*The Annals of Statistics*
1977, Vol. 5, No. 4, 658–670

# THE RATE OF CONVERGENCE OF SIMPLE LINEAR RANK STATISTICS UNDER HYPOTHESIS AND ALTERNATIVES

BY MARIE HUŠKOVÁ

*Charles University, Prague and Mathematical Center, Amsterdam*

Convergence rates for distributions of simple linear rank statistics are investigated. Both the null hypothesis and near alternatives are considered. The method of proof consists in approximating the characteristic function of the statistic by that of a sum of independent random variables and then applying standard tools.

**1. Introduction.** Let $X_{jN}$, $1 \leqq j \leqq N$, be independent random variables with respective distribution functions $F(x, \theta_{jN})$, where $\theta_{jN}$ are unknown parameters. Consider a general linear rank statistic

$$S_N = \sum_{j=1}^N c_{jN} a_N(R_{jN}), \tag{1.1}$$

where $c_{1N}, \cdots, c_{NN}$ are the regression constants, $a_N(1), \cdots, a_N(N)$ are the scores and $R_{jN}$ is the rank of $X_{jN}$ among $X_{1N}, \cdots, X_{NN}$.

The purpose of the present paper is to investigate the convergence rate of the distribution function of $S_N$ both under the randomness hypothesis as well as "near" alternatives. Throughout the paper, the following assumptions are adopted:

(I) the regression constants $c_{1N}, \cdots, c_{NN}$ fulfill:

$$\sum_{j=1}^N c_{jN} = 0, \qquad \sum_{j=1}^N c_{jN}^2 = 1, \qquad \max_{1 \leqq j \leqq N} |c_{jN}| \leqq \tfrac{1}{2};$$

(II) the scores $a_N(1), \cdots, a_N(N)$ are given in either of the following ways:

$$a_N(j) = \varphi(j/(N+1)), \qquad 1 \leqq j \leqq N, \tag{1.2}$$

$$a_N(j) = E\varphi(U_N^{(j)}), \qquad 1 \leqq j \leqq N, \tag{1.3}$$

with $U_N^{(j)}$ denoting the $j$th order statistic from a rectangular (0, 1) population;

(III) $\varphi$ is a nonconstant function on (0, 1), $\int_0^1 \varphi(u)\,du = 0$, its first derivative $\varphi'$ being absolutely continuous and the second one $\varphi''$ square integrable over (0, 1).

The asymptotic normality of $S_N$ has been established under very general conditions (Hájek (1968), Dupač–Hájek (1969)). Recent research has been focused on the rate of convergence and on the results concerning Edgeworth expansions (see Bickel (1974) for a review). Results concerning Edgeworth expansions deal with more or less special cases up to now: some two-sample rank statistics (Prášková (1974)), one-sample rank statistics (Albers, Bickel, van Zwet (1976)),

Received October 1975; revised December 1976.

*AMS* 1970 *subject classifications.* Primary 60F05; Secondary 62G10.

*Key words and phrases.* Simple linear rank statistics, rate of convergence, distribution free tests, contiguous alternatives.

two-sample rank statistics (Bickel, van Zwet (1973)). Obtaining the Edgeworth expansion for general linear rank statistics remains an open problem.

The convergence rate of $S_N$ was investigated by Jurečková (1973) and Koul (1976). Under assumptions (I), (II),

(III′) $\quad \varphi$ is nonconstant, $\int_0^1 \varphi(u)\,du = 0$, $\varphi'$ bounded on $(0, 1)$,

and (IV), given in Section 2 below, Jurečková found the convergence rate as $\sum_{j=1}^N |c_{jN}|^3 N^\delta$, $\delta > 0$ arbitrary.

It is natural to expect the convergence rate under the null hypothesis is equal to that of $\sum_{j=1}^N c_{jN} Y_{jN}$, with $Y_{jN}$ independent uniformly bounded random variables, i.e., equal to $\sum_{j=1}^N |c_{jN}|^3$ (see Feller (1971)).

In the present paper, this conjecture will be proved, provided that (III′) is replaced by the stronger (III). Under additional assumptions, the convergence rate under alternatives will be proved as $\sum_{j=1}^N (|c_{jN}|^3 + |\theta_{jN}|^3)$.

Our proving method is closely related to the method utilized by Bjerve (1973); cf. also Bickel (1974). The characteristic function of $S_N$ is replaced by the one of the sum $T_N$ of suitable chosen independent random variables. The Berry–Esseen argument is applied to the latter, whereas a Taylor expansion is made use of to estimate the difference of both.

**2. Rate of convergence under hypothesis.** In this section we shall assume that the distribution of $(X_{1N}, \cdots, X_{NN})$ satisfy:

(IV) the distribution function of the vector $(X_{1N}, \cdots, X_{NN})'$ is of the form $\prod_{j=1}^N F(x_j, 0)$ with $F(x, 0)$ continuous.

The main assertion of this section is the following:

THEOREM 2.1. *Let the assumptions* (I—IV) *be satisfied. Then there exists a constant* $A_1$ (*not depending on* $N$) *such that*

$$\sup_x |P(S_N < x(\textstyle\int_0^1 \varphi^2(u)\,du)^{\frac{1}{2}}) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-y^2/2}\,dy| \leqq A_1 \sum_{j=1}^N |c_{jN}|^3, \tag{2.1}$$

*where* $S_N$ *is given by* (1.1).

COROLLARY 2.2. *Consider the two-sample linear rank statistic*

$$S_N = \sum_{j=1}^m a_N(R_{jN}) - \bar{a}_N m,$$

*where* $\bar{a}_N = (1/N) \sum_{j=1}^N a_N(j)$. *Under assumptions* (II—IV) *there exists a constant* $A_1^*$ *such that*

$$\sup_x |P(S_N < x[m(N-m) \textstyle\int_0^1 \varphi^2(u)\,du/N]^{\frac{1}{2}}) - \Phi(x)| \leqq A_1^*[\max(m, N-m)]^{-\frac{1}{2}},$$

*where* $\Phi(x)$ *is the standardized normal distribution function.*

Theorem 2.1 will be implied by several lemmas. In the rest of this section we omit indices $N$ in $c_{jN}$, $R_{jN}$, etc. Denote by

$$T_N = \sum_{j=1}^N c_j \varphi(F(X_j, 0)), \tag{2.2}$$

$$T_N^* = \sum_{j=1}^N c_j \varphi'(F(X_j, 0))(R_j - E(R_j \mid X_j))/(N+1), \tag{2.3}$$

$$\sup_{u\in(0,1)}|\varphi(u)| = D_0\,, \qquad \sup_{u\in(0,1)}|\varphi'(u)| = D_1\,, \tag{2.4}$$
$$\textstyle\int_0^1 \varphi^2(u)\,du = \bar\varphi^2\,, \qquad \int_0^1 \varphi^3(u)\,du = \bar\varphi^3\,.$$

