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ON INVARIANT TESTS OF UNIFORMITY FOR DIRECTIONS AND ORIENTATIONS

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The very general results of Beran and Giné on invariant tests of uniformity are applied to S_p, the surface of the unit hypersphere, and H_p, the surface of the unit hypersphere with antipodes identified, to give a class of invariant tests of uniformity for signed and unsigned directional data in $(p+1)$-dimensions. The $(p+1)$-dimensional analogues of the test statistics due to Rayleigh, Bingham, Ajne, and Giné are constructed as the simplest examples, and corresponding methods are derived for particular orientation statistics as examples on H_3.

1. Introduction. There is a considerable body of literature relating to the statistical analysis of both signed and unsigned directions in the plane (points on the circle S_1, and circle with antipodes identified, H_1) and in three dimensions (points on the sphere, S_2, and sphere with antipodes identified, H_2). An exhaustive review is given by Mardia (1972). Most of the statistical theory of the analysis of directional data is parametric, although a certain amount of nonparametric methodology exists (e.g., Watson (1961), Brunk (1962)). Only Beran (1968) and Giné (1975) have developed invariant tests of uniformity for directional data in three dimensions, and there is currently no literature on the general case, $(p+1)$-dimensions. The results presented here may be regarded as applications to S_p of those of Giné (1975).

Consideration of the spectral decomposition of homogeneous random processes (Yaglom (1961)) on S_p, the surface of the unit $(p+1)$-sphere, leads in Section 2 to explicit forms for Beran's class of tests of uniformity (Beran 1968) on S_p and H_p. Giné (1975) obtained these results only for $p \leqq 2$. The null distributions of a number of test statistics on S_p and H_p are obtained in Section 3. One readily apparent application of the results of Section 2 is described in Section 4. The results of Deltheil (1926) (see also Miles (1965)) on the invariant measure for 3×3 orthogonal matrices are used to derive a class of invariant tests of uniformity for 3×3 or equivalently 3×2 orientation statistics (Downs 1972). Approximations to some sampling distributions are calculated in Section 5.

2. Beran's tests on the hypersphere. Given a random sample of $(p+1)$-vectors $\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_n$ on S_p, we consider the statistic

$$T_{p,n}(\{a_q\}) = (nV)^{-1} \int_{\mathbf{x}'\mathbf{x}=1} \left[\sum_{i=1}^{n} (f(\mathbf{x}'\boldsymbol{\sigma}_i) - 1)\right]^2 d\mathbf{x} \tag{2.1}$$

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where

$$V = \int_{\mathbf{x}'\mathbf{x}=1} 1\, d\mathbf{x} = 2\pi^{1+\alpha}/\Gamma(\alpha+1)\,,$$

$$f(z) = 1 + \sum_{q=1}^{\infty} (1 + q/\alpha) a_q C_q^{\alpha}(z)\,, \qquad \alpha = \tfrac{1}{2}(p-1)\,,$$

$$(2.2) \qquad 0 < \sum_{q=1}^{\infty} a_q^2 \nu(p, q) < \infty\,; \qquad \text{where } \nu(p,q) = \binom{p+q-2}{p-1} + \binom{p+q-1}{p-1}\,,$$

$$C_q^{\alpha}(z) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(q+2\alpha)(1-z^2)^{\frac{1}{2}-\alpha}}{\Gamma(q+\alpha+\frac{1}{2})\Gamma(2\alpha)q!\,(-2)^q} \frac{d^q}{dz^q}[(1-z^2)^{q+\alpha-\frac{1}{2}}]\,,$$

the Gegenbauer polynomial (zonal ultraspherical harmonic) of index α and order q, and

$$a_q = (1+q/\alpha)^{-1}\left(\int_{-1}^{+1} f(z) C_q^{\alpha}(z)(1-z^2)^{\alpha-\frac{1}{2}}\, dz\right)\Big/\left(\int_{-1}^{+1} (C_q^{\alpha}(z))^2 (1-z^2)^{\alpha-\frac{1}{2}}\, dz\right)$$
$$= q!\, 2^{2\alpha-1}(\Gamma(\alpha+1))^2\left(\int_{-1}^{+1} f(z) C_q^{\alpha}(z)(1-z^2)^{\alpha-\frac{1}{2}}\, dz\right)\Big/[\pi\alpha\Gamma(q+2\alpha)].$$

