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SEQUENTIAL ESTIMATION IN BERNOULLI TRIALS

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The sequential estimation of p, the probability of success in a sequence of Bernoulli trials, is considered for the case where loss is taken to be symmetrized *relative* squared error of estimation, plus a fixed cost c per observation. Using s_n/n as a terminal estimator of p, where s_n is the number of successes in n trials, a heuristic rule is derived and shown to perform well for any fixed $0 < p < 1$ as $c \to 0$. However for any fixed $c > 0$, this rule performs badly for p close to 0 or 1. To overcome this difficulty a uniform prior on p is introduced, and the optimal Bayes procedure is shown to exist and to have bounded sample size. The optimal Bayes risk is shown to be $\sim 2\pi c^{\frac{1}{2}}$ as $c \to 0$, and is computed for various values of c, along with the expected loss for various values of p.

0. Introduction. Let $x_1, x_2, \cdots$ be a sequence of independent identically distributed random variables, with $P(x_i = 1) = p$, $P(x_i = 0) = q = 1 - p$. The problem of estimating an unknown $0 < p < 1$ by some function δ_n of $x_1, \cdots, x_n$ with a loss structure

$$L(n, \delta_n, p) = L(|\delta_n - p|) + nc$$

where $0 < c < 1$ is some constant, has been approached by many authors. The case where

$$L(|\delta_n - p|) = (\delta_n - p)^2$$

has been considered as a special case of a more general problem by, in particular, Wald [13], Bickel and Yahav [3], and Alvo [1]. In all of these cases, either a heuristic *stopping rule* is proposed and its properties investigated, or a prior distribution on p is assumed, and a Bayes terminal estimator and an optimal stopping rule are found. The difficulty with the latter approach is that, for example, if the prior distribution on p is taken to be the beta, the optimal strategy can then only be expressed in terms of a backward induction equation, and thus cannot readily be applied, nor can it be compared to a heuristic rule for the same problem.

The case where

$$L(|\delta_n - p|) = \frac{(\delta_n - p)^2}{pq}$$

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has been dealt with by Whittle and Lane [14], through the use of beta priors. It is shown that the optimal rule is a fixed sample rule.

In the following we deal with what turns out to be a more amenable loss structure for this problem.

Consider the following hypothetical situation in medical trials. The probability p that a drug will cure a particular ailment is to be estimated sequentially with a cost $c > 0$ per observation. If this value of p is large, the drug will tend to be called a "cure," and research will shift to other ailments. If p is small, the drug will be discarded and more money and time will be invested in the problem. However, if p is close to one-half, no dramatic change will occur, that is, the drug will continue to be administered and research will continue in the same direction. Since in both extreme cases a dramatic decision will result, greater accuracy of estimation is demanded. A loss function which reasonably satisfies such a requirement is the following.

Let

$$L(n, \delta_n, p) = \left(\frac{\delta_n - p}{pq}\right)^2 + nc \,. \tag{1}$$

In Sections 1 and 2, δ_n is taken to be

$$\bar{\delta}_n = s_n/n \,, \tag{2}$$

where $s_n = x_1 + \cdots + x_n$, and a heuristic stopping rule $\bar{N}$ is considered. In Section 3 it is assumed that p has a uniform prior on (0, 1) and the Bayes terminal estimator $\delta_n{}^*$ and the optimal stopping rule N^* are found. In Section 4 numerical examples of the perfomance of $(N^*, \delta^*_{N^*})$ are given. Section 5 gives the proofs of various assertions and in particular that the Bayes risk of $(N^*, \delta^*_{N^*})$ is $\sim 2\pi c^{\frac{1}{2}}$ as $c \to 0$. Finally, Section 6 outlines the results obtained when p is assumed to have a more general beta prior.

An outline of parts of this problem has appeared in [4].

1. A heuristic stopping rule using the sample mean. The following discussion is modeled on that of [10], which deals with the sequential estimation of the mean of a normal distribution with unknown variance. For fixed n and p the expected loss using $\bar{\delta}_n = s_n/n$ is

$$E_p L(n, \bar{\delta}_n, p) = (npq)^{-1} + nc \,. \tag{3}$$

The value of n which minimizes (3) is

$$n(p) = (cpq)^{-\frac{1}{2}} \tag{4}$$

and for this n the expected loss is

$$E_p L(n(p), \bar{\delta}_{n(p)}, p) = 2(c/pq)^{\frac{1}{2}} \,. \tag{5}$$

Although the pair $(n(p), \bar{\delta}_{n(p)})$ is unavailable for statistical purposes, properties (4) and (5) provide a standard of comparison for sequential procedures when p

is unknown. Since $s_n/n \approx p$ and $(n - s_n)/n \approx q$, a sequential analogue of $n(p)$ may be defined by

$$\bar{N} = \text{first} \quad n \geqq 1 \quad \text{such that} \quad n \geqq \left(c \frac{s_n}{n} \cdot \frac{(n - s_n)}{n}\right)^{-\frac{1}{2}} . \tag{6}$$