REMARK 2.3. Notice that (I) implies

$$\textstyle\sum_{j=1}^N |c_j^3| \geqq N^{-\frac12}\,, \qquad \log\,(\sum_{j=1}^N |c_j^3|)^{-1} \leqq 2(\max_{1\le j\le N}|c_j^2|)^{-1}\,. \tag{2.5}$$

LEMMA 2.4. *Under assumptions* (I—IV) *there exist constants* $B_1$, $B_2$ *such that*

$$\textstyle P(|\sum_{j=1}^N c_j(a_N(R_j) - \varphi(R_j/(N+1))| > N^{-\frac12}) \leqq B_1 N^{-1}\,, \tag{2.6}$$

$$\textstyle P(|\sum_{j=1}^N c_j\varphi(R_j/(N+1)) - (T_N + T_N^*)| > 3(\bar\varphi^2)^{\frac12}N^{-\frac12}) \leqq B_2 N^{-\frac12}\,, \tag{2.7}$$

*where* $a_N(j) = E\varphi(U_N^{(j)})$.

PROOF. Making use of the Chebyshev inequality, the Taylor expansion for $\varphi$ and the formulas for moments of order statistics we obtain (2.6).

A similar argument leads to (2.7):

$$\begin{aligned}
&\textstyle P(|\sum_{j=1}^N c_j\varphi(R_j/(N+1)) - (T_N + T_N^*)| > 3(\bar\varphi^2)^{\frac12}N^{-\frac12})\\
&\quad\textstyle \leqq N(9\bar\varphi^2)^{-1}E\{\sum_{j=1}^N c_j(R_j/(N+1) - F(X_j,0))\\
&\qquad\textstyle \times \int_0^1[\int_{F(X_j,0)}^{(1-\lambda)R_j/(N+1)+\lambda F(X_j,0)} \varphi''(u)\,du]\,d\lambda\}^2\\
&\quad\textstyle \leqq N(9\bar\varphi^2)^{-1}\int_0^1(\varphi''(u))^2\,du\,\{E[R_1/(N+1) - F(X_1,0)]^4\}^{\frac12}\\
&\quad\textstyle \leqq (9\bar\varphi^2)^{-1}\int_0^1(\varphi''(u))^2\,du\,(32 + 70N^{-1} + 16N^{-2})N^{-\frac12}\,. \qquad \square
\end{aligned}$$

The following lemma (having probably a broader field of applications) will be the main tool of our proof.

LEMMA 2.5. *Let the assumption* (I) *be satisfied and let* $U_1, \cdots, U_N$ *be random variables such that for any permutation* $(j_1, \cdots, j_N)$ *of* $(1, \cdots, N)$

$$\textstyle E\prod_{i=1}^N U_i^{\alpha_i} = E\prod_{i=1}^N U_{j_i}^{\alpha_i}\,, \tag{2.8}$$

*where* $\sum_{i=1}^N \alpha_i = 2k$, $\alpha_i \geqq 0$ *integers. Then, for* $2k \leqq d(\max_{1\le i\le N}|c_i|)^{-1}$,

$$\textstyle E(\sum_{i=1}^N c_i U_i)^{2k} \leqq k^k(4e)^{2k+1}d^{2k}EU_1^{2k}\,. \tag{2.9}$$

PROOF. Using the multinomial expansion we get

$$\begin{aligned}
E(\textstyle\sum_{i=1}^N c_i U_i)^{2k} &= \textstyle\sum_{\alpha=1}^{2k}\sum_{(k_1,\cdots,k_\alpha)\in A(\alpha)} \dfrac{(2k)!}{\prod_{v=1}^\alpha (v!)^{k_v}\prod_{v=1}^\alpha (k_v)!}\\
&\quad\times \textstyle\sum_{(i_1,\cdots,i_{\sum_{v=1}^\alpha k_v})\in B(\sum_{v=1}^\alpha k_v)} E\prod_{j=1}^{k_1}(c_{i_j}U_{i_j})\\
&\quad\times \textstyle\prod_{j=k_1+1}^{k_1+k_2}(c_{i_j}U_{i_j})^2 \cdots \prod_{j=\sum_{v=1}^{\alpha-1}k_v+1}^{\sum_{v=1}^{\alpha}k_v}(c_{i_j}U_{i_j})^\alpha\,,
\end{aligned} \tag{2.10}$$

where $A(\alpha) = \{(k_1, \cdots, k_\alpha);\, k_v \geqq 0$ integer, $\sum_{v=1}^\alpha vk_v = 2k\}$ and $B(\sum_{v=1}^\alpha k_v) = \{(i_1, \cdots, i_{\sum_{v=1}^\alpha k_v});\, 1 \leqq i_j \leqq N$ integer, $i_j \neq i_v$ for $v \neq j\}$.

First, we estimate $EZ_N(k_1, \cdots, k_\alpha)$, where

$$\begin{aligned}
&Z_N(k_1, \cdots, k_\alpha)\\
&\quad= \textstyle\sum_{(i_1,\cdots,i_{\sum_{v=1}^\alpha k_v})\in B(\sum_{v=1}^\alpha k_v)} \prod_{j=1}^{k_1}(c_{i_j}U_{i_j}) \cdots \prod_{j=\sum_{v=1}^{\alpha-1}k_v+1}^{\sum_{v=1}^{\alpha}k_v}(c_{i_j}U_{i_j})^\alpha\,.
\end{aligned}$$

The Jensen inequality and (2.8) imply

$$|EZ_N(k_1, \cdots, k_\alpha)| \leqq EU_1^{2k}|V(k_1, \cdots, k_\alpha)| , \tag{2.11}$$

where

$$V(k_1, \cdots, k_\alpha) = \sum_{(i_1,\cdots,i_{\sum_{v=1}^{\alpha} k_v}) \in B(\sum_{v=1}^{\alpha} k_v)} \prod_{j=1}^{k_1} c_{i_j} \cdots \prod_{j=\sum_{v=1}^{\alpha-1} k_v+1}^{\sum_{v=1}^{\alpha} k_v} (c_{i_j} U_{i_j})^\alpha .$$

By induction on $k_1$ we shall prove that the inequality

$$V(k_1, \cdots, k_\alpha) \leqq 2^{k_1} d^{k_1} (k_1/2)! \, (\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v} \tag{2.12}$$

holds for any integers $(k_1, \cdots, k_\alpha)$, $k_v \geqq 0$, $\sum_{v=1}^{\alpha} vk_v = 2k$. Using assumption (I) we get for $k_1 = 0$ and any nonnegative integers $k_2, \cdots, k_\alpha$,

$$V(0, k_2, \cdots, k_\alpha) = (\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v} .$$

