y_1, y_2 \in (0, 1)$ *and with γ as in* (3.2).

PROOF OF LEMMA 1. (3.2) implies

$$\left|\frac{d}{dy} \log f(F^{-1}(y))\right| \leqq \gamma(y(1-y))^{-1} = \gamma \frac{d}{dy} \log \frac{y}{1-y}.$$

Whence, if $y_1 > y_2$, then

$$\log \frac{f(F^{-1}(y_1))}{f(F^{-1}(y_2))} \leqq \gamma \log \frac{y_1}{1-y_1} - \gamma \log \frac{y_2}{1-y_2} = \gamma \log \frac{y_1}{y_2} \frac{1-y_2}{1-y_1}$$

and, if $y_1 < y_2$, then

$$\log \frac{f(F^{-1}(y_1))}{f(F^{-1}(y_2))} \leqq \gamma \log \frac{y_2}{1-y_2} - \gamma \log \frac{y_1}{1-y_1} = \gamma \log \frac{y_2}{y_1} \frac{1-y_1}{1-y_2}.$$

Hence (3.6) is proved.

PROOF OF THEOREM 3. For $(k-1)/n < y \leqq k/n$,

$$\begin{aligned} f(F^{-1}(y))q_n(y) &= n^{\frac{1}{2}} f(F^{-1}(y))(F^{-1}(U_{k:n}) - F^{-1}(y)) \\ &= n^{\frac{1}{2}} f(F^{-1}(y))(F^{-1}(y + n^{-\frac{1}{2}}u_n(y)) - F^{-1}(y)) \\ &= u_n(y) - \tfrac{1}{2} n^{-\frac{1}{2}} u_n^2(y) f(F^{-1}(y)) \frac{f'(F^{-1}(\xi))}{f^3(F^{-1}(\xi))}, \end{aligned} \tag{3.7}$$

where ξ is between y and $U_{k:n} = y + n^{-\frac{1}{2}}u_n(y)$, i.e., $|\xi - y| \leqq n^{-\frac{1}{2}}|u_n(y)|$. Hence

$$(3.8) \qquad |f(F^{-1}(y))q_n(y) - u_n(y)| \leqq \tfrac{1}{2}n^{-\frac{1}{2}}u_n^2(y)f(F^{-1}(y))\frac{|f'(F^{-1}(\xi))|}{f^3(F^{-1}(\xi))}.$$

Theorem 2 implies that, uniformly for $\delta_n \leqq y \leqq 1 - \delta_n$, the right-hand side of the above inequality is majorized by

$$(3.9) \qquad 8n^{-\frac{1}{2}}(\log\log n)y(1-y)f(F^{-1}(y))\frac{|f'(F^{-1}(\xi))|}{f^3(F^{-1}(\xi))}$$
$$= 8n^{-\frac{1}{2}}(\log\log n)\left[\frac{y(1-y)}{\xi(1-\xi)}\right]\left[\xi(1-\xi)\frac{|f'(F^{-1}(\xi))|}{f^2(F^{-1}(\xi))}\right]\left[\frac{f(F^{-1}(y))}{f(F^{-1}(\xi))}\right]$$

with $|\xi - y| \leqq 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}$.

First we estimate $y(1-y)/\xi(1-\xi)$. Since $\xi > y - 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}$ and $y \geqq \delta_n$, we have

$$\frac{y}{\xi} \leqq 1 + \frac{4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}}{y - 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}}$$
$$= 1 + \frac{4(y^{-1}(1-y)n^{-1}\log\log n)^{\frac{1}{2}}}{1 - 4(y^{-1}(1-y)n^{-1}\log\log n)^{\frac{1}{2}}} \leqq 1 + \frac{\frac{4}{5}}{1-\frac{4}{5}} = 5.$$