PROPOSITION 2.1.

$$T_{p,n}(\{a_q\}) = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} h(\boldsymbol{\sigma}_i'\boldsymbol{\sigma}_j)$$

where

$$h(z) = \sum_{q=1}^{\infty} (1+q/\alpha) a_q^2 C_q^{\alpha}(z)\,.$$

PROOF. We consider only the case $p \geqq 2$, since this result coincides with that of Beran (1968) when $p = 1$.

We may parameterize S_p as the set of all p-tuples $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$, $0 \leqq \theta_p < 2\pi$, $0 \leqq \theta_j \leqq \pi$ $(1 \leqq j \leqq p-1)$, with surface element $dS_p = U_p(\boldsymbol{\theta})\, d\boldsymbol{\theta}$ where

$$U_p(\boldsymbol{\theta}) = \tfrac{1}{2}\Gamma(\alpha+1) \prod_{j=1}^{p} (\sin\theta_j)^{p-j}/\pi^{\alpha+1} = \prod_{j=1}^{p} u_{p,j}(\theta_j)\,,$$

$$u_{p,p}(\theta) = (2\pi)^{-1} \qquad 0 \leqq \theta < 2\pi\,,$$

and

$$(2.3) \qquad u_{p,j}(\theta) = \Gamma(\alpha - \tfrac{1}{2}(j-3))(\sin\theta)^{p-j}/(\pi^{\frac{1}{2}}\Gamma(\alpha - \tfrac{1}{2}(j-2)))\,,$$
$$0 \leqq \theta \leqq \pi,\ 1 \leqq j \leqq p-1\,.$$

The sample $\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_n$ may be reparameterized as $\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_n$, $\boldsymbol{\theta}_i = (\theta_{i,1}, \cdots, \theta_{i,p})$, where $\sin^2\theta_{i,j} = \sum_{k=j+1}^{p+1} \sigma_{i,k}^2$. We may write $\boldsymbol{\sigma}_i'\boldsymbol{\sigma}_j = \cos\psi_{i,j} = \cos\theta_{i,1}\cos\theta_{j,1} + \sum_{k=1}^{p} (\prod_{l=1}^{k} \sin\theta_{i,l}\sin\theta_{j,l}) \cos\theta_{i,k+1}\cos\theta_{j,k+1}$ where $\theta_{i,p+1} \equiv 0$ for all i, and $\psi_{i,j}$ is the angular separation (great circle distance) between $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$.

Now $T_{p,n}(\{a_q\}) = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} h_{i,j}$ where

$$h_{i,j} = \int_{S_p} (f(\cos\lambda_i) - 1)(f(\cos\lambda_j) - 1) u_p(\boldsymbol{\theta})\, d\boldsymbol{\theta}\,, \qquad \text{and} \qquad \cos\lambda_i = \mathbf{x}'\boldsymbol{\sigma}_i\,.$$

Choosing a new coordinate system with $\boldsymbol{\theta}_i$ transferred to the pole and $\boldsymbol{\theta}_j$ transferred to $(\psi_{i,j}, 0, \cdots, 0)$, we obtain

$$h_{i,j} = \pi^{-1}\alpha \int_0^{\pi} (\sin\theta_1)^{p-1}\left[\sum_{q=1}^{\infty} (1+q/\alpha) a_q C_q^{\alpha}(\cos\theta_1)\right]$$
$$\times \left(\int_0^{\pi} (\sin\theta_2)^{p-2}\left[\sum_{q=1}^{\infty} (1+q/\alpha) a_q C_q^{\alpha}(z)\right] d\theta_2\right) d\theta_1$$

where $z = \cos\lambda_j = \cos\theta_1\cos\psi_{i,j} + \sin\theta_1\cos\theta_2\sin\psi_{i,j}$. Applying the addition formula for Gegenbauer polynomials (Whittaker and Watson (1929), page 335, Example 42) and integrating, we obtain

$$h_{i,j} = h(\cos\psi_{i,j}) = \sum_{q=1}^{\infty} (1+q/\alpha) a_q^2 C_q^{\alpha}(\cos\psi_{i,j})\,,$$