This is equivalent to

$$\bar{N} = \text{first} \quad n \geqq 1 \quad \text{such that} \quad s_n(n - s_n) \geqq 1/c . \tag{7}$$

For convenience define

$$f_n = (n - s_n) .$$

It is easily seen that $\{\bar{N} > n\} = \{s_n f_n < 1/c\}$, $P(\bar{N} < \infty) = 1$ for any $0 < p < 1$, and $\bar{N} \geqq 2/c^{\frac{1}{2}}$. Modeling the discussion on that of [11], $\bar{N}$ has the following properties as $c \to 0$ for fixed $0 < p < 1$.

$$\bar{N}(cpq)^{\frac{1}{2}} \to 1 \quad \text{almost surely (a.s.)} \tag{8}$$

$$E_p\{\bar{N}(cpq)^{\frac{1}{2}}\}^k \to 1 \quad k = 1, 2, \cdots \tag{9}$$

$$(pq/c)^{\frac{1}{2}} \left(\frac{\bar{\delta}_{\bar{N}} - p}{pq}\right)^2 \to Z^2 \quad \text{in law} \tag{10}$$

$$(pq/c)^{\frac{1}{2}} E_p \left\{\left(\frac{\bar{\delta}_{\bar{N}} - p}{pq}\right)^2\right\} \to 1 , \tag{11}$$

where Z is a normally distributed random variable with mean 0 and variance 1. From (9) and (11) it follows that as $c \to 0$

$$\frac{E_p L(\bar{N}, \bar{\delta}_{\bar{N}}, p)}{E_p L(n(p), \bar{\delta}_{n(p)}, p)} = \frac{E_p\{(\bar{\delta}_{\bar{N}} - p)^2/(p^2q^2)\} + cE_p \bar{N}}{2(c/pq)^{\frac{1}{2}}} \to 1 , \tag{12}$$

so that $(\bar{N}, \bar{\delta}_{\bar{N}})$ is asymptotically as good as $(n(p), \bar{\delta}_{n(p)})$.

Proof of (8)—(11). Writing $\bar{N} = N$ throughout the rest of this section, it can be seen that (8) holds from

$$\frac{1}{c} \leqq s_N f_N < s_{N-1} f_{N-1} + N < \frac{1}{c} + N , \tag{13}$$

and the fact that $s_N/N \to p$ and $f_N/N \to q$ a.s. as $c \to 0$.

The proof of (9) is based on showing that for p fixed the convergence of (8) is dominated for all $0 < c < 1$. To this end define

$$M_1 = M_1(p) = \text{first} \quad n \geqq 2 \quad \text{such that} \quad s_i/i \geqq p/2 \quad \text{for all} \quad i \geqq n ,$$
$$M_2 = M_2(p) = \text{first} \quad n \geqq 2 \quad \text{such that} \quad f_i/i \geqq q/2 \quad \text{for all} \quad i \geqq n .$$

From Theorem 2 of [12], $E_p M_j^k < \infty$, $j = 1, 2$, and $k = 1, 2, \cdots$. Further, if $n \geqq \max(M_1, M_2, 2/(cpq)^{\frac{1}{2}})$ then $s_n f_n \geqq n^2pq/4 \geqq 1/c$, so that for $0 < c < 1$,

$$N(cpq)^{\frac{1}{2}} \leqq (M_1 + M_2)(cpq)^{\frac{1}{2}} + 2 < (M_1 + M_2)/2 + 2 .$$

Thus, by (8) and the dominated convergence theorem, (9) holds.

To prove (10) we note that

$$(pq/c)^{\frac{1}{2}}\left(\frac{\bar{\delta}_N - p}{pq}\right)^2 = \frac{(s_N - Np)^2}{Npq} \cdot \frac{1}{N(pqc)^{\frac{1}{2}}} \, .$$

Thus (10) follows from Anscombe's theorem [8, page 197] and (8).

Since $N \geqq 2/c^{\frac{1}{2}}$,

$$(pq/c)^{\frac{1}{2}}\left(\frac{\bar{\delta}_N - p}{pq}\right)^2 \leqq (c/pq)^{\frac{1}{2}} \cdot \frac{(s_N - Np)^2}{4pq}$$

and thus (11) follows from (10) if $(s_N - Np)^2(c/pq)^{\frac{1}{2}}$ is uniformly integrable for $0 < c < 1$. To prove the latter we make use of Wald's lemma for second moments [7], together with (9), to obtain as $c \to 0$

$$E_p(s_N - Np)^2 \cdot (c/pq)^{\frac{1}{2}} = E_p N(cpq)^{\frac{1}{2}} \to 1 \, . \tag{14}$$

Further, (8) and Anscombe's theorem yield

$$(s_N - Np)^2(c/pq)^{\frac{1}{2}} = \frac{(s_N - Np)^2}{Npq} N(cpq)^{\frac{1}{2}} \to Z^2 \quad \text{in law.} \tag{15}$$

The uniform integrability follows from the convergence theorem of [9, page 183].