Now applying the inequality $\xi < y + 4(y(1-y)n^{-1}\log\log n)^{\frac{1}{2}}$, where $y \leqq 1 - \delta_n$, a similar computation yields that $(1-y)/(1-\xi) \leqq 5$. Hence the first bracketed term of the right-hand side of (3.9) is bounded above by 5, while the second bracketed term of that is assumed to be bounded by γ (cf. (3.2)). Now we observe that Lemma 1 reduces the problem of estimating $f(F^{-1}(y))/f(F^{-1}(\xi))$ (the third bracketed term of the right-hand side of (3.9)) to that of estimating $(\xi(1-y)/y(1-\xi)) + y(1-\xi)/\xi(1-y)$; whence the former is majorized by 10^γ. From these statements and from (3.9) it follows then that the left-hand side of (3.8) is bounded above by $8\gamma 5 \cdot 10^\gamma n^{-\frac{1}{2}}(\log\log n)$ and (3.3), the first statement of Theorem 3, follows.

In order to prove (3.5), it suffices to show that

$$\sup_{0<y\leqq\delta_n} |f(F^{-1}(y)q_n(y) - u_n(y)| \qquad \text{and}$$
$$\sup_{1-\delta_n\leqq y<1} |f(F^{-1}(y))q_n(y) - u_n(y)|$$

are $=_{\text{a.s.}} O(\cdot)$ as indicated on the right-hand side of (3.5). We demonstrate this only for the first one of these sups since, for the second one, a similar argument holds. First of all we note that

$$(3.10) \qquad \sup_{0\leqq y\leqq\delta_n} |u_n(y)| \leqq 45n^{-\frac{1}{2}}\log\log n \quad \text{a.s.},$$

and the proof of (3.10) is as follows: for $0 \leqq y \leqq \delta_n$

$$(3.11) \qquad |u_n(y)| = n^{\frac{1}{2}}|U_n(y) - y| \leqq n^{\frac{1}{2}}y \leqq 25n^{-\frac{1}{2}}\log\log n,$$

whenever $y \geqq U_n(y)$, and

$$|u_n(y)| = n^{\frac{1}{2}}|U_n(y) - y| \leqq n^{\frac{1}{2}}U_n(y) \leqq n^{\frac{1}{2}}U_{[\delta_n:n]}$$

whenever $y \leqq U_n(y)$. In the latter case we consider

$$n^{\frac{1}{2}}(U_{[\delta_n:n]} - \delta_n) + n^{\frac{1}{2}}\delta_n \leqq 4(\delta_n \log\log n)^{\frac{1}{2}} + 25n^{-\frac{1}{4}} \log\log n = 45n^{-\frac{1}{4}} \log\log n \quad \text{a.s.}, \tag{3.12}$$

where the above a.s. inequality follows from Theorem 2. Now (3.11) and (3.12) combined imply (3.10).

Restricting attention then to the region $0 < y \leqq \delta_n$, we assume that $f(F^{-1}(y))$ is nondecreasing on an interval to the right of a (cf. (3.4)). Let $(k-1)/n < y \leqq k/n$. If $U_{k:n} \geqq y$,

$$|f(F^{-1}(y))q_n(y)| = n^{\frac{1}{2}} \int_y^{U_{k:n}} \frac{f(F^{-1}(y))}{f(F^{-1}(u))}\, du \leqq u_n(y), \tag{3.13}$$

where the inequality on the right-hand side results from the assumption that $f(F^{-1}(y))$ is nondecreasing on an interval to the right of a. If $U_{k:n} < y$

$$\begin{aligned} |f(F^{-1}(y))q_n(y)| &= n^{\frac{1}{2}} \int_{U_{k:n}}^{y} \frac{f(F^{-1}(y))}{f(F^{-1}(u))}\, du \\ &\leqq n^{\frac{1}{2}} \int_{U_{k:n}}^{y} \left(\frac{y(1-u)}{u(1-y)}\right)^{\gamma} du, \quad \text{by (3.6)} \\ &\leqq 2^{\gamma} n^{\frac{1}{2}} \int_{U_{k:n}}^{y} \left(\frac{y}{u}\right)^{\gamma} du \\ &\leqq \frac{2^{\gamma}}{1-\gamma} n^{\frac{1}{2}} y \qquad \text{if } \gamma < 1 \\ &\leqq \frac{2^{\gamma}}{\gamma - 1} n^{\frac{1}{2}} y^{\gamma} U_{k:n}^{-(\gamma-1)} \qquad \text{if } \gamma > 1 \\ &\leqq 2n^{\frac{1}{2}} y \log \frac{y}{U_{k:n}} \qquad \text{if } \gamma = 1 . \end{aligned} \tag{3.14}$$

Hence (3.13) (with the help of (3.10)) and (3.14) (via $0 < y \leqq \delta_n$ and in view of $\liminf U_{1:n} \cdot n(\log n)^{1+\varepsilon} = \infty$ for every $\varepsilon > 0$) together imply (3.5). This also completes the proof of Theorem 3.