For $k_1 = 1$ we have

$$V(1, k_2, \cdots, k_\alpha) = -\sum_{v=2}^{\alpha-1} k_v V(0, k_2, \cdots, k_v - 1, k_{v+1} + 1, \cdots, k_\alpha) - k_\alpha V(0, k_2, \cdots, k_\alpha - 1, 1)$$

and, then, by assumption (I) and for $2k \leqq d(\max_{1\leqq i\leqq N} |c_i|)^{-1}$

$$|V(1, k_2, \cdots, k_\alpha)| \leqq \sum_{v=2}^{\alpha} k_v (\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v+1} \leqq \tfrac{1}{2} d(\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v} .$$

Assuming that (2.12) is true for $k_1 = r$ we obtain for $k_1 = r + 1$

$$\begin{aligned} |V(r + 1, k_2, \cdots, k_\alpha)| &\leqq |\sum_{v=2}^{\alpha-1} k_v V(r, k_2, \cdots, k_v - 1, k_{v+1} + 1, \cdots, k_\alpha) \\ &\quad + k_\alpha V(r, k_2, \cdots, k_\alpha - 1, 1) + rV(r - 1, k_2 + 1, \cdots, k_\alpha)| \\ &\leqq d^{r+1} 2^{r+1} ((r + 1)/2)! \, (\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v} . \end{aligned}$$

If we use (2.10—2.12) and the polynomial expansion

$$(2k)^{2k} = \sum_{\alpha=1}^{2k} \sum_{(k_1,\cdots,k_\alpha) \in A(\alpha)} \frac{(2k)!}{\prod_{v=1}^{\alpha} (v!)^{k_v} \prod_{v=1}^{\alpha} (k_v)!} \cdot \frac{(2k)!}{(2k - \sum_{v=1}^{\alpha} k_v)!}$$

then to get (2.9) it suffices to show that

$$(2d)^{k_1} (k_1/2)! \, (\max_{1\leqq i\leqq N} |c_i|)^{\sum_{v=3}^{\alpha} (v-2)k_v} \leqq (4e)^{2k+1} d^{2k} k^k (2k)^{-2k} \frac{(2k)!}{(2k - \sum_{v=1}^{\alpha} k_v)!} . \tag{2.13}$$

In view of the assumption $d(\max_{1\leqq i\leqq N} |c_i|)^{-1} \geqq 2k$, inequality (2.13) will be implied by the following one:

$$(2d)^{k_1} (k/2)! (d/2k)^{\sum_{v=3}^{\alpha} (v-2)k_v} (2k - \sum_{v=1}^{\alpha} k_v)! \, ((2k)!)^{-1} \leqq d^{2k} (4e)^{2k+1} k^k (2k)^{-2k} . \tag{2.14}$$

Using the Stirling inequality (see Feller (1971)),

$$(2.15)\qquad (2\pi)^{\frac12}(n+\tfrac12)^{n+\frac12}\exp\{-(n+\tfrac12)\tfrac{25}{24}\} < n! < (2\pi)^{\frac12}(n+\tfrac12)^{n+\frac12}\exp\{-(n+\tfrac12)\},$$

we obtain after some calculations that the left-hand side of inequality (2.14) is smaller than or equal to

$$d^{2k}(4e)^{2k+1}(2k)^{-2k}k^k .$$

Thus inequality (2.14) holds and Lemma 2.5 is proved. □

LEMMA 2.6. *Under conditions of Theorem* 2.1 *there exists a constant* $B_3(d)$ *(not depending on N and k) such that for* $2k \leqq d(\max_{1\leqq i\leqq N}|c_i|)^{-1}$, $0 \leqq d \leqq 1$,

$$E(T_N^*)^{2k} \leqq (2k)^{2k}N^{-k}(B_3(d))^{2k} ,$$

*where* $T_N^*$ *is given by* (2.3).

PROOF. Putting $U_i = (N+1)^{-1}(R_i - E(R_i\,|\,X_i))\varphi'(F(X_i, 0))$, $1 \leqq i \leqq N$, we have

$$E(T_N^*)^{2k} \leqq k^kN^{-2k}d^{2k}(4e)^{2k+1}E[(R_1 - E(R_1\,|\,X_1))\varphi'(F(X_1, 0))]^{2k}$$

and then applying Lemma 6.1 (Bickel (1974)) for $\xi_j = (u(X_1 - X_j) - F(X_1, 0))\varphi'(F(X_1, 0))$, $1 \leqq j \leqq N$, $u(x) = 1$, $x \geqq 0$, $u(x) = 0$, $x < 0$, we obtain the assertion. □

LEMMA 2.7. *Under assumptions* (I—IV) *there exist constants* $D^*$, $B_4$ *(not depending on N) such that*

$$\int_{|t|\leqq\{\log(\sum_{j=1}^N|c_j|^3)^{-1}\}D^*}|E\exp\{itT_N\}T_N^*|\,dt \leqq B_4\max(N^{-1}, \textstyle\sum_{j=1}^N|c_j|^5) .$$

PROOF. $ET_N^*\exp\{itT_N\}$ can be written in the following form:

$$\begin{aligned}(N+1)ET_N^*\exp\{itT_N\} &= E\exp\{itT_N\}\textstyle\sum_{v=1}^N\sum_{j=1;v\neq j}^N c_vE(u(X_v - X_j) - F(X_v, 0))\\ &\quad\times\varphi'(F(X_v, 0))\exp\{it(c_v\varphi(F(X_v, 0)) + c_j\varphi(F(X_j, 0)))\}\\ &\quad\times\{E\exp\{it[c_v\varphi(F(X_v, 0)) + c_j\varphi(F(X_j, 0))]\}\}^{-1} .\end{aligned}$$

For the characteristic function of $c_j\varphi(F(X_j, 0))$, $1 \leqq j \leqq N$, the relation

$$(2.16)\qquad E\exp\{itc_j\varphi(F(X_j, 0))\} = 1 - c_j^2t^2\bar\varphi^2/2 + (3!)^{-1}|t|^3|c_j^3||\bar\varphi|^3\eta_j ,\quad |\eta_j| \leqq 1 ,$$

holds and thus for $|t| \leqq -(\log(\sum_{j=1}^N|c_j|^3))\frac12[\bar\varphi^2 + 1/3|\bar\varphi|^3]^{-1}$, the Taylor expansion for $\log E[\exp\{itc_j\varphi(F(X_j, 0))\}]$ can be established and after some calculations we arrive at the following:

$$(2.17)\qquad E\exp\{itT_N\} = \exp\{-t^2\bar\varphi^2/2\}[1 + 2t\textstyle\sum_{j=1}^N|c_j^3||\bar\varphi|^3(3\bar\varphi^2)^{-1}\eta_j^*] ,\quad |\eta_j^*| \leqq 1,\ 1 \leqq j \leqq N .$$