The remainder of this section deals with the asymptotic distribution of N as $c \to 0$, and is not used in the rest of the paper. As with many other sequential problems dealing with Bernoulli trials, for example [2] and [11], this distribution is different when $p = \frac{1}{2}$ and $p \neq \frac{1}{2}$.

CASE 1. If $p \neq \frac{1}{2}$, then as $c \to 0$

$$\frac{2(pq)^{\frac{3}{4}}(N - 1/(cpq)^{\frac{1}{2}})c^{\frac{1}{4}}}{|p - q|} \to Z \quad \text{in law.} \tag{16}$$

PROOF. We introduce the identity

$$s_n(n - s_n) = -(s_n - np)^2 + n(q - p)(s_n - np) + n^2pq \, , \tag{17}$$

substitute N for n, and use (13) to obtain for $p < \frac{1}{2}$ the inequality

$$\begin{aligned}\frac{c^{-1} - N^2pq}{N^{\frac{3}{2}}(q - p)(pq)^{\frac{1}{2}}} &\leqq -\frac{(s_N - Np)^2}{Npq}\left(\frac{pq}{N}\right)^{\frac{1}{2}} \cdot \frac{1}{(q - p)} + \frac{s_N - Np}{(Npq)^{\frac{1}{2}}} \\ &\leqq \frac{c^{-1} - N^2pq}{N^{\frac{3}{2}}(q - p)(pq)^{\frac{1}{2}}} + \frac{1}{(Npq)^{\frac{1}{2}}(q - p)} \, .\end{aligned}$$

As $c \to 0$, the previous results show that

$$\frac{N^2pq - c^{-1}}{N^{\frac{3}{2}}|p - q|(pq)^{\frac{1}{2}}} \to Z \quad \text{in law.}$$

Similarly the same result holds for $p > \frac{1}{2}$. From (8) it follows that

$$\frac{(pq)^{\frac{5}{4}}(N^2 - (cpq)^{-1})c^{\frac{3}{4}}}{|p - q|} \to Z \quad \text{in law.} \tag{18}$$

Further, since

$$\left(N + \frac{1}{(cpq)^{\frac{1}{2}}}\right)(cpq)^{\frac{1}{2}} \to 2 \quad \text{a.s., then (16)}$$

follows from (18).

In addition, (16) remains valid with $1/(cpq)^{\frac{1}{2}}$ replaced by E_pN. To prove this it is sufficient to show that

$$(19) \qquad (pq)^{\frac{1}{2}}E_pN - \frac{1}{c^{\frac{1}{2}}} = o(c^{-\frac{1}{4}}) .$$

PROOF OF (19). By Wald's lemma and the Schwartz inequality,

$$|E_p\{N(s_N - Np)\}| \leqq \left(E_p\{Npq\}E_p\left\{\left(N - \frac{1}{(cpq)^{\frac{1}{2}}}\right)^2\right\}\right)^{\frac{1}{2}} = o(c^{-\frac{3}{4}}) .$$

Taking expectations in (17), with n replaced by N and using (13), yields the fact that

$$pqE_pN^2 - c^{-1} = o(c^{-\frac{3}{4}}) .$$

Thus (19) follows from

$$(pq)^{\frac{1}{2}}E_pN - \frac{1}{c^{\frac{1}{2}}} \leqq \frac{pqE_pN^2 - c^{-1}}{(pqE_pN^2)^{\frac{1}{2}} - 1/c^{\frac{1}{2}}} = o(c^{-\frac{1}{4}}) ,$$

and the observation that $(cpq)^{\frac{1}{2}}E_pN \geqq 1$.