(with convergence assured by 2.2) whence

$$T_{p,n}(\{a_q\}) = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} h(\boldsymbol{\sigma}_i'\boldsymbol{\sigma}_j)\,.$$

PROPOSITION 2.2. *On the null hypothesis of uniformity, $T_{p,n}(\{a_q\})$ is distributed as $\mathscr{L}[\sum_{q=1}^{\infty} a_q^2 K_{\nu(p,q)}]$, where the $K_{\nu(p,q)}$ are independent random variables distributed as $\chi^2_{\nu(p,q)}$.*

PROOF. From Vilenkin (1968, page 468), or equivalently Yaglom (1961, page 600), in the terminology of Giné (1975), an orthonormal basis for E_q, the qth eigenspace of the Laplacian, is

$$\{f_q^{(\mathbf{k})}; \mathbf{k} = (k_2, \cdots, k_{p-2}, \pm k_{p-1}), q \geqq k_2 \geqq \cdots \geqq k_{p-1} \geqq 0\}$$

where $f_q^{(\mathbf{k})}$ is expressible as a product of harmonics on hyperspheres of lower orders. The dimensionality of E_q thus satisfies the relationship

$$\nu(p, q) = \sum_{k=0}^{q} \nu(p-1, k)\,, \quad \text{where} \quad \nu(p, 0) \equiv 1\,, \qquad \text{and}$$
$$\nu(1, q) = 2\,, \quad q > 0\,.$$

Hence $\nu(p, q) = \binom{p+q-2}{p-1} + \binom{p+q-1}{p-1}$.

Since S_p is a two-point homogeneous compact Riemannian manifold, and $\int_{-1}^{+1} u_{p,1}(\theta)[C_q^{\alpha}(\cos\theta)]^2\, d\theta = h_q^{\alpha} = (1 + q/\alpha)^{-1}\binom{q+p-2}{p-2}$, whence $\nu(p, q)/h_q^{\alpha} = (1 + q/\alpha)^2$, the result is an immediate consequence of Giné's Proposition 5.2 (1975, page 1257).

From Giné's Theorem 5.3 (ibid., page 1258), $T_{p,n}(\{\alpha_q\})$ is most powerful invariant except for terms of order $O(\alpha^3)$ against the family of densities

$$f_{\alpha,\mathbf{p}}(\mathbf{x}) = (1 - \alpha) + \alpha f(\mathbf{x}'\mathbf{p}) \quad (\mathbf{p} \text{ unknown})\,.$$

By analogy with Beran's Theorem 3 (1969, page 1200), it may be shown that on the general local alternative (with arbitrary choice of origin)

$$g(\mathbf{x}) = 1 + n^{-\frac{1}{2}} \sum_{q=1}^{\infty} \sum_{\mathbf{k}} b_{q,\mathbf{k}} f_q^{(\mathbf{k})}(\mathbf{x})(\nu(p, q))^{\frac{1}{2}}\,,$$

the asymptotic distribution of $T_{p,n}(\{a_q\})$ is $\mathscr{L}(\sum_{q=1}^{\infty} a_q^2 H_{\nu(p,q)}(b_q^2))$, where $b_q^2 = \sum_{\mathbf{k}} b_{q,\mathbf{k}}^2$, and the $H_\nu(\lambda)$ are independent random variables distributed as noncentral chi-squared.

