The Annals of Statistics
1977, Vol. 5, No. 2, 385-393

ON ASYMPTOTICALLY OPTIMAL TESTS

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Sequences of tests with error $\exp(-nA)$ of the first type are investigated. It is shown that the error of the second type of such a sequence of tests is bounded by $\exp(-nB)$ where B is determined by the Kullback-Leibler information distance of the hypotheses tested. The information distance between the empirical measure and the null-hypothesis on a finite partition of the sample space is proposed to use as a test statistic. A sufficient condition is given which ensures that this test has error of the second type about $\exp(-nB)$ with the best possible B. The exact Bahadur slope of the proposed statistic is investigated.

1. Introduction. Let $S = (X, \mathscr{A}, \mathscr{P})$ be a statistical space, and $S_n = (X^{(n)}, \mathscr{A}^{(n)}, \mathscr{P}^{(n)})$ its nth power. Suppose that $\mathscr{P}$ consists of all probability measures on $(X, \mathscr{A})$, and it is desired to test the null-hypothesis $H_0: P \in \mathscr{P}_0$ on S_n, where $\mathscr{P}_0$ is an arbitrary nonempty subset of $\mathscr{P}$. Let $\varphi_n = \varphi_n(x_1, \cdots, x_n)$ denote a randomized test function (φ_n is the probability of the rejection of H_0), and let

$$(1.1) \qquad \alpha_n(P) = \int_{X^{(n)}} \varphi_n \, dP^{(n)} , \qquad \beta_n(P) = 1 - \alpha_n(P) , \qquad \alpha_n = \sup_{P \in \mathscr{P}_0} \alpha_n(P) .$$

Several different ways have been proposed for defining the asymptotic optimality of a sequence of tests φ_n. In some of them a sequence of alternatives Q_n is chosen in such a way that the probabilities $\beta_n(Q_n)$ are bounded away from 0 and 1. The speed of Q_n approaching $\mathscr{P}_0$ is measured somehow and this speed is taken as a criterion of optimality. Another possibility is to fix an alternative Q and consider the rate of $\beta_n(Q)$ tending to 0. The first method was investigated by Neyman (1937), the second by Bahadur (1960), Hoeffding (1965) and Brown (1971). The following definition is motivated by their results.

DEFINITION 1. A sequence of tests is of rate A, $0 \leqq A \leqq \infty$, if

$$(1.2) \qquad \limsup_{n\to\infty} \alpha_n < 1 , \qquad \text{in case} \quad A = 0 ,$$

$$(1.3) \qquad \limsup_{n\to\infty} \frac{1}{n} \log \alpha_n \leqq -A , \qquad \text{in case} \quad A > 0 .$$

Let Φ_A be the set of sequences of tests of rate A, and let

$$(1.4) \qquad B(A, Q, \mathscr{P}_0) = -\inf_{\varphi_n \in \Phi_A} \liminf_{n\to\infty} \frac{1}{n} \log \beta_n(Q) .$$

Received November 1974; revised September 1976.
AMS 1970 *subject classifications.* Primary 62G20, Secondary 62B10.
Key words and phrases. Asymptotic optimality, goodness-of-fit, exact Bahadur slope, likelihood ratio, sample entropy.

A sequence of tests $\varphi_n \in \Phi_A$ is called exponential rate optimal (ERO) at Q, if

(1.5) $$\lim_{n\to\infty} \frac{1}{n} \log \beta_n(Q) = -B(A, Q, \mathscr{P}_0) .$$

A sequence of statistics T_n is ERO at Q, if the sequence of tests

(1.6) $$\begin{aligned} \varphi_n &= 0 \quad \text{if} \quad T_n < A \\ &= 1 \quad \text{if} \quad T_n \geqq A \end{aligned}$$

is of rate A and ERO at Q for any continuity point $A > 0$ of $B(A, Q, \mathscr{P}_0)$. □

Although the above definition has a local character, our basic aim is to give statistics which depend only on $\mathscr{P}_0$ and are ERO at as many alternatives Q as possible. Brown (1971) gives sufficient conditions ensuring that the likelihood ratio is an ERO statistic. His conditions are of topological character, and are not fulfilled if $\mathscr{P}$ consists of all probability measures on $(X, \mathscr{A})$-except for the multinomial case, investigated by Hoeffding (1965). We shall extend Hoeffding's result using finite partitions of $(X, \mathscr{A})$.