REMARK 4. Using Theorems 2 and 3, a Theorem 2-type analogue of Theorem D could be proved also for the general quantile process $f(F^{-1}(y))q_n(y)$. It would be more desirable, however, first to produce complete analogues for the uniform quantile process $u_n(y)$ à la Csáki (1977) and then to use these exact analogues, instead of our Theorem 2, in combination with Theorem 3 to prove the same complete Csáki-type analogues for the general quantile process $f(F^{-1}(y))q_n(y)$. That is to say Theorem 3 may be viewed and can be used as a strong invariance theorem for studying the problem of what kind of a.s., in-probability and in-distribution properties of $u_n(y)$ should be inherited by $f(F^{-1}(y))q_n(y)$. For example, it follows from (1.5) and (3.5) that $f(F^{-1}(y))q_n(y) \to_D B(y)$, given the conditions (3.2) and (3.4). Some further examples are given in the next two sections.

Random notes re: the proof of Theorem 3. The Associate Editor has pointed out that γ in (3.5) (but not in the constants in (3.3) and (3.5)) can be replaced by, for instance,

$$\gamma' = \sup_{x\in(a,a')\cup(b',b)} F(x)(1 - F(x)) \left|\frac{f'(x)}{f^2(x)}\right|$$

for any $a < a'$, $b' < b$. This is so because the second part of the proof (the proof of 3.5)) involves only the tails (cf. (3.4)). An advantage of doing this is that γ' may be clearly smaller than γ, and then the first and best $O(\cdot)$ rate of (3.5) would prevail longer (typically, though, one would think that $\gamma' = \gamma$). The same Associate Editor has noted that one can also replace γ by

$$\gamma'' = \max\left\{\lim_{x\downarrow a} \sup F(x)(1-F(x))\left|\frac{f'(x)}{f^2(x)}\right|, \lim_{x\uparrow b} \sup F(x)(1-F(x))\left|\frac{f'(x)}{f^2(x)}\right|\right\},$$

noting that then, of course (3.5) is no longer correct for $\gamma'' = 1$ if $\gamma'' < \gamma'$. That is to say in the latter case the third $O(\cdot)$ rate of (3.5) would prevail.

4. An extension of Kiefer's result on the distance between the empirical and quantile processes and a law of iterated logarithm for the latter. Extending a result of Bahadur (1966), Kiefer (1970) proved the following:

THEOREM E (Kiefer (1970)). *Let* $X_1, X_2, \cdots$ *be* i.i.d. rv *with a twice differentiable distribution function* F *on the unit interval. If* $\inf_{0\le x\le 1} f(x) > 0$ *and* $\sup_{0\le x\le 1} f'(x) < \infty$, *then*

$$\limsup_{n\to\infty} \frac{n^{\frac{3}{4}}}{(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}} \times \sup_{0\le y\le 1} |(F_n(F^{-1}(y)) - y) - (F^{-1}(y) - Q_n(y))f(F^{-1}(y))| =_{\text{a.s.}} 2^{-\frac{1}{4}}. \tag{4.1}$$

Given this theorem and Theorem 3 we immediately get the following extension of the former.

THEOREM 4. $X_1, X_2, \cdots$ *be* i.i.d. rv *with a continuous distribution function* F *which is also twice differentiable on* (a, b), *where* $a = \sup\{x : F(x) = 0\}$, $b = \inf\{x : F(x) = 1\}$ *and* $F' = f \neq 0$ *on* (a, b). *Assume that* F *also satisfies conditions* (3.2) *and* (3.4) *of Theorem* 3. *Then the statement of* (4.1) *is still true.*

REMARK 5. We emphasized here that the conditions (3.2) and (3.4) of Theorem 3 together are much weaker than those of Theorem E. Especially it is not assumed here that a and b are necessarily finite.