By analogy with Beran's Theorem 1 (1969, page 1198), it may be shown that on the general distant alternative probability density $g(\mathbf{x})$, if we define

$$b(\mathbf{x}) = \int_{\mathbf{y}'\mathbf{y}=1} (f(\mathbf{y}'\mathbf{x}) - 1) g(\mathbf{y})\, d\mathbf{y}$$

and

$$B(\mathbf{y}_1, \mathbf{y}_2) = \int_{\mathbf{x}'\mathbf{x}=1} (f(\mathbf{y}_1'\mathbf{x}) - 1)(f(\mathbf{y}_2'\mathbf{x}) - 1) g(\mathbf{x})\, d\mathbf{x} - b(\mathbf{y}_1) b(\mathbf{y}_2)\,,$$

then

$$n^{-\frac{1}{2}}[T_{n,p}(\{a_q\}) - n \int_{\mathbf{x}'\mathbf{x}=1} b^2(\mathbf{x})\, d\mathbf{x}]$$

is asymptotically normal with mean 0 and variance $\sigma^2 = 4 \int_{\mathbf{x}'\mathbf{x}=1} \int_{\mathbf{y}'\mathbf{y}=1} B(\mathbf{x}, \mathbf{y}) \times b(\mathbf{x}) b(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$, and the convergence is uniform if $\sigma^2 > 0$.

The imposition of the restriction that $a_{2q+1} = 0$, all $q \geqq 0$, throughout this section gives a corresponding class of test statistics and distributional results on H_p (see also Giné (1975), Corollary 6.2, page 1261).

3. Examples. The simplest possible example on S_p is provided by $h(z) = (1 + 1/\alpha)C_1^\alpha(z)$. Thus $n^{-1}(p+1)\mathbf{r}'\mathbf{r}$, where $\mathbf{r} = \sum_{i=1}^n \boldsymbol{\sigma}_i$ is Rayleigh's resultant, is asymptotically distributed as χ^2_{p+1}.

The simplest example on H_p is the $(p+1)$-dimensional analogue of Bingham's test statistic. Thus, if $h(z) = (1 + 2/\alpha)C_2^\alpha(z)$, then we obtain

$$(2n)^{-1}(p+1)(p+3)\,\text{trace}\,(T^2)\,, \qquad \text{where} \quad T = \textstyle\sum_{i=1}^n (\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i' - (p+1)^{-1}I)\,,$$

asymptotically distributed as $\chi^2_{\frac{1}{2}p(p+3)}$.

The $(p+1)$-dimensional analogue of Ajne's statistic (Mardia (1972), page 191 and page 282) is

$$A_{p,n} = (nV)^{-1} \int_{\mathbf{x}'\mathbf{x}=1} (N(\mathbf{x}) - \tfrac{1}{2}n)^2\, d\mathbf{x}\,,$$

where $N(\mathbf{x})$ is the number of data points $\boldsymbol{\sigma}_i$ such that $\mathbf{x}'\boldsymbol{\sigma}_i \geqq 0$. We may write $A_{p,n} = n/4 - (\pi n)^{-1} \sum\sum_{i<j} \psi_{i,j}$, where $\psi_{i,j} = \cos^{-1}(\boldsymbol{\sigma}_i'\boldsymbol{\sigma}_j)$. Since

$$\int_0^1 C_{2q}^\alpha(x)(1 - x^2)^{\alpha - \frac{1}{2}}\, dx = 0$$

and

$$\int_0^1 C_{2q+1}^\alpha(x)(1 - x^2)^{\alpha-\frac{1}{2}}\, dx = \frac{(-1)^q\Gamma(q + \alpha + 1)}{(2q+1)(q + \alpha + \frac{1}{2})q!\,\Gamma(\alpha)}$$

(Abramowitz and Stegun (1965), page 785, 22.13.2), it follows from the results of Section 2 that the asymptotic null distribution of $A_{p,n}$ is

$$\mathscr{L}\left(\textstyle\sum_{q=1}^\infty a_{2q-1}^2 K_{\nu(p,2q-1)}\right),$$

$$\text{where} \quad a_{2q-1} = \frac{(-1)^{q-1}2^{p-1}\Gamma(\alpha+1)\Gamma(q+\alpha)(2q-2)!}{\pi(q-1)!\,(2q+p-2)!}\,.$$

This result reduces to that of Watson (1968) on S_1, and Beran (1968) on S_2. On S_3 we obtain the asymptotic distribution of $A_{3,n}$ as $\mathscr{L}[\sum_{q=1}^\infty [(2q^2 - \frac{1}{2})\pi]^{-2}K_{4q^2}]$.