Definition 2. The empirical measure p_n is defined by

(1.7) $$p_n = p_n(Y; x_1, \cdots, x_n) = \frac{1}{n} \sum_{i=1}^n \mathrm{X}_Y(x_i) ,$$

where $\mathrm{X}_Y(x) = 1$, if $x \in Y$, and $\mathrm{X}_Y(x) = 0$ otherwise. For measuring the distance between p_n and $\mathscr{P}_0$ we shall use a partition $\mathscr{B}_n$ of $(X, \mathscr{A})$. A sub-σ-algebra $\mathscr{B}_n$ is called a partition of $(X, \mathscr{A})$ if it is generated by disjoint sets $Y_1, \cdots, Y_m \in \mathscr{A}$ whose union is the whole space X. Throughout this paper, the sequence $m = m(n)$ denotes the number of atoms in the partition $\mathscr{B}_n$. The Kullback–Leibler information number for $P, Q \in \mathscr{P}$ on $\mathscr{B}_n$ is

(1.8) $$K_n(P, Q) = \sum_{i=1}^m P(Y_i) \log \frac{P(Y_i)}{Q(Y_i)} ,$$

where $\frac{0}{0} = 0$, $0 \log 0 = 0$; and for any $\mathscr{Q} \subset \mathscr{P}$

(1.9) $$K_n(P, \mathscr{Q}) = \inf_{Q \in \mathscr{Q}} K_n(P, Q) ; \qquad K_n(\mathscr{Q}, P) = \inf_{Q \in \mathscr{Q}} K_n(Q, P) .$$

Let L_n denote the statistic

(1.10) $$L_n = K_n(p_n, \mathscr{P}_0) .$$ □

Definition 3. Given a family $\mathscr{B} \subset \mathscr{A}$ of sets in X, we say that the sequence $P_n \in \mathscr{P}$ converges to the set function ν on $\mathscr{B}$ if $\lim_{n\to\infty} P_n(Y) = \nu(Y)$ for any $Y \in \mathscr{B}$. A monotone sequence of partitions $\mathscr{B}_n$ is said to be adequate (wrt $\mathscr{P}_0$), if $\mathscr{P}_0$ is closed under convergence on $\mathscr{B} = \bigcup_{n=1}^\infty \mathscr{B}_n$, and $\mathscr{B}$ generates $\mathscr{A}$. □

In Section 2 we investigate the function $B(A, Q, \mathscr{P}_0)$. Section 3 contains our main result (cf. Corollary 2): if the sequence $\mathscr{B}_n$ is adequate, and the sequence $m(n)$ satisfies

(1.11) $$\lim_{n\to\infty} m(n) \frac{\log n}{n} = 0 ,$$

then the sequence of statistics L_n is ERO. (It is easy to see that any monotone $\mathscr{B}_n$ generating $\mathscr{A}$ is adequate, if $\mathscr{P}_0$ is dominated and the set of densities is compact in L_1-norm. This condition fulfills trivially if $\mathscr{P}_0$ is simple, or at least finite.) The relation of ERO statistics and the exact slope introduced by Bahadur is investigated in Section 4. Some concluding remarks, examples and problems are collected in Section 5.

This work was initiated by a survey paper of Bahadur (1971). We shall mostly use the notation introduced there, and refer to known results through that paper.

2. The best exponent. The investigation of the function $B(A, Q, \mathscr{P}_0)$ (cf. Definition 1) will be based on a lemma of Stein (cf. Lemma 6.1 in Bahadur (1971)). This lemma states that

$$B(0, Q, \{P\}) = K(P, Q)\,, \tag{2.1}$$

for any $P, Q \in \mathscr{P}$, where K is defined by

$$\begin{aligned} K(P, Q) &= \textstyle\int_X \log \frac{dP}{dQ}\, dP \qquad \text{if} \quad P \ll Q\,, \\ &= \infty \qquad\qquad\qquad\quad \text{otherwise.} \end{aligned} \tag{2.2}$$

For any $\mathscr{Q} \subset \mathscr{P}$ let

$$K(P, \mathscr{Q}) = \inf_{Q \in \mathscr{Q}} K(P, Q)\,; \qquad K(\mathscr{Q}, P) = \inf_{Q \in \mathscr{Q}} K(Q, P)\,. \tag{2.3}$$