Using the Taylor expansion for $\exp\{it[c_v\varphi(F(X_v, 0)) + c_j\varphi(F(X_j, 0))]\}$ and using (2.16) for $(E\exp\{itc_v\varphi(F(X_v, 0))\})^{-1}$ and then again the Taylor expansion, we get

(after long calculations)

$$\int_{|t|\leq -D^*\log(\sum_{v=1}^N |c_v|^3)} |ET_N{}^* \exp\{itT_N\}|\, dt$$
$$\leq 2^{\frac{1}{2}}(\bar\varphi^2)^{-\frac{1}{2}} D_1 D_0{}^4 3 \sum_{v=1}^N |c_v|^5 + N^{-\frac{1}{2}} \sum_{v=1}^N |c_v|^3 D_0{}^3 \tfrac{1}{3}4 + D_0 N^{-1}$$
$$+ O(\textstyle\sum_{v=1}^N c_v{}^6), \qquad \text{where } D^* = \tfrac{1}{2}\{\bar\varphi^2 + \tfrac{1}{3}|\bar\varphi|^3\}^{-1}. \qquad \square$$

LEMMA 2.8. *If assumptions* (I—IV) *are satisfied then*

$$|ET_N{}^{*k} \exp\{itT_N\}| \leq D_1{}^k \max\,((|t|D_0)^k, 4^k) \exp\{-t^2(1 - 2k \max_{1\leq v\leq N} c_v{}^2)\bar\varphi^2/2\}$$

*for* $|t| \leq 3/2\bar\varphi^2(|\bar\varphi|^3 \sum_{\nu=1}^N |c_\nu|^3)^{-1}$ *and* $k = 1, 2, 3, \cdots$. *Moreover, if* $0 \leq 2k \leq -1/8 \log(\sum_{\nu=1}^N |c_\nu|^3)$ *then*

$$|ET_N{}^{*k} \exp\{itT_N\}| \leq D_1{}^k \max\,((|t|D_0)^k, 4^k) \exp\left\{-\frac{t^2}{4}\bar\varphi^2\right\}. \tag{2.18}$$

PROOF. Denoting $h(v_1, \cdots, v_k, j_1, \cdots, j_k) = \prod_{\alpha=1}^k c_{v_\alpha}\{u(X_{v_\alpha} - X_{j_\alpha}) - F(X_{v_\alpha}, 0)\}\varphi'(F(X_{v_\alpha}, 0)) \exp\{itT_N\}$ we can write

$$ET_N{}^{*k} \exp\{itT_N\} \leq (N+1)^{-k} \sum_{v_1=1}^N \cdots \sum_{v_k=1}^N \sum_{j_1=1}^N \cdots \sum_{j_k=1}^N |Eh(v_1, \cdots, v_k, j_1, \cdots, j_k)|. \tag{2.19}$$

Now, decomposing the set $(j_1, \cdots, j_k)$ into three subsets $B_1$, $B_2$, $B_3$, where $B_1 = \{j_i; j_i \neq j_\alpha, \alpha = 1, \cdots, k, \alpha \neq i, j_i \neq v_\alpha, \alpha = 1, \cdots, k\}$, $B_2 = \{j_i, j_i \neq j_\alpha, \alpha = 1, \cdots, k, \alpha \neq i, j_i \notin B_1\}$, and $B_3$ denotes the complement to $B_1 \cup B_2$, the right-hand side of (2.19) can be rewritten as follows:

$$(N+1)^{-k} \sum_{v_1=1}^N \cdots \sum_{v_k=1}^N \sum_{(p_1,p_2,p_3)\in A} \frac{k!}{p_1!\,p_2!\,p_3!}$$
$$\sum_{(j_1,\cdots,j_{p_1})\in B_1} \sum_{(j_{p_1+1},\cdots,j_{p_2+p_1})\in B_2} \sum_{(j_{p_1+p_2+1},\cdots,j_k)\in B_3} |Eh(v_1, \cdots, v_k, j_1, \cdots, j_k)|, \tag{2.20}$$

where $A = \{(p_1, p_2, p_3); p_1 + p_2 + p_3 = k, p_i \geq 0 \text{ integers}\}$. By (2.16) and some elementary considerations we have for $|t| \leq 3/2\bar\varphi^2(|\bar\varphi|^3 \sum_{j=1}^N |c_j|^3)^{-1}$

$$|E \exp\{it \textstyle\sum_{j\in B} c_j \varphi(F(X_j, 0))\}| \leq \exp\left\{\dfrac{t^2}{2}\bar\varphi^2(1 - \max_{1\leq j\leq N} c_j{}^2 \# B)\right\}, \tag{2.21}$$

where $B$ is a subset of $\{1, \cdots, N\}$ and $\# B$ denotes its cardinal number.

Further, by a Taylor expansion we obtain

$$|E\{u(X_v - X_j) - F(X_v, 0)\} \exp\{itc_j\varphi(F(X_j, 0))\}| \leq |t|D_0 c_j, \qquad v \neq j,\ 1 \leq v, j \leq N. \tag{2.22}$$

Now the independence of $X_1, \cdots, X_N$, (2.20—2.21) and the last inequality imply

$$|Eh(v_1, \cdots, v_k, j_1, \cdots, j_k)|$$
$$= |E \exp\{it \textstyle\sum_{j\in B} c_j \varphi(F(X_j, 0))\} E \exp\{it \sum_{j\notin B} c_j \varphi(F(X_j, 0)\}$$
$$\times \textstyle\prod_{\alpha=1}^k c_{v_\alpha}[u(X_{v_\alpha} - X_{j_\alpha}) - F(X_{v_\alpha}, 0)]\varphi'(F(X_{v_\alpha}, 0))|$$
$$\leq \exp\left\{-\frac{t^2}{2}\bar\varphi^2(1 - 2k \max_{1\leq j\leq N} c_j{}^2)\right\} \textstyle\prod_{\alpha=1}^k |c_{v_\alpha}| D_1{}^k$$
$$\times \textstyle\prod_{j_\nu\in B_1} |c_{j_\nu}|(|t|D_0)^{\# B_1},$$

where $B = \{j;\ 1 \leqq j \leqq N, j \neq j_\alpha, j \neq v_\alpha, \alpha = 1, \cdots, k\}$. Then the right-hand side of (2.20) is smaller than or equal to

$$(N+1)^{-k} \exp\left\{-\frac{t^2}{2}\bar{\varphi}^2(1 - 2k \max\nolimits_{1 \leqq j \leqq N} c_j^2)\right\}$$
$$\times \left(\sum\nolimits_{\nu=1}^N |c_\nu|\right)^k D_1^k \left(1 + \sum\nolimits_{\nu=1}^N |c_\nu||t| D_0 + N^{\frac{1}{2}}\right)^k .$$