Giné's statistic (Giné (1975), page 1262) generalizes to

$$G_{p,n} = \tfrac{1}{2}n - (p/2n)[\Gamma(\alpha + \tfrac{1}{2})/\Gamma(\alpha+1)]^2 \textstyle\sum\sum_{i<j} \sin\psi_{i,j}\,.$$

It may be shown that

$$\int_0^\pi (\tfrac{1}{2} - \tfrac{1}{2}p[\Gamma(\alpha+\tfrac{1}{2})/\Gamma(\alpha+1)]^2 \sin\theta)C_{2q+1}^\alpha(\cos\theta)(\sin\theta)^{p-1}\, d\theta = 0$$

and

$$\int_0^\pi (\tfrac{1}{2} - \tfrac{1}{2}p[\Gamma(\alpha+\tfrac{1}{2})/\Gamma(\alpha+1)]^2 \sin\theta)C_{2q}^\alpha(\cos\theta)(\sin\theta)^{p-1}\, d\theta$$
$$= \frac{p(2q-1)(2q+p-2)!}{2^p(2q)!\,(p-1)(2q+p)}\left[\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(q-\frac{1}{2})}{\Gamma(q+\alpha+\frac{1}{2})\Gamma(\alpha)}\right]^2$$

whence

$$\tfrac{1}{2} - \tfrac{1}{2}p[\Gamma(\alpha+\tfrac{1}{2})/\Gamma(\alpha+1)]^2\sin\theta$$
$$= \sum_{q=1}^\infty \frac{p(2q-1)(4q+p-1)}{(p-1)(2q+p)8\pi}\left[\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(q-\frac{1}{2})}{\Gamma(q+\alpha+\frac{1}{2})}\right]^2 C_{2q}^\alpha(\cos\theta)$$

and $G_{p,n}$ has asymptotic null distribution

$$\mathscr{L}[\sum_{q=1}^{\infty} a_{2q}^2 K_{\nu(p,2q)}], \qquad \text{where} \quad a_{2q}^2 = \frac{p(2q-1)}{8\pi(2q+p)}\left[\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(q-\frac{1}{2})}{\Gamma(q+\alpha+\frac{1}{2})}\right]^2.$$

The asymptotic null distribution of $G_{2,n}$ is incorrectly stated by Giné (1975, page 1263). The correct result is

$$\mathscr{L}\left[\sum_{q=1}^{\infty} \frac{(q-\frac{1}{2})(\Gamma(q-\frac{1}{2}))^2}{4\pi q!\,(q+1)!} K_{4q+1}\right],$$

or in Giné's notation,

$$\mathscr{L}[\sum_{k=1}^{\infty} 2^{-2k-1}(2k-1)^{-1}(k+1)^{-1}((2k-1)!!/k!)^2 H_{2k}],$$

which is compatible with Giné's (6.10) (ibid., page 1262).

On H_3, we obtain

$$G_{3,n} = n/2 - (3\pi/8n)\sum\sum_{i<j} \sin \psi_{i,j},$$

with asymptotic distribution

$$\mathscr{L}\left[\sum_{q=1}^{\infty} \frac{3}{2(2q-1)(2q+1)^2(2q+3)} K_{(2q+1)^2}\right]. \tag{3.1}$$

From the work of Giné (1975) (Theorem 4.4, page 1254), it follows that since all even spectral coefficients a_{2q} of $G_{p,n}$ are nonzero, $G_{p,n}$ is consistent against all alternatives to uniformity on H_p. Also, $A_{p,n}$ is consistent against alternatives on S_p with at least one nonvanishing odd spectral coefficient a_{2q-1}. Clearly any weighted sum of $G_{p,n}$ and $A_{p,n}$ is consistent against all alternatives to uniformity on S_p (e.g., Giné (1975), Proposition 6.3, page 1261, on S_2).