For any $0 < A \leqq \infty$, let

$$\mathscr{P}_A = \{Q : K(Q, \mathscr{P}_0) < A\}\,, \tag{2.4}$$

$$\mathscr{P}_{A,n} = \{Q : K_n(Q, \mathscr{P}_0) < A\}\,. \tag{2.5}$$

THEOREM 1. *For any* $\mathscr{P}_0 \subset \mathscr{P}$, $Q \in \mathscr{P}$, $0 \leqq A \leqq \infty$

$$B(A, Q, \mathscr{P}_0) \leqq K(\mathscr{P}_A, Q)\,. \tag{2.6}$$

PROOF. First we prove that for any sequence $\varphi_n \in \Phi_A$, $0 < A \leqq \infty$

$$\lim_{n\to\infty} \alpha_n(R) = 0 \tag{2.7}$$

holds true for any $R \in \mathscr{P}_A$. If we had a $R \in \mathscr{P}_A$ such that

$$\limsup_{n\to\infty} \alpha_n(R) > 0\,, \qquad \text{i.e.,} \qquad \lim_{k\to\infty} \beta_{n_k}(R) < 1$$

for some subsequence n_k, then the sequence

$$\tilde{\varphi}_n = 1 - \varphi_{n_k} \qquad \text{for} \quad n_k \leqq n < n_{k+1}$$

would be of rate 0 for testing the simple hypothesis that the actual measure is R. Hence Stein's lemma would imply that

$$\begin{aligned} \liminf_{k\to\infty} \frac{1}{n_k} \log \alpha_{n_k}(P_0) &= \liminf_{n\to\infty} \frac{1}{n} \log \tilde{\beta}_n(P_0) \\ &\geqq -B(0, P_0, \{R\}) = -K(R, P_0) \end{aligned} \tag{2.8}$$

for any $P_0 \in \mathscr{P}$. ($\tilde{\beta}_n$ denotes the error of second type of $\tilde{\varphi}_n$.) But there is a $P_0 \in \mathscr{P}_0$ such that $K(R, P_0) < A$ since $R \in \mathscr{P}_A$; consequently (2.8) contradicts (1.3).

The obtained (2.7) implies that given $R \in \mathscr{P}_A$, any sequence $\varphi_n \in \Phi_A$ is of rate 0 for testing the simple hypothesis that the actual measure is R. Hence $B(A, Q, \mathscr{P}_0) \leqq B(0, Q, \{R\}) = K(R, Q)$ for any $R \in \mathscr{P}_A$, which proves (2.6). □

THEOREM 2. *The sequence of tests*

$$\varphi_n = 0 \quad \text{if} \quad L_n < A \qquad (2.9)$$
$$= 1 \quad \text{if} \quad L_n \geqq A$$

is of rate A for any $0 < A \leqq \infty$, where L_n is defined by Definition 2, *and $\mathscr{B}_n$ is any sequence of partitions for which $m(n)$ satisfies* (1.11). *For this sequence of tests*

$$\limsup_{n\to\infty} \frac{1}{n} \log \beta_n(Q) \leqq -\liminf_{n\to\infty} K_n(\mathscr{P}_{A,n}, Q) \qquad (2.10)$$

holds true for any $Q \in \mathscr{P}$, where $\mathscr{P}_{A,n}$ is defined by (2.5).

COROLLARY 1. $B(A, Q, \mathscr{P}_0) \geqq \liminf_{n\to\infty} K_n(\mathscr{P}_{A,n}, Q)$.

PROOF. It is easy to see (cf. (5.33) in Bahadur (1971)) that

$$P_0(K_n(p_n, \mathscr{P}_0) \geqq A) \leqq (n+1)^m e^{-nA} \qquad (2.11)$$

holds true for any $P_0 \in \mathscr{P}_0$; hence (1.11) implies $\varphi_n \in \Phi_A$. Let the numbers $K_n(\mathscr{P}_{A,n}, Q)$ be denoted by B_n. The set $\mathscr{P}_{A,n}$ of measures is contained in the set $\{R: K_n(R, Q) \geqq B_n\}$, hence

$$Q(K_n(p_n, \mathscr{P}_0) < A) \leqq Q(K_n(p_n, Q) \geqq B_n) .$$

On applying the same argument as in (2.11) we get

$$Q(K_n(p_n, Q) \geqq B_n) \leqq (n+1)^m e^{-nB_n} ;$$

consequently (1.11) implies (2.10), too. □

3. Adequate sequences of partitions.

THEOREM 3. *If the sequence of partitions $\mathscr{B}_n$ is adequate* (cf. Definition 3), *then*

$$\lim_{n\to\infty} K_n(\mathscr{P}_{a,n}, Q) \geqq K(\mathscr{P}_A, Q) \qquad (3.1)$$

holds true for any $0 \leqq a < A \leqq \infty$, $Q \in \mathscr{P}$, $\mathscr{P}_0 \subset \mathscr{P}$ (cf. Definition 2, (2.4) and (2.5)).