PROOF. Let $k_n = n^{\frac{3}{4}}(\log n)^{-\frac{1}{2}}(\log\log n)^{-\frac{1}{4}}$ and $R_n = \sup_{0<y<1} (F_n(F^{-1}(y)) - y) - (F^{-1}(y) - Q_n(y))f(F^{-1}(y))|$. Let $U_{k:n} = F(X_{k:n})$ and define $U_n(y)$ in terms of these uniform order statistics. Consider

$$\begin{aligned}
&\sup_{0<y<1} |(F_n(F^{-1}(y)) - y) - (y - U_n(y))|k_n \\
&\quad - \sup_{0<y<1} |(y - U_n(y)) - (F^{-1}(y) - Q_n(y))f(F^{-1}(y))|k_n \\
&\qquad \le k_n R_n \le \sup_{0<y<1} |(F_n(F^{-1}(y)) - y) - (y - U_n(y))|k_n \\
&\qquad\qquad + \sup_{0<y<1}|(y - U_n(y)) - (F^{-1}(y) - Q_n(y))f(F^{-1}(y))|k_n .
\end{aligned}$$

Let $O(g_1(n))$, $O(g_2(n))$ and $O(g_3(n))$ stand for the respective $O(\cdot)$ terms on the right-hand side of (3.5). Taking lim sup as $n \to \infty$ on the above inequalities we get

$$2^{-\frac{1}{4}} - n^{-\frac{1}{2}}O(g_r(n))k_n \leqq \limsup_{n\to\infty} k_n R_n \leqq 2^{-\frac{1}{4}} + n^{-\frac{1}{2}}O(g_r(n))k_n ,$$

since Kiefer's result (Theorem E) holds for $\sup_{0<y<1} |(F_n(F^{-1}(y)) - y) - (y - U_n(y))|k_n$, and (3.5) of Theorem 3 holds. This, in turn, also completes the proof of Theorem 4.

Consider now the sequence

$$\eta_n(y) = \frac{n(F_n(F^{-1}(y)) - y)}{(2n \log\log n)^{\frac{1}{2}}} ,$$

where F is a continuous distribution function. Let $C = C(0, 1)$ be the space of continuous real valued functions endowed with the supremum norm. Let $K \subset C$ be the set of absolutely continuous (with respect to Lebesgue measure) functions for which $f(0) = f(1) = 0$ and $\int_0^1 (f'(y))^2\, dy \leqq 1$. Finkelstein (1971) proved the following Strassen-type theorem for the above defined sequence $\eta_n(y)$:

THEOREM F (Finkelstein (1971)). *The set of the limit points* (*with respect to the* sup *norm*) *of the sequence* $\eta_n(y)$ *is* a.s. K.

We can now combine Theorem F with Theorem 4 and get immediately:

THEOREM 5. *Let* $X_1, X_2, \cdots$ *be* i.i.d. rv *with a continuous distribution function* F *which is also twice differentiable on* (a, b), *where* $a = \sup\{x : F(x) = 0\}$, $b = \inf\{x : F(x) = 1\}$ *and* $F' = f \neq 0$ *on* (a, b). *Assume that* F *also satisfies conditions* (3.2) and (3.4) *of Theorem* 3. *Let*

$$\xi_n(y) = \frac{nf(F^{-1}(y))(Q_n(y) - F^{-1}(y))}{(2n \log\log n)^{\frac{1}{2}}} .$$

Then the set of the limit points (*with respect to the* sup *norm*) *of the sequence* $\xi_n(y)$ *is* a.s. K.