Both the assertions of our lemma can be concluded from the last relation, (2.19—2.21) and Remark 2.3. □

Proof of Theorem 2.1. In view of (2.6) it suffices to consider $S_N$ with scores given by (1.2). According to (2.7) we have

$$P(S_N < x(\bar{\varphi}^2)^{\frac{1}{2}}) \leqq P(T_N + T_N^* < (x + 3N^{-\frac{1}{2}})(\bar{\varphi}^2)^{\frac{1}{2}}) + B_2 N^{-\frac{1}{2}}$$

and

$$P(S_N < x(\bar{\varphi}^2)^{\frac{1}{2}}) \geqq P(T_N + T_N^* < (x - 3N^{-\frac{1}{2}})(\bar{\varphi}^2)^{\frac{1}{2}} - B_2 N^{-\frac{1}{2}} .$$

Now, we shall proceed as Bickel (1974) in the proof of Theorem 4.1. Thus it suffices to show that there exist constants $\varepsilon_1$, $\varepsilon_2$, $D_1^*$, $D_2^*$ such that

$$\int_{|t| \leqq \varepsilon_1 (\sum_{v=1}^N |c_v|^3)^{-1}} |E \exp\{itT_N\} - \exp\{-t^2\bar{\varphi}^2/2\}||t|^{-1}\, dt \leqq D_1^* \sum\nolimits_{v=1}^N |c_v|^3 \tag{2.23}$$

and

$$\begin{aligned}\int_{|t| \leqq \varepsilon_2 (\sum_{v=1}^N |c_v|^3)^{-1}} &|E \exp\{it(T_N + T_N^*)\} - E\exp\{itT_N\}||t|^{-1}\, dt \\ &\leqq D_2^* \sum\nolimits_{v=1}^N |c_v|^3 .\end{aligned} \tag{2.24}$$

Inequality (2.23) follows from Feller (1971), for $T_N$ is a sum of independent random variables. As for (2.24), we use the Taylor expansion

$$E \exp\{it(T_N + T_N^*)\} = E \sum\nolimits_{v=0}^{2k-1} \frac{(itT_N^*)^v}{v!} \exp\{itT_N\} + \frac{(tT_N^*)^{2k}}{(2k)!}\eta_N , \quad |\eta_N| \leqq 1 .$$

Denote $\sum_{v=1}^N |c_v|^3 = \rho_3$. Then, making use of Lemmas 2.6, 2.7, 2.8 for $k = 1$, we obtain

$$\begin{aligned}&\int_{|t| \leqq \rho_3^{-\frac{1}{2}}} |E \exp\{it(T_N + T_N^*)\} - E\exp\{itT_N\}||t|^{-1}\, dt \\ &\leqq \int_{|t| \leqq D^* \log(\rho_3)^{-1}} |E \exp\{itT_N\}|T_N^*|\, dt \\ &\quad + \int_{D^* \log(\rho_3^{-1}) \leqq |t| \leqq \rho_3^{-\frac{1}{2}}} |ET_N^* \exp\{itT_N\}\, dt + \int_{|t| \leqq (\rho_3)^{-\frac{1}{2}}} \tfrac{1}{2}|t| ET_N^{*2}\, dt \\ &\leqq B_4 \max\left(\sum\nolimits_{v=1}^N |c_v|^5, N^{-1}\right) + 2D_1 \max(D_0 \rho_3^{-1}, 4\rho_3^{-\frac{1}{2}}) \\ &\qquad \times \exp\left\{-\frac{D^{*2}\bar{\varphi}^2}{4}(\log \rho_3)^2\right\} + \tfrac{1}{2}B_3^2(2)\rho_3 .\end{aligned} \tag{2.25}$$

Denote $\varepsilon_2^* = 3/2\bar{\varphi}^2(|\varphi|^3\rho_3)^{-1}$ and

$$C = \left[\tfrac{1}{16} \min{}^* \left(1, \frac{\bar{\varphi}^2}{4}, \frac{\bar{\varphi}^2}{\log(\varepsilon_2^{*2} D_1 D_0)}, \frac{\bar{\varphi}^2}{\log(4D_1\varepsilon_2^*)}\right)\right],$$

where $[x]$ denotes the largest integer not exceeding $x$ and $\min^* \{x_i, i = 1, \cdots, 4\} = \min\{x_i;\ i = 1, \cdots, 4, x_i > 0\}$.

If $0 < \varepsilon_2 \leqq \varepsilon_2^*$ and $2k \leqq 1/8 \log \rho_3^{-1}$, by Lemma 2.8 we get

$$\int_{\rho_3^{-\frac{1}{2}} \leq |t| \leq \varepsilon_2(\rho_3)^{-1}} \sum_{j=1}^{2k-1} \frac{|t|^{j-1}}{j!} |E(T_N^*)^j \exp\{itT_N\}| \, dt$$
$$\leqq 2 \exp\left\{-\frac{\rho_3^{-1}}{4} \bar{\varphi}^2\right\} \sum_{j=1}^{2k-1} \max\{(D_1 D_0 \varepsilon_2^2 \rho_3^{-1})^j, (4D_1 \varepsilon_2 \rho_3^{-1})^j\}$$
$$\leqq 2 \exp\left\{-\frac{\rho_3^{-1}}{4} \bar{\varphi}^2\right\} \max\{(D_1 D_0 \varepsilon_2^2 \rho_3^{-1})^{4k}, (4D_1 \varepsilon_2 \rho_3^{-1})^{2k}, 2^{2k}\} .$$

Then putting $2k = C[\log \rho_3^{-1}]$ and making use of elementary inequalities

$$\tfrac{1}{2}(\log x)^2 \leqq x, \qquad x \geqq 1,$$
$$\log x \leqq x, \qquad x \geqq 0,$$

we can conclude

$$\int_{\rho_3^{-\frac{1}{2}} \leq |t| \leq \varepsilon_2 \rho_3^{-1}} \sum_{j=1}^{2k-1} \frac{|t|^{j-1}}{j!} |E(T_N^*)^j \exp\{itT_N\}| \, dt \leqq 2 \exp\left\{-\frac{\rho_3^{-1}}{8} \bar{\varphi}^2\right\} . \tag{2.26}$$

Remark 2.3, Lemma 2.6 and (2.15) imply

$$\int_{\rho_3^{-\frac{1}{2}} \leq |t| \leq \varepsilon_2^0 \rho_3^{-1}} \frac{|t|^{2k-1}}{(2k)!} ET_N^{*2k} \, dt = \frac{e^{\frac{25}{48}}}{(2\pi)^{\frac{1}{2}}} 2(\varepsilon_2^0 B_3(2) e^{\frac{25}{24}})^{2k} .$$

Then for $\varepsilon_2^0 = (B_3(2))^{-1} \exp\{+C^{-1} - \frac{25}{24}\}$ the relation

$$\int_{\rho_3^{-\frac{1}{2}} \leq |t| \leq \varepsilon_2^0 \rho_3^{-1}} \frac{|t|^{2k-1}}{(2k)!} ET_N^{*2k} \, dt \leqq \frac{e^{\frac{25}{24}}}{(2\pi)^{\frac{1}{2}}} 2\rho_3$$

holds.