COROLLARY 2. *If $\mathscr{B}_n$ is adequate and the sequence $m(n)$ satisfies* (1.11), *then the sequence of statistics L_n is* ERO (cf. Definition 1).

COROLLARY 3. *If $\mathscr{P}_0$ is such that there exists an adequate sequence of partitions, then for any $0 \leqq a < A \leqq \infty$, $Q \in \mathscr{P}$, $\mathscr{P}_0 \subset \mathscr{P}$*

$$K(\mathscr{P}_A, Q) \leqq B(a, Q, \mathscr{P}_0) \leqq K(\mathscr{P}_a, Q) . \qquad (3.2)$$

PROOF. Let the numbers $K_n(\mathscr{P}_{a,n}, Q)$ be denoted by B_n. The sequence B_n is monotone increasing, for $\mathscr{B}_n$ is monotone. Hence the limit on the left-hand side of (3.1) exists. (It is also easy to see that $B_n \leqq K(\mathscr{P}_a, Q)$.) It is enough to prove that for any B, such that $\lim_{n\to\infty} B_n < B < \infty$, there is an $R \in \mathscr{P}_A$ such that $K(R, Q) \leqq B$.

There are $P_n \in \mathscr{P}_0$, $R_n \in \mathscr{P}_a$ such that $K_n(R_n, Q) < B$, $K_n(R_n, P_n) < a$. For any j there is a convergent subsequence of P_n, R_n on $\mathscr{B}_j$, because $\mathscr{B}_j$ has finitely many elements. By using the diagonal method we can choose convergent subsequences of P_n, R_n on $\mathscr{B} = \bigcup_{j=1}^{\infty} \mathscr{B}_j$, too. Let P_{n_k}, R_{n_k} be such subsequences; then there is a $P_0 \in \mathscr{P}_0$ such that P_{n_k} tends to P_0 on $\mathscr{B}$ for $\mathscr{B}_n$ is adequate. Let the limit of R_{n_k} be denoted by ν:

$$\nu(Y) = \lim_{k\to\infty} R_{n_k}(Y) \qquad \text{for} \quad Y \in \mathscr{B}. \tag{3.3}$$

Now we prove that there is a probability measure R on $(X, \mathscr{A})$ such that its restriction to $(X, \mathscr{B})$ is ν.

Let $Y_1, \cdots, Y_m$ be the atoms of $\mathscr{B}_n$. It is easy to see that the sequence

$$r_n(x) = \frac{\nu(Y_i)}{Q(Y_i)} \tag{3.4}$$

if $x \in Y_i$ is a martingale with respect to $\mathscr{B}_n$ on $(X, \mathscr{A}, Q)$. The function $f(u, v) = u \log (u/v)$; $f(0, v) = 0$; $f(u, 0) = \infty$; $f(0, 0) = 0$ $(0 < u, v < 1)$ is lower semi-continuous, i.e., for any convergent sequences $0 \leqq u_i \leqq 1$; $0 \leqq v_i \leqq 1$

$$f(\lim_{i\to\infty} u_i, \lim_{i\to\infty} v_i) \leqq \liminf_{i\to\infty} f(u_i, v_i)\,, \tag{3.5}$$

consequently $\sum_{i=1}^{m} \nu(Y_i) \log (\nu(Y_i)/Q(Y_i)) \leqq B$. This implies that r_n is uniformly integrable on $(X, \mathscr{A}, Q)$, since

$$\int_{r_n > C} r_n \, dQ \leqq \frac{1}{\log C} \int_{r_n > C} r_n \log r_n \, dQ < \frac{B+1}{\log C}$$

for any $C > 1$ (note that $t \log t > -1$). Thus r_n is convergent Q a.s. and in L_1-norm on $(X, \mathscr{A}, Q)$ (cf. Theorem VII.4.1 in Doob (1953)), and the measure defined by

$$R(Y) = \int_Y \lim_{n\to\infty} r_n(x) \, dQ \tag{3.6}$$

for any $Y \in \mathscr{A}$ is the desired limit of R_n.