Let $D_n = \sup_{-\infty<x<+\infty} |F_n(x) - F(x)|$, where $F(\cdot)$ is a continuous distribution function. Recently Mogul'skiĭ (1977) proved

$$\liminf_{n\to\infty} D_n(n \log\log n)^{\frac{1}{2}} =_{\text{a.s.}} \pi(8^{-\frac{1}{2}}) . \tag{4.2}$$

Earlier Kiefer (1970) proved that, under the conditions of Theorem E, we have

$$\lim_{n\to\infty} P\{n^{\frac{3}{4}}(\log n)^{-\frac{1}{2}}R_n > t\} = 2 \sum_{m=1}^{\infty} (-1)^{m+1}e^{-2m^2t^4} , \qquad t > 0 , \tag{4.3}$$

with R_n as in the proof of Theorem 4.

(4.3) states that, under the conditions of Theorem E, $n^{\frac{3}{4}}(\log n)^{-\frac{1}{2}}R_n$ has the same limiting distribution function as $(n^{\frac{1}{2}}D_n)^{\frac{1}{2}}$. Indeed, Kiefer (1970) proved (4.3) via the more fundamental

THEOREM G (Kiefer (1970)). *Under the conditions of Theorem E, as* $n \to \infty$,

$$n^{\frac{3}{4}}R_n/(n^{\frac{1}{2}}D_n \log n)^{\frac{1}{2}} \to_P 1 . \tag{4.4}$$

The latter theorem implies (4.3) at once. Kiefer (1970) also noted that (4.4) was actually true with probability one.

A combination of (4.2), (4.4) and (3.5) of Theorem 3 now yields that (4.4), and hence also (4.3), are also true under the conditions (3.2) and (3.4) of Theorem 3, which together are weaker than those Theorem E. Namely we have

THEOREM 4*. *Let* $X_1, X_2, \cdots$ *be* i.i.d. rv *with a continuous distribution function* F *which is also twice differentiable on* (a, b), *where* $-\infty \leqq a = \sup\{x: F(x) = 0\}$, $+\infty \geqq b = \inf\{x: F(x) = 1\}$ *and* $F' = f \neq 0$ *on* (a, b). *Assume that* F *also satisfies conditions* (3.2) *and* (3.4) *of Theorem* 3. *Then the statement of* (4.4), *and hence also that of* (4.3), *are still true.*

5. Approximation of the general quantile process by Brownian bridges and a Kiefer process. Our main result in this section is an analogue of Theorem B for the general quantile process $q_n(y)$:

THEOREM 6. *Let* $X_1, X_2, \cdots$ *be* i.i.d. rv *with a continuous distribution function* F *which is also twice differentiable on* (a, b), *where* $a = \sup\{x: F(x) = 0\}$, $b = \inf\{x: F(x) = 1\}$ *and* $F' = f \neq 0$ *on* (a, b). *One can then define a Brownian bridge* $\{B_n(y); 0 \leqq y \leqq 1\}$ *for each n and a Kiefer process* $\{K(y, t); 0 \leqq y \leqq 1, 0 \leqq t\}$ *such that if condition* (3.2) *of Theorem* 3 *is assumed then*

$$\sup_{\delta_n \leqq y \leqq 1-\delta_n} |f(F^{-1}(y))q_n(y) - B_n(y)| =_{\text{a.s.}} O(n^{-\frac{1}{2}} \log n) \tag{5.1}$$

and

$$\sup_{\delta_n \leqq y \leqq 1-\delta_n} |n^{\frac{1}{2}}f(F^{-1}(y))q_n(y) - K(y, n)| =_{\text{a.s.}} O((n \log\log n)^{\frac{1}{4}}(\log n)^{\frac{1}{2}})\,, \tag{5.2}$$

where δ_n *is as in Theorem* 2.

If, in addition to (3.2), *condition* (3.4) *of Theorem* 3 *is also assumed, then*

$$\begin{aligned}\sup_{0<y<1} &|f(F^{-1}(y))q_n(y) - B_n(y)| \\ &=_{\text{a.s.}} O(n^{-\frac{1}{2}} \log n) && \textit{if} \quad \gamma < 2 \\ &=_{\text{a.s.}} O(n^{-\frac{1}{2}}(\log\log n)^{\gamma}(\log n)^{(1+\varepsilon)(\gamma-1)}) && \textit{if} \quad \gamma \geqq 2\,,\end{aligned} \tag{5.3}$$

where γ *is as in* (3.2) *and* $\varepsilon > 0$ *is arbitrary*; *also*

$$\sup_{0<y<1} |n^{\frac{1}{2}}f(F^{-1}(y))q_n(y) - K(y, n)| =_{\text{a.s.}} O((n \log\log n)^{\frac{1}{4}}(\log n)^{\frac{1}{2}})\,. \tag{5.4}$$