Inequality (2.24) follows from (2.25), (2.26), and the last one, if we choose $\varepsilon_2 = \min(\varepsilon_2^*, \varepsilon_2^0)$. □

**3. Rate of convergence under "near" alternatives.** In this section we shall assume that

(V) $X_{1N}, \cdots, X_{NN}$ are independent random variables, $X_{jN}$ has a density $f(x, \theta_{jN}) \in \mathscr{F}$, where $\theta_{jN}$ are unknown parameters and $\mathscr{F}$ is a family of densities $f(x, \theta)$, $\theta \in J$ ($J$ is an open interval containing zero) satisfying

a. $f(x, \theta)$ is absolutely continuous;

b. the limit

$$\dot{f}(x, 0) = \lim_{\theta \to 0} \theta^{-1}(f(x, \theta) - f(x, 0))$$

exists for almost every $x$;

c. there exist $\theta_0$ and a constant $C$ such that for all $|\theta| \leqq \theta_0$

$$\int_{-\infty}^{+\infty} \frac{(\dot{f}(x, \theta))^2}{f(x, 0)} \, dx \leqq C .$$

REMARK 3.1. Notice that under (V), $\int_{-\infty}^{+\infty} \dot{f}(x, 0) \, dx = 0$.

Further, unknown parameters $\theta_{1N}, \cdots, \theta_{NN}$ are assumed to satisfy:

$$\sum_{j=1}^{N} \theta_{jN}^2 = 1 , \qquad \sum_{j=1}^{N} \theta_{jN} = 0 . \tag{VI}$$

In the following we shall denote by $E_H$ and $E_A$ the expectation under hypothesis (IV) and alternative (V), respectively (similarly, $\mathrm{Var}_A P_A$, $\mathrm{Var}_H P_H$ and so on). Moreover, $E_A{}^0 g(X_{1N}, \cdots, X_{NN})$ will denote the integral with respect to the measure $P_A{}^0$ which is a restriction of $P_A$ to the set $\prod_{j=1}^N f(X_{jN}, 0) \neq 0$, i.e.,

$$(3.1) \qquad E_A{}^0 g(X_{1N}, \cdots, X_{NN}) = \int_{\prod_{j=1}^N f(x_{jN},0)\neq 0} g(x_1, \cdots, x_N) \prod_{j=1}^N f(x_j, \theta_{jN})\, dx_1, \cdots, dx_N .$$

The main assertion of this section:

THEOREM 3.2. *Consider statistics* $S_N$ *given by* (1.1). *Then under assumptions* (I—III) *and* (V—VI) *there exist constants* $A_2$ *and* $\theta_0$ (*not depending on* $N$) *such that for* $\max_{1\leq j\leq N} |\theta_{jN}| \leqq \theta_0$

$$(3.2) \qquad \sup_x |P_A(S_N - \mu_N < x(\bar{\varphi}^2)^{\frac{1}{2}}) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-y^2/2}\, dy| \leqq A_2 \sum_{j=1}^N (|c_{jN}| + |\theta_{jN}|)^3 ,$$

*where* $\mu_N = \sum_{j=1}^N c_{jN} \int_{f(x_j,0)\neq 0} \varphi(F(x_j, 0)) f(x_j, \theta_{jN})\, dx$.

REMARK 3.3. If assumptions (I—III), (V—VI) are fulfilled and moreover:

a′. $\dot{f}(x, \theta)$ exists and is absolutely continuous;

b′. the limit

$$\ddot{f}(x, 0) = \lim_{\theta\to 0} \theta^{-1}(\dot{f}(x, \theta) - \dot{f}(x, 0))$$

exists for almost every $x$;

c′. there exist $\theta_1{}^*$ and $C^*$ such that

$$\int_{-\infty}^{+\infty} |\ddot{f}(x, \theta)|\, dx \leqq C^* \qquad \text{for all} \quad |\theta| \leqq \theta_1{}^*$$

then the assertion of Theorem 3.2 remains true if we replace in (3.2)

$$\mu_N \quad \text{by} \quad \sum_{j=1}^N \theta_{jN} c_{jN} \int \dot{f}(x_j, 0)\varphi(F(x_j, 0))\, dx .$$

COROLLARY 3.4. *Two-sample case*: *Let assumptions* (II—III) *and* (V) *be satisfied and let* $\theta_{iN} = (N - m)^{\frac{1}{2}}(mN)^{-\frac{1}{2}}, 1 \leqq i \leqq m, \theta_{iN} = -m^{\frac{1}{2}}(N(N - m))^{-\frac{1}{2}}\ m < i \leqq N$; *then there exist constants* $A_2{}^*$ *and* $\theta_0{}^{**} > 0$ *such that*

$$\sup_x \left| P_A\left(\sum_{j=1}^m a_N(R_{jN}) - m\bar{a}_N - \mu_N < x\left(\frac{m(N-m)}{N}\,\bar{\varphi}^2\right)^{\frac{1}{2}}\right) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-y^2/2}\, dy \right| \leqq A_2{}^*\{\max(m, N - m)\}^{-\frac{1}{2}}$$

*if* $[\max(m/(N - m), (N - m)/m)]^{\frac{1}{2}} \leqq \theta_0{}^{**} N^{\frac{1}{2}}$ *and, where*

$$\mu_N = \left\{\frac{(N-m)m}{N}\right\}^{\frac{1}{2}} \int_{f(x,0)\neq 0} \varphi(F(x, 0)) \left\{ f\left(x, \left(\frac{N-m}{mN}\right)^{\frac{1}{2}}\right) - f\left(x, -\left(\frac{m}{(N-m)N}\right)^{\frac{1}{2}}\right)\right\} dx .$$

We shall prove Theorem 3.2, using the same method as in Section 2. The present proof can be simplified by the following two lemmas (we shall write $c_j, \theta_j, X_j$ instead of $c_{jN}, \theta_{jN}, X_{jN}$):

LEMMA 3.5. *Under assumptions* (V—VI) *there exist constants* $C_1$ *and* $\theta_1^{**}$ (*not depending on* $N$) *such that*

$$P_A(\prod_{j=1}^N f(X_j, 0) = 0) \leqq C_1 \sum_{j=1}^N |\theta_j|^3 \tag{3.3}$$

*for* $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_1^{**}$.