We have seen that $K_n(R, Q) \leqq B$. Similarly $K_n(R, P_0) \leqq a < A$, and the proof is completed by the well-known fact that K_n tends to K if $\mathscr{B}_n$ is monotone and generates $\mathscr{A}$ (cf. Theorem 2.4.2 in Pinsker (1964)). □

4. The exact slope of ERO-tests. Let T_n be a sequence of statistics for testing $H_0: P \in \mathscr{P}_0$, large values of T_n being significant, and let the function $G_n(t)$ be defined by

$$G_n(t) = \sup_{P \in \mathscr{P}_0} P(T_n \geqq t)\,. \tag{4.1}$$

The sequence T_n has exact slope c when Q obtains if

$$\lim_{n\to\infty} \frac{1}{n} \log G_n(T_n) = -\tfrac{1}{2}c(Q) \quad Q \quad \text{a.s.} \tag{4.2}$$

This definition is due to Bahadur (1960). He proved that

$$\liminf_{n\to\infty} \frac{1}{n} \log G_n(T_n) \geqq -K(Q, \mathscr{P}_0) \quad Q \quad \text{a.s.} \tag{4.3}$$

(cf. Theorem 7.5 in Bahadur (1971)). He also proved that under suitable compactification conditions the exact slope of the likelihood ratio statistic is $2K(Q, \mathscr{P}_0)$.

THEOREM 4. *If the sequence of partitions $\mathscr{B}_n$ is adequate* (cf. Definition 3), *and the sequence $m(n)$ satisfies* (1.11), *then the exact slope of the statistic L_n is $2K(Q, \mathscr{P}_0)$* (cf. Definition 2).

PROOF. It is enough to prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log H_n(L_n) \leqq -K(Q, \mathscr{P}_0) \quad Q \quad \text{a.s.}, \tag{4.4}$$

where

$$H_n(t) = \sup_{P\in\mathscr{P}_0} P(L_n \geqq t)\,.$$

Let $0 < a < A < K(Q, \mathscr{P}_0)$ be arbitrary. Theorem 2 implies

$$\limsup_{n\to\infty} \frac{1}{n} \log H_n(a) \leqq -a\,;$$

thus it is enough to prove that

$$\liminf_{n\to\infty} L_n \geqq a \quad Q \quad \text{a.s.}$$

Theorems 2 and 3 imply that

$$\limsup_{n\to\infty} \frac{1}{n} \log Q(L_n < a) \leqq -K(\mathscr{P}_A, Q)\,;$$

hence, by the Borel–Cantelli lemma, it is enough to prove that $K(\mathscr{P}_A, Q) > 0$. We have seen in the proof of Theorem 3 that

$$\lim_{n\to\infty} K_n(Q, \mathscr{P}_0) = K(Q, \mathscr{P}_0)\,, \tag{4.5}$$

thus $A < K_n(Q, \mathscr{P}_0)$ for some n. The m-dimensional set of probabilities $\{R(Y_i);\ i = 1, \cdots, m\}$ of those R-s for which $K_n(R, \mathscr{P}_0) \leqq A$, is closed, and does not contain the vector $\{Q(Y_i), i = 1, \cdots, m\}$, hence $K_n(\mathscr{P}_{A,n}, Q) > 0$. This completes the proof, because $K_n(\mathscr{P}_{A,n}, Q) \leqq K(\mathscr{P}_A, Q)$. □

5. Remarks, examples, problems. First of all it should be noted that the adequateness of $\mathscr{B}_n$ actually is a property of the system $\{\mathscr{P}_0, \mathscr{B}_n, n = 1, 2, \cdots\}$. The convergence on $\mathscr{B} \subset \mathscr{A}$ is a weaker version of the weak convergence of probability measures defined by

$$P(Y) = \lim_{n\to\infty} P_n(Y) \qquad \text{for all} \quad Y \in \mathscr{A}; \tag{5.1}$$

consequently the weak closedness is a necessary condition for the existence of an adequate sequence of partitions. Given any convergent sequence R_{n_k} on $\mathscr{B}$ we can define ν by (3.3) and r_n by (3.4) (by choosing any dominating Q). Moreover r_n will still be a martingale with respect to $\mathscr{B}_n$. Hence it is convergent Q a.s.; but it is no longer convergent in L_1 norm, unless r_n is uniformly integrable. In any case, if $\mathscr{P}_0$ is dominated, and its densities form a compact set with respect to L_1-norm, then the limit measure (3.6) belongs to $\mathscr{P}_0$.