CASE 2. For $p = \frac{1}{2}$, put

$$n = [2/c^{\frac{1}{2}}] + j , \qquad j = 1, 2, \cdots ,$$

where $[a]$ denotes the greatest integer $\leqq a$. Since

$$P(N \leqq n) = P\left(s_nf_n \geqq \frac{1}{c}\right) = P\left(\frac{(s_n - n/2)^2}{n/4} \leqq n - \frac{4}{nc}\right),$$

it follows that

$$P(N - [2/c^{\frac{1}{2}}] \leqq j) = P\left(\frac{(s_n - n/2)^2}{n/4} \leqq j + [2/c^{\frac{1}{2}}] - \frac{4}{c([2/c^{\frac{1}{2}}] + j)}\right).$$

As $c \to 0$ the central limit theorem shows that, uniformly in $j = 1, 2, \cdots$,

$$P(N - [2/c^{\frac{1}{2}}] \leqq j) - P\left(Z^2 \leqq j + [2/c^{\frac{1}{2}}] - \frac{4}{c([2/c^{\frac{1}{2}}] + j)}\right) \to 0 .$$

Since as $c \to 0$

$$\frac{4}{c[2/c^{\frac{1}{2}}]} - \frac{4}{c([2/c^{\frac{1}{2}}] + j)} \to j ,$$

it follows that

$$P(N - [2/c^{\frac{1}{2}}] \leqq j) - P\left(Z^2 \leqq j + [2/c^{\frac{1}{2}}] - \frac{4}{c[2/c^{\frac{1}{2}}]}\right) \to 0 ,$$

uniformly in $j = 1, 2, \cdots$. The term $[2/c^{\frac{1}{2}}] - 4/(c[2/c^{\frac{1}{2}}])$ oscillates finitely between -1 and 0 as $c \to 0$.

2. The Bayes risk for $(\bar{N}, \bar{\delta}_{\bar{N}})$. By (12) the procedure $(\bar{N}, \bar{\delta}_{\bar{N}})$, with $\bar{N}$ defined by (7), has a risk function that is equivalent to that of $(n(p), \bar{\delta}_{n(p)})$ as $c \to 0$ for any fixed $0 < p < 1$. The Bayes risk of $(n(p), \bar{\delta}_{n(p)})$ with respect to a uniform prior is

$$\int_0^1 2c^{\frac{1}{2}}(pq)^{-\frac{1}{2}}\, dp = 2\pi c^{\frac{1}{2}} .$$

However, the Bayes risk of $(\bar{N}, \bar{\delta}_{\bar{N}})$ is infinite, and thus the asymptotic equivalence of the two risks is *not* uniform in $0 < p < 1$. In fact, both of the integrals

$$\int_0^1 (pq)^{-2}E_p(\bar{\delta}_{\bar{N}} - p)^2\, dp , \qquad \int_0^1 E_p\bar{N}\, dp$$

are infinite.

Indeed, given $x_1 = 1$, $E_p\bar{N} \geqq (1 + 1/q)$ and given $x_1 = 0$, $E_p\bar{N} \geqq (1 + 1/p)$. Hence

$$E_p\bar{N} \geqq p\left(1 + \frac{1}{q}\right) + q\left(1 + \frac{1}{p}\right) = \frac{1}{pq} - 1 .$$

Moreover if an integer $m \geqq 1/c$ and $s_m = m$, $x_{m+1} = 0$, then $\bar{\delta}_N = m/(m+1)$, so that

$$E_p\left(\frac{\bar{\delta}_{\bar{N}} - p}{pq}\right)^2 \geqq \left\{p^m q\left(\frac{m}{m+1} - p\right)^2 + q^m p\left(\frac{1}{m+1} - p\right)^2\right\}\Big/ p^2q^2$$
$$= q^{-1}p^{m-2}\left(\frac{m}{m+1} - p\right)^2 + p^{-1}q^{m-2}\left(\frac{1}{m+1} - p\right)^2 .$$

In the following sections we shall find the Bayes procedure $(N^*, \delta^*_{N^*})$ for which

$$\int_0^1 E_p L(N, \delta_N, p)\, dp$$

is a minimum with respect to all stopping rules N and terminal estimators δ_n of p. This Bayes procedure will be shown to satisfy properties (8)—(12) and in addition to have Bayes risk $\sim 2\pi c^{\frac{1}{2}}$ as $c \to 0$. We remark in passing that it is possible to find certain ad hoc modifications of $(\bar{N}, \bar{\delta}_{\bar{N}})$ which satisfy (8)—(12) and have *finite* Bayes risk. One such procedure is $(\tilde{N}, \bar{\delta}_{\tilde{N}})$, where

$$\tilde{N} = \text{first} \quad n \geqq 1 \quad \text{such that} \quad |(s_n - 1)(f_n - 1)| \geqq \frac{1}{c} .$$