PROOF. Obviously,

$$P_A(\prod_{j=1}^N f(X_j, 0) = 0) \leqq P_A\left(\prod_{j=1}^N \frac{f(X_j, \theta_j)}{f(X_j, 0)} \geqq K\right) \tag{3.4}$$

for $K > 0$ arbitrary. Put, in accordance with Hájek–Šidák (1967),

$$\begin{aligned} L_N &= \prod_{j=1}^N \frac{f(X_j, \theta_j)}{f(X_j, 0)} && \text{if } \prod_{j=1}^N f(X_j, 0) > 0\,, \\ &= 1 && \text{if } \prod_{j=1}^N f(X_j, 0) = 0 = \prod_{j=1}^N f(X_j, \theta_j) \\ &= +\infty && \text{if } \prod_{j=1}^N f(X_j, 0) = 0 < \prod_{j=1}^N f(X_j, \theta_j)\,. \end{aligned}$$

Le Cam's third lemma (VI.1.4) and Theorem VI.2.2 (Hájek–Šidák (1967)) imply that under assumptions of our lemma for $\max_{1\leqq j\leqq N} |\theta_j| \to 0$

$$\sup_x |P_A(\log L_N - b^2/2 < xb) - \Phi(x)| \to 0\,,$$

where $b^2 = \int_{-\infty}^{+\infty} ((\dot{f}(x, 0))^2/f(x, 0))\,dx$. Thus there exists $\theta_1^{**}$ such that for $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_1^{**}$

$$\sup_x |P_A(\log L_N - b^2/2 < bx) - \Phi(x)| \leqq e^{-\frac{1}{2}}$$

and then according to Lemma V.4.9 in Petrov (1972) we have

$$|P_A(\log L_N - b^2/2 < xb) - \Phi(x)| \leqq (C_p e^{-\frac{1}{2}}/2 + \lambda_p)(1 + |x|^p)^{-1} \tag{3.5}$$

for all $x$, where $p > 0$, $C_p > 0$ is a constant not depending on $N$ and

$$\lambda_p = \left|E_A\left|\frac{\log L_N - b^2/2}{b}\right|^p - \int |x|^p\, d\Phi(x)\right|\,, \qquad p > 0\,.$$

Choosing in (3.5) $p = 1$ and $x = (\sum_{j=1}^N |\theta_j|^3)^{-1}$ we obtain that there exists a constant $C_1$ such that

$$P_A(\log L_N > b(\sum_{j=1}^N |\theta_j|^3)^{-1} + b^2/2) \leqq C_1 \sum_{j=1}^N |\theta_j|^3$$

if $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_1^{**}$. The assertion (3.3) can be concluded from (3.4) and the last inequality. □

LEMMA 3.6. *Let* $Y_N$ *be a measurable function of* $(X_1, \cdots, X_N)$. *Then there exist constants* $C_2$ *and* $\theta_2^*$ (*not depending on* $N$) *such that*

$$E_A^0 Y_N^{2k} = (E_H Y_N^{4k})^{\frac{1}{2}} C_2 \qquad \text{if } \max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_2^*\,. \tag{3.6}$$

PROOF. Using Hölder's inequality we have

$$E_A^0 Y_N^{2k} = E_H Y_N^{2k} \prod_{j=1}^N \frac{f(X_j, \theta_j)}{f(X_j, 0)} \leqq \left\{E_H Y_N^{4k} E_H \left(\prod_{j=1}^N \frac{f(X_j, \theta_j)}{f(X_j, 0)}\right)^2\right\}^{\frac{1}{2}} \tag{3.7}$$

and hence it suffices to prove that $E_H \prod_{j=1}^N \{f(X_j, \theta_j)/f(X_j, 0)\}^2$ is bounded from above.

Obviously,

$$E_H \left\{\prod_{j=1}^N \frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\}^2 = \prod_{j=1}^N E_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\}^2 . \tag{3.8}$$

Because of assumption (V.c) and the absolute continuity of $f(x, \theta)$ in $\theta$ we can proceed in a similar way as Hájek–Šidák (1967) and Jurečková (1971) and get

$$E_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\}^2 = \operatorname{Var}_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\} + \left\{E_H \frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\}^2 , \tag{3.9}$$

$$E_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\} = \int_{f(x,\theta)\neq 0} f(x, \theta_j)\, dx \leqq 1 , \tag{3.10}$$

$$\begin{aligned} &\operatorname{Var}_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\} \\ &= E_H \left\{\frac{f(X_j, \theta_j) - f(X_j, 0)}{f(X_j, 0)} - \int_0^{\theta_j} \int_{f(x,0)\neq 0} \dot{f}(x, \theta)\, d\theta\, dx\right\}^2 \\ &\leqq 2E_H \left\{\int_0^{\theta_j} \frac{\dot{f}(X_j, \theta)}{f(X_j, 0)}\, d\theta\right\}^2 + 2|\theta_j| \int_0^{\theta_j} \left\{\int \frac{(\dot{f}(x, \theta))^2}{f(x, 0)}\, dx\right\} d\theta \leqq 4\theta_j^2 C . \end{aligned} \tag{3.11}$$

The last inequality follows from the Schwarz inequality and assumption (V.c). Relations (3.9—3.11) imply

$$E_H \left\{\frac{f(X_j, \theta_j)}{f(X_j, 0)}\right\}^2 \leqq 1 + 4\theta_j^2 C .$$

Assertion (3.6) can now be concluded from (3.7—3.8), assumption (VI) and the last inequality. □

In view of Lemmas 3.5 and 3.6 we need not prove the lemmas analogous to Lemma 2.4 and Lemma 2.6. It suffices to prove only lemmas analogous to Lemmas 2.7 and 2.8, a sketch of their proofs is given (emphasizing differences). But first we shall prove the following assertion:

LEMMA 3.7. *Under assumptions* (I—III) *and* (V—VI) *there exist* $\theta_3^* > 0$, $C_3$ *and* $\varepsilon_3 > 0$ *(not depending on* $N$*) such that*

$$\begin{aligned} &\int_{|t|\leqq \varepsilon_3(\sum_{j=1}^N |c_j|^3)^{-1}} |E_A{}^0 \exp\{it(T_N - E_A{}^0 T_N)\} - \exp\{-t^2\bar{\varphi}^2/2\}||t|^{-1}\, dt \\ &\qquad \leqq C_3 \sum_{j=1}^N (|c_j| + |\theta_j|)^3 , \end{aligned} \tag{3.12}$$

*if* $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_3^*$.