4. Orientation statistics. The results of Section 2 may also be applied to a particular type of constrained multivariate directional data known as orientations (Downs, 1972). An orientation is a rigid configuration of p distinguishable signed directions in m dimensions ($m \geqq p$). Hence we consider the sample space $S_{m,p}(C)$ of all $m \times p$ matrices X such that $X'X = C$, where C is a known $p \times p$ symmetric positive definite matrix specifying the angles between every pair of the p distinguishable directions. The specific application by Downs (1972) to vectorcardiography involves $S_{3,2}(I)$, or equivalently $S_{3,3}(I) = O(3)$, the class of all proper 3×3 orthogonal matrices, since any two orthogonal directions in three dimensions define a third direction uniquely by the right-hand rule.

It may be shown that if $\mathbf{x}$ is a 4-vector uniformly distributed on H_3, then

$$X(\mathbf{x}) = X(-\mathbf{x}) = \begin{pmatrix} 2(x_1^2+x_2^2)-1, & 2(x_2x_3 - x_1x_4), & 2(x_2x_4 + x_1x_3) \\ 2(x_2x_3 + x_1x_4), & 2(x_1^2+x_3^2)-1, & 2(x_3x_4 - x_1x_2) \\ 2(x_2x_4 - x_1x_3), & 2(x_3x_4 + x_1x_2), & 2(x_1^2+x_4^2)-1 \end{pmatrix}$$

is uniformly distributed (has the invariant Haar measure) on $O(3)$ (see Deltheil (1926), Miles (1965)). Since trace $(X(\mathbf{x})X(\mathbf{y})) = 4(\mathbf{x}'\mathbf{y})^2 - 1$, it follows that we may restate the results of Section 2 for $O(3)$ as follows.

Given a random sample $S_1, S_2, \cdots, S_n$ on $O(3)$, Beran's test statistics are of the form

$$W_n(\{a_{2q}\}) = (nV)^{-1} \int_{O(3)} [\sum_{i=1}^n f(\text{trace}(X'S_i)) - 1)]^2 \, dX$$

where $f(z) = 1 + \sum_{q=1}^\infty a_{2q}(2q+1)U_{2q}(\frac{1}{2}(1+z)^{\frac{1}{2}})$, and U_{2q} is a Chebyshev polynomial of the second kind, an even zonal harmonic on the 4-sphere. Also, $W_n(\{a_{2q}\})$ may be expressed as

$$W_n(\{a_{2q}\}) = n^{-1} \sum_{i=1}^n \sum_{j=1}^n h(\text{trace}(S_i'S_j)) ,$$

where $h(z) = \sum_{q=1}^\infty a_{2q}^2(2q+1)U_{2q}(\frac{1}{2}(1+z)^{\frac{1}{2}})$, and has asymptotic null distribution $\mathscr{L}[\sum_{q=1}^\infty a_{2q}^2 K_{(2q+1)^2}]$.

Since $U_2(\frac{1}{2}(1+z)^{\frac{1}{2}}) = z$, it follows that $3n^{-1}\,\text{trace}(R'R)$, where $R = \sum_{i=1}^n S_i$, is asymptotically distributed as $\chi_9{}^2$ on the null hypothesis of uniformity. This result may also be obtained from other considerations (e.g., Downs (1972), page 672). Also, the analogue on H_3 of Giné's statistic, consistent against all alternatives to uniformity, reduces on $O(3)$ to

$$G_{3,n} = n/2 - (3\pi/16n) \sum\sum_{i<j} [\text{trace}(I - S_i'S_j)]^{\frac{1}{2}} ,$$

asymptotically distributed as (3.1) on the null hypothesis of uniformity. Similarly, on $S_{3,3}(C)$ we obtain

$$G_{3,n} = n/2 - (3\pi/16n) \sum\sum_{i<j} [\text{trace}(I - S_i'C^{-1}S_j)]^{\frac{1}{2}} .$$