3. The Bayes procedure for the uniform prior. The Bayes risk for procedure (N, δ_N), when p has a prior density f on $(0, 1)$, is given by

$$B(N, \delta_N) = \int_0^1 E_p\left(\left(\frac{\delta_N - p}{pq}\right)^2 + cN\right) f(p)\, dp \tag{20}$$
$$= \sum_{n=1}^{\infty} \sum_{\{N=n\}} \left\{\int_0^1 \left(\left(\frac{\delta_n - p}{pq}\right)^2 + cn\right) f(p \mid x_1, \cdots, x_n)\, dp\right\}$$
$$\times P(x_1, \cdots, x_n)$$

where

$$P(x_1, \cdots, x_n) = \int_0^1 p^{s_n}q^{f_n}f(p)\, dp \tag{21}$$

and

$$f(p \mid x_1, \cdots, x_n) = \frac{p^{s_n} q^{f_n} f(p)}{P(x_1, \cdots, x_n)} . \tag{22}$$

Let

$$f(p; a, b) = \frac{1}{B(a, b)} p^{a-1} q^{b-1} \qquad a, b > 0 \tag{23}$$

denote the beta prior on $0 < p < 1$, for which

$$Ep = a/(a + b) , \qquad \operatorname{Var} p = ab/\{(a + b + 1)(a + b)^2\} . \tag{24}$$

Consider the case where $a = b = 1$, that is the uniform prior distribution. For this case, (22) becomes

$$f(p \mid x_1, \cdots, x_n) = \frac{p^{s_n} q^{f_n}}{B(s_n + 1, f_n + 1)} = f(p; s_n + 1, f_n + 1) , \tag{25}$$

and (21) becomes

$$P(x_1, \cdots, x_n) = B(s_n + 1, f_n + 1) = \frac{1}{(n + 1)\binom{n}{s_n}} . \tag{26}$$

For a given N, the Bayes estimator $\delta_n^* = \delta_n^*(x_1, \cdots, x_n)$ that minimizes the integral

$$\int_0^1 \left(\frac{\delta_n - p}{pq}\right)^2 f(p; s_n + 1, f_n + 1)\, dp , \tag{27}$$

is found to be

$$\begin{aligned} \delta_n^* &= \frac{s_n - 1}{n - 2} \qquad \text{if} \quad 1 \leqq s_n \leqq n - 1 , \quad n \geqq 3 \\ &= \frac{s_n}{n} \qquad \text{if} \quad s_n = 0 \quad \text{or} \quad n , \quad n \geqq 2 . \end{aligned} \tag{28}$$

With $\delta_n = \delta_n^*$, (27) becomes

$$\begin{aligned} H_n(s_n) &= \frac{n(n + 1)}{(n - 2)s_n f_n} \qquad \text{if} \quad 1 \leqq s_n \leqq n - 1 , \quad n \geqq 3 \\ &= \frac{n + 1}{n - 1} \qquad \text{if} \quad s_n = 0 \quad \text{or} \quad n , \quad n \geqq 2 \\ &= +\infty \qquad \text{otherwise} \end{aligned} \tag{29}$$

and thus the uniform prior Bayes risk (20) for an arbitrary stopping rule N and the best estmator δ_n^*, may be witten as

$$\sum_{n=2}^{\infty} \sum_{\{N=n\}} (H_n(s_n) + cn) P(x_1, \cdots, x_n) = E\{H_N(s_N) + cN\} . \tag{30}$$

Note that $\{N = 2\}$ is understood to be some subset of $\{s_2 = 0 \text{ or } 2\}$. The Bayes procedure for a uniform prior is $(N^*, \delta_{N^*}^*)$, where N^* is the solution, (if one exists), to the *optimal stopping problem* of finding the N which minimizes (30). To treat this problem we make use of the general theory of optimal stopping as found in [6]. A simplification of the problem is achieved by using the fact that, at stage n, all conditions on $x_1, \cdots, x_n$ may be replaced by conditions on s_n. That is we are in the Markov case [6, page 102]. Thus for $n = 0, 1, \cdots$

$$(31)\qquad P(s_{n+1} = s_n + 1 \mid x_1, \cdots, x_n) = P(s_{n+1} = s_n + 1 \mid s_n) = \frac{s_n + 1}{n + 2},$$
$$P(s_{n+1} = s_n \mid x_1, \cdots, x_n) = P(s_{n+1} = s_n \mid s_n) = \frac{f_n + 1}{n + 2}$$

with $s_0 = 0$.

The remaining part of this section is spent first, in showing that for this problem the optimal rule N^* is *bounded* by

$$(32)\qquad I(c) = \text{smallest} \quad n \geqq 3 \quad \text{such that} \quad \frac{n(n+1)}{(n-1)^2(n-2)} \leqq c,$$

and second, in finding the explicit form of this rule. To this end, define

$$(33)\qquad W_n(s) = H_n(s) + cn .$$

To find optimal rule N_I^* in the class of rules $N \leqq I$, $I = 1, 2, \cdots$, define

$$(34)\qquad w_I{}^I(s) = W_I(s), \qquad \text{for} \quad s = 0, 1, \cdots, I,$$

and by backward induction,

$$(35)\qquad w_n{}^I(s_n) = \min \{W_n(s_n), E(w_{n+1}^I(s_{n+1}) \mid x_1, \cdots, x_n)\},$$

for $n = I - 1, I - 2, \cdots, 1, 0$.

