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ON A CONJECTURE ABOUT THE LIMITING MINIMAL EFFICIENCY OF SEQUENTIAL TESTS

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For use in comparisons of sequential and nonsequential tests Berk (*Ann. Statist.* 3 991-998) has defined the limiting relative efficiency of sequential tests as the limiting ratio of the expected sample size under the null hypothesis and the supremum over the parameter set of the expected sample size. He has proved that for the symmetric binomial case the limiting relative efficiency of a class of SPR type tests coincides with a related quantity for SPR tests of drift in a Wiener process. He has also conjectured that this result applies to a more general class. In this note we prove that it holds for exponential families satisfying some mild regularity conditions.

1. Introduction and result. Berk (1975) discusses comparisons between sequential and nonsequential tests. He considers random variables $X_1, X_2, \cdots$ that are i.i.d. copies of a random variable X having a pdf $f(x \mid \theta)$ with respect to some fixed measure λ, where θ is a parameter belonging to Θ, a subinterval of R, and he studies tests of the hypothesis $H_1: \theta = \theta^*$ against $H_2: \theta > \theta^*$ with a prescribed level α and a prescribed expectation ν of the stopping time under H_1. The locally most powerful test in this situation has a stopping time of the form

$$N = \inf \{n : S_n \notin (-a_1, a_2)\}$$

where

$$S_n = \sum_{i=1}^n \left[\frac{\partial \log f(X_i \mid \theta)}{\partial \theta}\right]_{\theta=\theta^*}.$$

H_1 is rejected if $S_N \geqq a_2$ and a_1 and a_2 are chosen to satisfy the constraints $P_{\theta^*}(S_N \geqq a_2) = \alpha$ and $E_{\theta^*}(N) = \nu$. The asymptotic behaviour of this test (and especially its expected stopping time in relation to the sample size of a locally most powerful nonsequential test with the same slope at $\theta = \theta^*$) is given when $a = a_1 \wedge a_2$ tends to ∞, i.e., ν tends to ∞.

The limiting minimal efficiency of a sequential test of the above type is defined by $1/\eta(\alpha)$, where

$$\eta(\alpha) = \lim_{a \to \infty} \frac{\sup_{\theta \in \Theta} E_\theta[N]}{E_{\theta^*}[N]}.$$

Berk shows that for the symmetric binomial problem with $\theta = p$, $\theta^* = \frac{1}{2}$

$$\eta(\alpha) = \sup_{\mu \in R} \frac{E_\mu[\tau]}{E_0[\tau]},$$

where $E_\mu[\tau]$ is the expectation of the hitting time τ of the borders $1 - \alpha$ and

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$-\alpha$ of a standard Wiener process with drift μ. He also conjectures that this result applies to a greater class of problems. In this note it is shown that the conjecture is true when $\{f(x \mid \theta) : \theta \in \Theta\}$ is an exponential class satisfying some mild regularity conditions.

Suppose that $\{f(x \mid \theta) : \theta \in \Theta\}$ is an exponential class, i.e., that

$$f(x \mid \theta) = c(\theta) e^{\theta T(x)} .$$

This means that θ is the natural parameter in the class. If the original parameter is not the natural one, it is always possible to make a reparametrization and a common factor (depending on x only) could be included in the measure λ. Now

$$S_n = \left[\sum_{i=1}^{n} \frac{\partial \log f(X_i \mid \theta)}{\partial \theta} \right]_{\theta=\theta^*} = \sum_{i=1}^{n} Z_i$$

where

$$Z_i = T(X_i) - \left[\frac{\partial}{\partial \theta} \ln c(\theta) \right]_{\theta=\theta^*} .$$

From Theorem 1 in Holm (1975) follows that a test based on the above random sums S_n with continuation region

$$-\alpha M < S_n < (1-\alpha)M$$

has the limiting level α as M tends to ∞, i.e., satisfies

$$\lim_{M \to \infty} P(S_N \geqq (1-\alpha)M) = \alpha .$$

Our theorem will be formulated for this type of test. For the theorem we also need the following two conditions.

(1) To each interior point of Θ there exists a neighbourhood such that the functions

$$\gamma_1(\theta) = \inf_{r>0} E_\theta[Z_1 + r \mid Z_1 \leqq -r < 0]$$

and

$$\gamma_2(\theta) = \sup_{r>0} E_\theta[Z_1 - r \mid Z_1 \geqq r > 0]$$

are bounded in this neighbourhood. (If the conditioning event has probability 0, the expectation should be interpreted as 0.)

(2) If θ^+ and θ^- are the upper and lower boundary points of Θ then

$$\limsup_{\theta \uparrow \theta^+} \frac{\max(-\gamma_1(\theta); \gamma_2(\theta))}{|E_\theta[Z_1]|} < \infty$$

and

$$\limsup_{\theta \downarrow \theta^-} \frac{\max(-\gamma_1(\theta); \gamma_2(\theta))}{|E_\theta[Z_1]|} < \infty .$$

THEOREM. *If* $\{f(x \mid \theta) : \theta \in \Theta\}$ *is a one parameter exponential class satisfying conditions* (1) *and* (2), *then the abovementioned sequence of tests with asymptotic level* α *has a limiting minimal efficiency* $1/\eta(\alpha)$, *determined by*

$$\eta(\alpha) = \sup_{\mu \in R} \frac{E_\mu[\tau]}{E_0[\tau]} ,$$

where $E_{\mu}[\tau]$ is the expectation of the hitting time τ of the borders $1 - \alpha$ and $-\alpha$ of a standard Wiener process with drift μ.