EXAMPLE 1. Let S and $Q \in \mathscr{P}$ be arbitrary, and let

$$\mathscr{P}_0 = \{P: K(Q, P) \geqq A\}. \tag{5.2}$$

Then $Q \notin \mathscr{P}_A$, but in most cases the closure of $\mathscr{P}_0$ is equal to $\mathscr{P}$ for any $\mathscr{B}_n$, hence there is no adequate sequence of partitions. On the other hand, one can prove that if a sequence of partitions is adequate with respect to $\mathscr{P}_0$, then it is adequate with respect to $\mathscr{P}_A$ too. Consequently if $\mathscr{A}$ is countably generated, then any monotone sequence of partitions is adequate with respect to $\mathscr{P}_0 = \{P: K(P, Q) < A\}$.

EXAMPLE 2. Let $(X, \mathscr{A})$ be the real line with the σ-algebra of Borel-measurable sets, and let $\mathscr{P}_0$ be the collection of all probability measures concentrated on the dyadic rationals in $[0, 1)$. Let Q be the uniform distribution on $(0, 1)$. Let $\mathscr{B}_n$ be the partition of $[0, 1)$ consisting of the half-open intervals $[(i-1)/2^n, i/2^n)$. Then, for any $A < \infty$, $K_n(\mathscr{P}_{A,n}, Q) = 0$ and $K(\mathscr{P}_A, Q) = \infty$. Hence (3.1) does not hold, and $\mathscr{B}_n$ is not adequate.

EXAMPLE 3. Let X be arbitrary, and let $\mathscr{A}$ be a countably generated σ-algebra (e.g., X is the Euclidean space and $\mathscr{A}$ is the σ-algebra of Borel-measurable sets). Let $\mathscr{P}_0 = \{P_0\}$, where P_0 is arbitrary. Then any monotone sequence of partitions generating $\mathscr{A}$ is adequate. Consequently

$$B(A, Q, \{P_0\}) = K(\mathscr{P}_A, Q) \tag{5.3}$$

for any continuity point A of the function $B(A) = B(A, Q, \{P_0\})$. This function is convex, because $K(R, P)$ is convex in R: for any R_1, R_2, $0 < t < 1$ and $R = tR_1 + (1-t)R_2$ we have

$$K(R, P) \leqq tK(R_1, P) + (1-t)K(R_2, P),$$

(which in turn is a consequence of the convexity of $u \log u$). Hence the only discontinuity that may occur is at

$$A_\infty = \sup\{A: K(\mathscr{P}_A, Q) = \infty\}. \tag{5.4}$$

(5.3) has been obtained by Blahut (1972) in the case when $\mathscr{A}$ is finitely generated. The majority of tests of rate $A = 0$ for this problem are ERO at any Q (e.g., the Kolmogorov–Smirnov test in a Euclidean space). This however is true only for $A = 0$, and one can expect that any ERO statistic for testing goodness-of-fit is asymptotically equal to L_n.

EXAMPLE 4. Let $(X, \mathscr{A})$ be the k-dimensional Euclidean space with the σ-algebra of Borel-measurable sets, and let $\mathscr{P}_0$ be the set of normal distributions. It is easy to see that there is no adequate sequence of partitions in this case. This is partly the consequence of using only finite partitions. The finiteness of the partitions was used in an essential manner in (2.11), but there is no evidence that it would really be needed for the validity of Theorem 2. Vasicek (1976) introduced a test for normality based on sample entropy. It would be worthwhile to determine whether his statistic is ERO or not.

Another limitation of our results is the lack of error terms. The statistic L_n is ERO, but for a reasonable sample size it may be very poor. This problem consists of two parts. One is the error term in Stein's lemma and in Theorem 1. The question is the following: are there any sequences $\varepsilon_n \to 0$ depending on $\mathscr{P}_0$, Q only through the function $B(A, Q, \mathscr{P}_0)$ such that for any sequences φ_n for which

$$\frac{1}{n} \log \alpha_n \leqq -A - \varepsilon_n \tag{5.5}$$

we have

$$\frac{1}{n} \log \beta_n(Q) \geqq -B(A, Q, \mathscr{P}_0) - \varepsilon_n \, . \tag{5.6}$$

The other side of this problem is the speed of convergence of the sequence B_n. It is very likely that its investigation needs different methods from those used in the proof of Theorem 3.

6. Acknowledgment. I am indebted to T. Nemetz and I. Csiszár who have given me great assistance throughout this work. I am also grateful to the referees for their encouraging comments and suggestions. Example 2 is due to one of them and this has led me to the concept of adequateness.

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