By (31), equation (35) may be rewritten as

$$(36)\qquad w_n{}^I(s) = \min \left\{ W_n(s), \frac{(s+1)w_{n+1}^I(s+1) + (n+1-s)w_{n+1}^I(s)}{n+2} \right\}$$

for $n = I - 1, I - 2, \cdots, 1, 0$ and $s = 0, 1. \cdots, n$. Therefore $N_I^* =$ smallest $n \geqq 0$ such that $w_n{}^I(s_n) = W_n(s_n)$. Since $s(n - s) \geqq n - 1$ for $1 \leqq s \leqq n - 1$, then

$$(37)\qquad 0 \leqq H_n(s) \leqq \frac{n(n+1)}{(n-1)(n-2)} \leqq 6 \qquad \text{for all} \quad n \geqq 3 .$$

Thus it follows from Theorems 4.4 and 4.5 of [6] that an optimal rule N^* exists and is given by

$$N^* = \lim_{I\to\infty} N_I^* .$$

To show that N^* is bounded by (32) we make use of the following

LEMMA. *If the integer I is such that*

$$(38)\qquad \frac{I(I-1)}{(I-2)^2(I-3)} \leqq c$$

then

$$(39)\qquad w_{I-1}^I(s) = w_{I-1}^{I-1}(s) \qquad \textit{for all} \quad s = 0, 1, \cdots, I - 1 .$$

PROOF. For $s = 0$ or $I - 1$, (39) is seen to hold for all $I \geqq 4$. For $n = I - 1$ and any $1 \leqq s \leqq I - 2$, (35) becomes

$$w_{I-1}^{I}(s) = \min\left\{\frac{I(I-1)}{s(I-s-1)(I-3)}, \frac{I}{(I-s-1)(I-2)} + \frac{I}{s(I-2)} + c\right\}.$$

Hence (39) will hold if

$$w_{I-1}^{I-1}(s) = \frac{I(I-1)}{s(I-s-1)(I-3)} \leqq \frac{I}{(I-2)}\left(\frac{1}{I-s-1} + \frac{1}{s}\right) + c$$
$$= \frac{I(I-1)}{s(I-s-1)(I-2)} + c ,$$

or equivalently, if

(40) $$\frac{I(I-1)}{s(I-s-1)(I-2)(I-3)} \leqq c .$$

Since $s(I - s - 1) \geqq I - 2$ for $1 \leqq s \leqq I - 2$, then (40) holds by (38). This completes the proof of the lemma.

It follows from the lemma, that whenever (38) holds, then $N_I{}^* = N_{I-1}^*$. By induction, $N_I{}^* = N_{I(c)}^*$ for all $I \geqq I(c)$, where $I(c)$ is defined in (32). Therefore, the optimal stopping rule is *the bounded rule*

(41) $$N^* = \lim_{I\to\infty} N_I{}^* = N_{I(c)}^* .$$

A further simplification may be had by observing that the problem exhibits a modified form of the monotone case property as discussed in [5] and [6]. If we stop at stage n, and if $1 \leqq s_n \leqq n - 1$, then the expected loss will be

(42) $$W_n(s_n) = \frac{n(n+1)}{(n-2)s_n f_n} + nc .$$

If on the other hand one more observation is taken and then we stop, the expected loss will be

(43) $$\frac{(s_n+1)W_{n+1}(s_n+1) + (f_n+1)W_{n+1}(s_n)}{(n+2)} = \frac{n(n+1)}{(n-1)s_n f_n} + (n+1)c .$$

The inequality (42) $\leqq$ (43) is seen to hold if and only if

(44) $$s_n f_n \geqq \frac{n(n+1)}{(n-1)(n-2)c} .$$

The left-hand side of (44) is increasing with n, while the right-hand side of (44) is decreasing with n. Thus, *once* (44) *holds, it will continue to hold thereafter.* Since the optimal rule N^* is bounded, it follows that it must be of the form:

(45) $$N^* = \text{first } n \geqq 3 \text{ such that (44) occurs if } 1 \leqq s_n \leqq n-1$$
$$= N_0 \text{ if } s_{N_0} = 0 \text{ or } N_0 ,$$

where $N_0 = N_0(c) \leqq I(c)$ is the smallest integer n such that $w_n^{I(c)}(0) = W_n(0) = (n+1)/(n-1) + nc$. The determination of N_0 requires that we go through the backward induction (36) for $I = I(c)$. The right-hand side of (44) is $\sim 1/c$ as $n \to \infty$, so that N^* is, in fact, a slightly modified version of the heuristic rule $\bar{N}$ of (7). In the next section we give numerical data, based on computations for various values of c, to compare the perfomance of $(N^*, \delta_{N^*}^*)$ and $(n(p), \bar{\delta}_{n(p)})$.