5. Numerical approximations to sampling distributions. A first-order approximation to the tail area of the distribution (2.4) may be obtained by the method of Blum et al. (1962) (see also Beran (1968), page 194). The asymptotic null hypothesis moment generating function of $T_{p,n}(\{a_q\})$ is

$$M(s) = \prod_{q=1}^\infty (1 - 2sa_q{}^2)^{-\frac{1}{2}\nu(p,q)} .$$

Let a_u denote the (unique) largest a_q. Then $M(s)$ may be approximated in the upper tail by

$$M^*(s) = \rho_T(1 - 2sa_u{}^2)^{-\frac{1}{2}\nu(p,u)} , \qquad \text{where} \quad \rho_T = \prod_{q\neq u} (1 - a_q{}^2/a_u{}^2)^{-\frac{1}{2}\nu(p,q)} ,$$

whence

$$\text{Prob}(T_{p,n}(\{a_q\}) > a_u{}^2x) \doteq \rho_T \, \text{Prob}(\chi^2_{\nu(p,u)} > x) .$$

For the statistics $A_{1,n}$ and $G_{1,n}$, since $\nu(1, q) \equiv 2$, we may invert $M(s)$ directly (Beran (1969), page 1201) to obtain

$$\text{(5.1)} \qquad \lim_{n\to\infty} \text{Prob}(A_{1,n} > x) = \sum_{q=1}^\infty \frac{4(-1)^{q-1}}{\pi(2q-1)} \exp(-\tfrac{1}{2}\pi^2(2q-1)^2x) ,$$

as in Watson (1967), and

$$\text{(5.2)} \qquad \lim_{n\to\infty} \text{Prob}(G_{1,n} > x) = \sum_{q=1}^\infty \frac{16q(-1)^{q-1}}{\pi(4q^2-1)} \exp(-(4q^2-1)x) .$$

I am grateful to a referee for drawing my attention to the work of Hoeffding

1964) which may be used to provide an improved asymptotic expansion for the ail area in the general case when $\nu(p, u) > 2$:

$$\lim_{n\to\infty} \text{Prob}\,(T_{p,n}(\{a_q\}) > a_u^2 x) \doteq \rho_T \textstyle\sum_{k\geq 0} (-1)^k g_k \,\text{Prob}\,(\chi^2_{\nu(p,u)-2k} > x) \tag{5.3}$$

where $g_0 = 1$, $g_1 = \frac{1}{2}\sum_{q\neq u} b_q \nu(p, q)$, $b_q = a_q^2/(a_u^2 - a_q^2)$, $g_2 = \frac{1}{2}(g_1^2 + \frac{1}{2}\sum_{q\neq u} b_q^2 \nu(p, q))$, and g_3, g_4 etc. may be obtained by further differentiation of $\log M(S)/M^*(S)$.

For the statistics $A_{2,n}$, $A_{3,n}$, $G_{2,n}$, $G_{3,n}$, the various infinite products and sums were found by numerical methods to 6 significant figures, and are quoted here to 4 significant figures only. We obtain

$$\lim_{n\to\infty} \text{Prob}\,(A_{2,n} > x/16) \doteq 1.652[\text{Prob}\,(\chi_3^2 > x) - 0.5155\,\text{Prob}\,(\chi_1^2 > x)]\,, \tag{5.4}$$

$$\begin{aligned} &\lim_{n\to\infty} \text{Prob}\,(A_{3,n} > 4x/9\pi^2) \\ &\quad \doteq 2.138[\text{Prob}\,(\chi_4^2 > x) - 0.7893\,\text{Prob}\,(\chi_2^2 > x)] \\ &\quad = (0.4505 + 1.069x)e^{-\frac{1}{2}x}\,, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \lim_{n\to\infty} \text{Prob}(G_{2,n} > x/16) \doteq 4.638[&\text{Prob}(\chi_5^2 > x) - 1.593\,\text{Prob}(\chi_3^2 > x) \\ &+ 1.323\,\text{Prob}\,(\chi_1^2 > x)] \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \text{Prob}\,(G_{3,n} > x/30) \\ &\quad \doteq 19.65[\text{Prob}\,(\chi_9^2 > x) - 3.112\,\text{Prob}\,(\chi_7^2 > x) \\ &\qquad + 4.903\,\text{Prob}\,(\chi_5^2 > x) - 5.217\,\text{Prob}\,(\chi_3^2 > x) \\ &\qquad + 4.216\,\text{Prob}\,(\chi_1^2 > x)]\,. \end{aligned} \tag{5.7}$$

Table 1 contains approximations to some common significance levels obtained from (5.1) to (5.7) inclusive.