PROOF. Let $U_{k:n} = F(X_{k:n})$ and define $u_n(y)$ in terms of these uniform order statistics. Let $B_n(y)$ and $K(y, t)$ be as in Theorem B. Then Theorem B holds for the thus defined $u_n(y)$ and combining it with Theorem 3 we get Theorem 6. The $=_{\text{a.s.}} O(n^{-\frac{1}{2}} \log n)$ rate of (5.3) for $\gamma < 2$ holds because of the first two $=_{\text{a.s.}} O(\cdot)$ rates of (3.5), and by taking $\varepsilon < 1/(\gamma - 1) - 1$ (if $1 < \gamma < 2$) in the $=_{\text{a.s.}} O(n^{-\frac{1}{2}}(\log\log n)^{\gamma}(\log n)^{(1+\varepsilon)(\gamma-1)})$ rate of (3.5).

REMARK 6. Theorem 5 could be also proved via first proving a similar statement for a Kiefer process and then using (5.4). Doing things this way, Theorem F would follow from Theorem 4 or from (1.4) of Theorem A.

REMARK 7. The nonuniform quantile process was also studied by Shorack

(1972a, 1972b). Under somewhat different conditions than ours he proves a number of results and the sharpest one of them reads as follows (Corollary 1, 1972b):

$$\sup_{n^{-1}\leq y\leq 1-n^{-1}} \frac{|f(F^{-1}(y))q_n(y) - B_n(y)|}{q(y)} = o(1)$$

for certain functions q.

Acknowledgments. The authors wish to thank an associate editor and the referee for pointing out a number of oversights in the original manuscript. In addition to this, Lemma 1 and the assumption (3.4) (which, with the help of our condition (3.2), replace some earlier and quite complicated sets of additional conditions) are due to the associate editor who, with the referee, had handled our paper with much insight and made many useful comments and suggestions. Indeed, their remarks have inspired us to formulate and prove our key theorems (Theorems 2 and 3), which were not even a part of our original manuscript. Our sincere thanks are due to them and recorded here with much appreciation.

REFERENCES

[1] BAHADUR, R. R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

[2] CSÁKI, E. (1977). The law of the iterated logarithm for normalized empirical distribution function. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **38** 147–167.

[3] CSÖRGŐ, M. and RÉVÉSZ, P. (1975). Some notes on the empirical distribution function and the quantile process. *Limit Theorems of Probability Theory* **11** 59–71 (Colloquia Mathematica Societatis János Bolyai, Keszthely, Hungary 1974). North Holland, Amsterdam.

[4] FINKELSTEIN, H. (1971). The law of the iterated logarithm for empirical distributions. *Ann. Math. Statist.* **42** 605–615.

[5] KIEFER, J. (1970). Deviations between the sample quantile process and the sample DF. *In Non-parametric Techniques in Statistical Inference* (Madan Lal Puri, ed.) 299–319. Cambridge Univ. Press.

[6] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent *RV*'s and the sample *DF*. I. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **32** 111–131.

[7] MOGUL'SKIĬ, A. A. (1977). Laws of iterated logarithm on function spaces (in Russian). Abstracts of Communications **2** 44–47, Second Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius—1977.

[8] SHORACK, G. R. (1972a). Functions of order statistics. *Ann. Math. Statist.* **43** 412–427.

[9] SHORACK, G. R. (1972b). Convergence of quantile and spacings processes with applications. *Ann. Math. Statist.* **43** 1400–1411.

DEPARTMENT OF MATHEMATICS
CARLETON UNIVERSITY
COLONEL BY DRIVE
OTTAWA K1S 5B6
CANADA

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
H-1053 BUDAPEST
REÁLTANODA U. 13–15
HUNGARY