4. Numerical examples. The computations required for this section were run on an IBM 360 computer using a Fortran IV program. For different values of c, Table 1 lists the corresponding values of N_0, N_l, $I(c)$, B_{N^*} and $B_{N^*}/c^{\frac{1}{2}}$, where N_l denotes the lower bound on N^* when $s_{N^*}f_{N^*} \geqq 1$, and B_{N^*} is the Bayes risk of the procedure $(N^*, \delta_{N^*}^*)$. In the next section it will be shown that $B_{N^*}/c^{\frac{1}{2}} \to 2\pi$ as $c \to 0$. The numerical evidence seems to indicate that this convergence is not monotone in c, since $B_{N^*}/c^{\frac{1}{2}}$ exceeds 2π for relatively small c. However, the possibility of computing errors for very small values of c cannot be discounted, and a better program than the one used here might give smaller values for the Bayes risk.

TABLE 1

c	N_l	$I(c)$	N_0	B_N^*	$B_N^*/c^{\frac{1}{2}}$
0.03	14	39	12	1.042360	6.01805
0.01	22	105	31	0.616042	6.16042
0.005	31	205	59	0.439559	6.21630
0.001	66	1005	280	0.198510	6.27745
0.0004	102	2505	695	0.125774	6.28871
0.0002	144	5005	1386	0.088983	6.29202
0.00015	166	6672	1847	0.077068	6.29261

The expected loss of the optimal rule $(N^*, \delta_{N^*}^*)$, here denoted by $R(p)$, was computed for a grid of values of p and for different values of c. The grid used is the following, with the quantity in brackets denoting the size of the jump in the value of p:

$$.000025 \to .0001\,(.000025)\,, \quad .0002 \to .001\,(.0001)\,, \quad .002 \to .01\,(.001)\,,$$
$$.0125 \to .02\,(.0025)\,, \quad .03 \to .1\,(.01)\,, \quad .125 \to .175\,(.025)\,, \quad .2 \to .5\,(.1)\,.$$

Figure 1 compares $R(p)/c^{\frac{1}{2}}$ to $E_p L(n(p), \bar{\delta}_{n(p)}, p)/c^{\frac{1}{2}} = 2/(pq)^{\frac{1}{2}}$, for $c = .005$ and $c = .0002$. These computations were also done for $c = .001$, but for reasons of clarity these points are not included in the graph. It can be seen that for values of p away from 0 (or 1), where $R(p)/c^{\frac{1}{2}}$ is greater than $2/(pq)^{\frac{1}{2}}$, the difference

TABLE 2

c	p_a	p_b	p_l	p^*	p_r
0.005	0.0125	0.015	0.002	0.003	0.004
0.001	0.003	0.004	0.0005	0.0006	0.0007
0.0002	0.0005	0.0006	0.000075	0.0001	0.0002

between the two is small, and decreases as c decreases. In conjunction with Figure 1, the following values of p are tabulated in Table 2:

p_b = the smallest grid value of p for which $R(p)/c^{\frac{1}{2}} \geqq 2/(pq)^{\frac{1}{2}}$

p_a = the largest grid value of p for which $R(p)/c^{\frac{1}{2}} < 2/(pq)^{\frac{1}{2}}$

p^* = the grid value of p for which $R(p)/c^{\frac{1}{2}}$ is greatest

p_l = the grid value of p immediately to the left of p^*

p_r = the grid value of p immediately to the right of p^*.