PROOF. For the characteristic function of the $j$th summands of $T_N - E_A{}^0 T_N$ we can write

$$\begin{aligned} &E_A{}^0 \exp\{itc_j(\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0)))\} \\ &\quad = 1 - (c_j^2 t^2/2) E_A{}^0(\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0)))^2 \\ &\qquad + \tfrac{1}{3}|c_j|^3 |t|^3 E_A{}^0|\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0))|^3 \delta_j , \quad |\delta_j| \leqq 1 . \end{aligned} \tag{3.13}$$

Noticing

$$|E_A{}^0(\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0)))^2 - \bar{\varphi}^2| \leqq |\theta_j| 2D_0{}^2C$$

we observe that there exist $\eta_3{}^*$, $\eta_3{}^{**}$ and $\varepsilon_3{}^*$, $\varepsilon_3{}^{**}$ such that for

$$|t| \leqq \varepsilon_3{}^* \log (\sum_{j=1}^N |c_j|^3)^{-1} \qquad \text{and} \qquad \max_{1\leqq j \leqq N} |\theta_j| \leqq \eta_3{}^*$$

$$\begin{aligned}(3.14)\qquad \log E_A{}^0 \exp\{itc_j(\varphi(F(X_j, 0)) &- E_A{}^0\varphi(F(X_j, 0))\} \\ &= -t^2c_j{}^2E_A{}^0(\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0)))^2/2 \\ &\quad + |tc_j{}^3|\delta_j{}^*/3\,, \quad |\delta_j{}^*| \leqq 8D_0{}^3\,,\end{aligned}$$

and for $|t| \leqq \varepsilon_3{}^{**}(\sum_{j=1}^N |c_j|^3)^{-1}$ and $\max_{1\leqq j\leqq N} |\theta_j| \leqq \eta_3{}^{**}$

$$(3.15)\qquad |E_A{}^0 \exp\{itc_j(\varphi(F(X_j, 0)) - E_A{}^0\varphi(F(X_j, 0)))\}| \leqq \exp\{-t^2c_j{}^2/8\}\,.$$

The assertion can now be concluded in a usual way. □

LEMMA 3.8. *Under assumptions* (I—III) *and* (V—VI) *there exist constants* $\varepsilon_4$, $\theta_4{}^*$ *and* $C_4$ (*not depending on* $N$) *such that for* $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_4{}^*$

$$\int_{|t|\leqq\{\log(\sum_{j=1}^N (|c_j^3|+|\theta_j|^3))^{-1}\}\varepsilon_4} |E_A{}^0 \exp\{it(T_N - E_A{}^0T_N)\}T_N{}^*|\, dt \leqq C_4 \sum_{j=1}^N (|c_j|^3 + |\theta_j|^3)\,.$$

The *proof* is the same as that of Lemma 2.7, only we use (3.13) or (3.14) instead of (2.16) or (2.17), respectively. □

LEMMA 3.9. *Under assumptions* (I—III) *and* (V—VI) *there exist constants* $\varepsilon_5$, $C_5$, $\theta_5{}^*$ (*not depending on* $N$) *such that*

$$|E_A{}^0T_N{}^{*m} \exp\{it(T_N - E_A{}^0T_N)\}| \leqq C_5 \sum_{j=1}^N (|c_j|^3 + |\theta_i|^3)\,,$$

*if* $|t| \leqq \varepsilon_5 \sum_{j=1}^N (|c_j|^3 + |\theta_j|^3)^{-1}$ *and* $\max_{1\leqq j\leqq N} |\theta_j| \leqq \theta_5{}^*$.

The *proof* runs in the same line as that of Lemma 2.8, only, the inequality

$$\begin{aligned}|E_A{}^0(u(X_v - X_\alpha) - F(X_v, 0)) \exp\{itc_\alpha(\varphi(F(X_\alpha, 0)) - E_A{}^0\varphi(F(X_\alpha, 0)))\}| \\ \leqq (1 + |tc_\alpha|(1 + |\theta_\alpha|))D_0D_1|\theta_\alpha|\end{aligned}$$

and (3.15) instead of (2.22) and (2.21), respectively, must be applied. □

PROOF OF THEOREM 3.2. By Lemma 3.5 one can write

$$|P_A(S_N - \mu_N < x(\bar{\varphi}^2)^{\frac{1}{2}}) - P_A{}^0(S_N - \mu_N < x(\bar{\varphi}^2)^{\frac{1}{2}})| \leqq C_1 \sum_{j=1}^N |\theta_j|^3\,.$$

The rest of the proof runs in the same way as that of Theorem 2.1. The lemmas analogous to Lemmas 2.4 and 2.6 follow from the mentioned ones and from 3.5 and 3.6. □

**Acknowledgments.** The author wishes to express many thanks to Professor W. R. van Zwet for suggesting the problem and his kind help during the preparation of this paper.

## REFERENCES


[1] ALBERS, W., BICKEL, P. J. and VAN ZWET, W. R. (1976). Asymptotic expansion for the power of distribution free tests in the one sample problem. *Ann. Statist.* **4** 108–156.

[2] BICKEL, P. J. and VAN ZWET, W. R. (1973). Asymptotic expansions for the power of the distribution free tests in the two sample problem. Unpublished.

[3] BICKEL, P. J. (1974). Edgeworth expansions in nonparametric statistics. *Ann. Statist.* **2** 1–20.

[4] BJERVE, S. (1973). Error bounds and asymptotic expansions for linear combinations of order statistics. Thesis.

[5] ERICKSON, R. V. and KOUL, H. L. (1976). Rates of convergence for linear rank statistics. *Ann. Statist.* **4** 771–774.

[6] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications.* Wiley, New York.

[7] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academia, Prague.

[8] JUREČKOVÁ, J. (1971). Almost sure uniform asymptotic linearity of rank statistics in regression parameter. *Trans. Sixth Prague Conf. Information Theor. Statist. Decis. F.* 305–313.

[9] JUREČKOVÁ, J. (1973). Order of normal approximation for rank statistics distribution. Inst. Math. Statist., Univ. of Copenhagen.

[10] PETROV, V. V. (1972). *The Sums of Independent Random Variables* (in Russian). Nauka, Moscow.

[11] PRÁŠKOVÁ-VÍZKOVÁ, Z. (1974). Local limit theorems for rank statistics. Ph. D. dissertation, Charles Univ., Prague.



DEPARTMENT OF STATISTICS
CHARLES UNIVERSITY, PRAGUE
CZECHOSLOVAKIA