TABLE 1

Approximate significance points of Ajne and Giné statistics

Statistic T	ρ_T	a_u^2	Significance level			
			0.100	0.050	0.010	0.001
$A_{1,n}$	1.273	0.1013	0.517	0.655	0.982	1.448
$A_{2,n}$	1.654	0.0625	7.1	8.7	12.2	17.2
$A_{3,n}$	2.138	0.0450	9.3	11.0	14.8	20.0
$G_{1,n}$	1.697	0.1667	0.944	1.175	1.712	2.489
$G_{2,n}$	4.638	0.0625	12.2	14.1	18.1	23.5
$G_{3,n}$	19.65	0.0333	20.5	22.7	27.7	33.0

6. Discussion. Since $C_n^0(z) = T_n(z)$, the Chebyshev polynomial of the first kind, and $C_n^{\frac{1}{2}}(z) = P_n(z)$, the Legendre polynomial, the results of Section 2 reduce to those of Beran ((1968), page 186) on S_1 and Giné ((1975), page 1261) on S_2. The relationship between $O(3)$ and H_3 (Section 4) provides one new application of these results in four dimensions. However, the distributional problems associated with the construction of invariant tests of uniformity based on Sobolev norms for general orientation statistics are difficult. The absence of explicit forms for the zonal spherical functions on the Stiefel manifold is one major difficulty. The corresponding functions on the unoriented Grassmann manifold have been obtained by James and Constantine (1974), and similar techniques are being applied to the Stiefel manifold. The analogues of Propositions 2.1 and 2.2 are rather more complicated.

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REFERENCES

ABRAMOWITZ, M. and STEGUN, I. A. (1965). *Handbook of Mathematical Functions.* Dover, New York.

BERAN, R. J. (1968). Testing for uniformity on a compact homogeneous space. *J. Appl. Probability* **5** 177–195.

BERAN, R. J. (1969). Asymptotic theory of a class of tests for uniformity of a circular distribution. *Ann. Math. Statist.* **40** 1196–1206.

BLUM, J. R., KIEFER, J. and ROSENBLATT, M. (1961). Distribution-free tests of independence based on the sample distribution function. *Ann. Math. Statist.* **32** 485–492.

BRUNK, H. D. (1962). On the range of the difference between hypothetical distribution functions and Pyke's modified empirical distribution function. *Ann. Math. Statist.* **33** 525–532.

DELTHEIL, R. (1926). *Probabilités Géométriques.* Traité du calcul des probabilités et de ses applications. Gauthier-Villars, Paris.

DOWNS, T. D. (1972). Orientation statistics. *Biometrika* **59** 665–676.

GINÉ, E. M. (1975). Invariant test of uniformity on compact Riemannian manifolds based on Sobolev norms. *Ann. Statist.* **3** 1243–1266.

HOEFFDING, W. (1964). On a theorem of V. M. Zolotarev. *Theor. Probability Appl.* **9** 89–91.

JAMES, A. T. and CONSTANTINE, A. G. (1974). Generalized Jacobi polynomials as spherical functions of the Grassmann manifold. *Proc. London Math. Soc.* **29** (Third Series) 174–192.

MARDIA, K. V. (1972). *Statistics of Directional Data.* Academic Press, New York.

MILES, R. E. (1965). On random rotations in R^3. *Biometrika* **52** 636–639.

VILENKIN, N. J. (1968). Special functions and the theory of group representations. American Mathematical Society. (Translated from the Russian by V. N. Singh. *Transl. Math. Monographs* **22**.)

WATSON, G. S. (1961). Goodness-of-fit tests on a circle I. *Biometrika* **48** 109–114.

WATSON, G. S. (1967). Another test for the uniformity of a circular distribution. *Biometrika* **54** 675–677.

WHITTAKER, E. T. and WATSON, G. N. (1929). *A Course of Modern Analysis.* Cambridge Univ. Press.

YAGLOM, A. M. (1961). Second-order homogeneous random fields. *Proc. Fourth Berkeley Symp. Math. Statist. Probability* **2** 593–622.

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