The conditions (1) and (2) are satisfied for all common one parameter exponential classes such as the classes of one parameter normal, exponential, gamma, binomial, negative binomial and Poisson distributions. The main use of the theorem is for asymptotic comparisons of the sample size of locally most powerful nonsequential tests and the maximal expected sample size of locally most powerful sequential tests with the same asymptotic level and the same asymptotic slope of the power function at the null hypothesis. It is also possible to use for asymptotic power comparisons of locally most powerful nonsequential and sequential tests with the same asymptotic level and the same asymptotic maximal expected sample size.

The function $\eta(\alpha)$ is the asymptotic ratio of the maximal expected sample size of the locally most powerful sequential test and its expected sample size under the null hypothesis. Numerical values of $\eta(\alpha)$ are given by Berk (1975). He has also given numerical values of the local relative efficiency $e(\alpha)$ at the null hypothesis, which is the asymptotic ratio of the sample size of a locally most powerful nonsequential test and the expected sample size under the null hypothesis of a locally most powerful sequential test with the same asymptotic level and the same asymptotic slope. The asymptotic ratio of the sample size of the locally most powerful nonsequential test and the maximal expected sample size of the locally most powerful sequential test with the same asymptotic level and the same asymptotic slope is then $e(\alpha)/\eta(\alpha)$. Berk (1975) has given numerical values of this ratio for $\alpha \leqq \frac{1}{2}$. For $\alpha = 0.05$ it takes the value 3.07 and it can be shown to be greater than or equal to $\pi/2$ for $\alpha \leqq 0.5$. This indicates high efficiency of the sequential test, although the asymptotic power functions are not the same and slightly favor the nonsequential test.

2. Proof of the theorem. Holm (1975) studied tests with continuation region

$$-\alpha_1 M < S_n < \alpha_2 M$$

where M tends to ∞. Theorem 1 of that paper says that

$$\lim_{M\to\infty} P_{\theta^*+\Delta/M}(S_N \geqq \alpha_2 M) = \frac{e^{2\alpha_1\Delta} - 1}{e^{2\alpha_1\Delta} - e^{-2\alpha_2\Delta}} \quad \text{for} \quad \Delta \neq 0 \tag{2.1}$$

$$= \alpha_1/(\alpha_2 + \alpha_1) \quad \text{for} \quad \Delta = 0$$

and

$$\lim_{M\to\infty} \frac{1}{M^2} E_{\theta^*+\Delta/M}(N)$$

$$= \frac{\alpha_2(e^{2\alpha_1\Delta} - 1) - \alpha_1(1 - e^{-2\alpha_2\Delta})}{\Delta\sigma_0^2(e^{2\alpha_1\Delta} - e^{-2\alpha_2\Delta})} \quad \text{for} \quad \Delta \neq 0 \tag{2.2}$$

$$= \alpha_1\alpha_2/\sigma_0^2 \quad \text{for} \quad \Delta = 0\,,$$

where $\sigma_0^2 = \mathrm{Var}_{\theta^*} T(X)$, and the convergence is uniform in Δ on any interval

$-d \leqq \Delta \leqq d$ if the regularity condition (1) is satisfied. Further by the proof of Lemma 3 in the same paper

$$E_\theta[N] \leqq \max(\alpha_2 M + \gamma_2(\theta); \alpha_1 M + \gamma_1(\theta))/E_\theta[|Z_1|]$$

which together with conditions (1) and (2) imply the existence of a constant c such that

$$E_\theta[N] \leqq \frac{cM}{|\theta - \theta^*|}.$$

Now first choose d so that

$$\frac{c}{d} \leqq \frac{1}{\sigma_0^2} \sup_{\mu \in R} \frac{E_\mu[\tau]}{E_0[\tau]}.$$

Next by the uniformity in the convergence in (2.2) for $-d \leqq \Delta \leqq d$ we see that for every $\varepsilon > 0$ there exists an $M_\varepsilon > 0$ such that

$$\left| E_{\theta^*}[N] - \frac{M^2}{\sigma_0^2} E_0[\tau] \right| \leqq \varepsilon M^2$$

and

$$\left| \sup_{\theta \in \Theta} E_\theta[N] - \frac{M^2}{\sigma_0^2} \sup_{\mu \in R} E_\mu[\tau] \right| \leqq \varepsilon M^2$$

for $M > M_\varepsilon$ since the right member of (2.2) is in fact

$$\frac{1}{\sigma_0^2} E_\Delta[\tau].$$

Thus for every $\varepsilon > 0$

$$\lim_{M \to \infty} \frac{\sup_{\theta \in \Theta} E_\theta[N]}{E_{\theta^*}[N]} \leqq \frac{\sup_{\mu \in R} E_\mu[\tau] + \sigma_0^2 \varepsilon}{E_0[\tau] - \sigma_0^2 \varepsilon}$$

and

$$\lim_{M \to \infty} \frac{\sup_{\theta \in \Theta} E_\theta[N]}{E_{\theta^*}[N]} \geqq \frac{\sup_{\mu \in R} E_\mu[\tau] - \sigma_0^2 \varepsilon}{E_0[\tau] + \sigma_0^2 \varepsilon}$$

which proves the theorem.

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