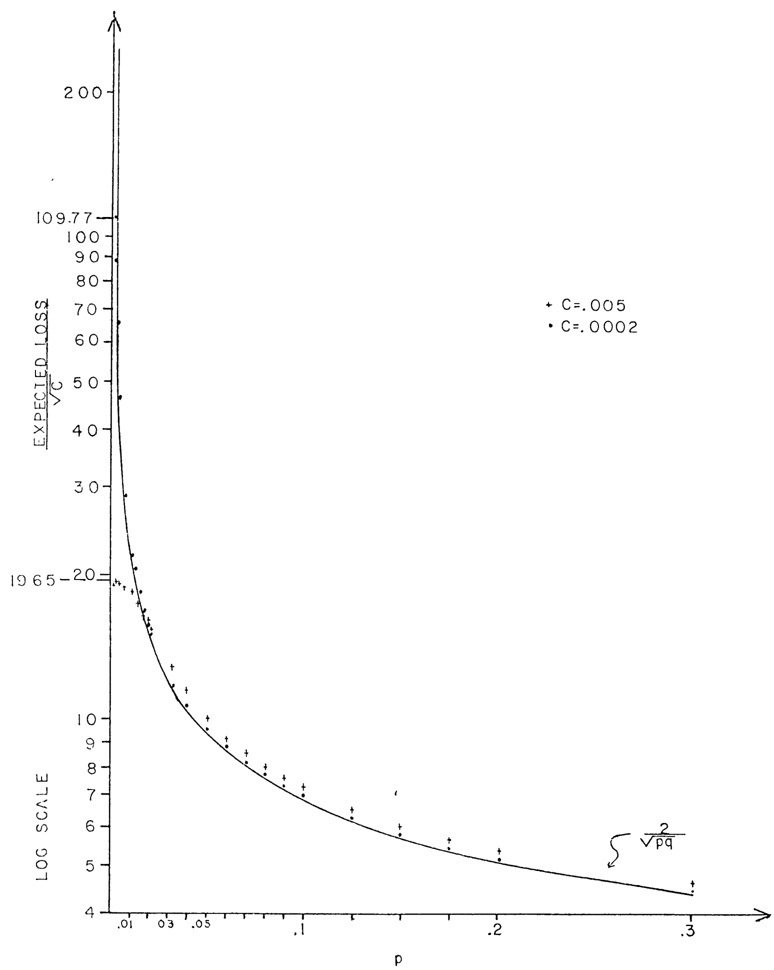

FIG. 1.

Table 3 lists the corresponding values of $R(p)/c^{\frac{1}{2}}$ and the value $R_0/c^{\frac{1}{2}} = (1 + N_0 c)/c^{\frac{1}{2}}$, the limiting value of $R(p)/c^{\frac{1}{2}}$ as $p \to 0$.

TABLE 3

c	$R(p_a)/c^{\frac{1}{2}}$	$R(p_b)/c^{\frac{1}{2}}$	$R(p_l)/c^{\frac{1}{2}}$	$R(p^*)/c^{\frac{1}{2}}$	$R(p_r)/c$	$R_0/c^{\frac{1}{2}}$
0.005	17.3247	16.5308	19.5722	19.6517	19.6081	18.316
0.001	36.1270	33.0271	42.4253	42.4889	42.4879	40.506
0.0002	89.0718	83.4370	109.693	109.776	106.925	90.581

5. Asymptotic properties of the Bayes rule. For convenience in this section N^* is written as N. For any fixed $0 < p < 1$, N satisfies the asymptotic property (9). That is, as $c \to 0$

(46) $$E_p\{(N(cpq)^{\frac{1}{2}})^k\} \to 1 \qquad k = 1, 2, \cdots .$$

This convergence is *not* uniform in p, so that, in order to find the asymptotic value of the Bayes risk of (N, δ_N^*), it is necessary to show that $c^{\frac{1}{2}}E_p N$ is uniformly integrable with respect to the distribution of p. This we proceed to show after proving (46).

PROOF OF (46). Define

(47) $$T = \text{first} \quad n \geqq 3 \quad \text{such that} \quad s_n f_n \geqq \frac{n(n+1)}{(n-1)(n-2)c} .$$

For all fixed $0 < p < 1$, it can be shown that as $c \to 0$

(48) $$T(cpq)^{\frac{1}{2}} \to 1 \quad \text{a.s.}$$

(49) $$E_p\{(T(cpq)^{\frac{1}{2}})^k\} \to 1 \qquad k = 1, 2, \cdots .$$

Statement (48) follows immediately from the inequality

$$\frac{1}{cT^2} \leqq \frac{s_T}{T}\frac{f_T}{T}\frac{(T-2)(T-1)}{T(T+1)} < \frac{1}{cT^2}\frac{(T-1)^2}{(T+1)(T-3)} + \frac{(T-2)(T-1)}{(T+1)T^2}$$

and the law of large numbers. By an argument similar to the proof of (9), statement (49) is seen to hold. Further,

(50) $$E_p N c^{\frac{1}{2}} = c^{\frac{1}{2}} N_0(p^{N_0} + q^{N_0}) + E_p T c^{\frac{1}{2}} - \int_{(s_{N_0}=0)} T c^{\frac{1}{2}}\, dP - \int_{(s_{N_0}=N_0)} T c^{\frac{1}{2}}\, dP.$$

Noting that $N_0 \to \infty$ as $c \to 0$, a fact which is proved later in this section, it follows that both $P_p(s_{N_0} = 0)$ and $P_p(s_{N_0} = N_0)$ converge to 0 as $c \to 0$. Thus the negative part of (50) goes to 0 as $c \to 0$, and (46) follows on noting that as $c \to 0$

$$c^{\frac{1}{2}} N_0(p^{N_0} + q^{N_0'}) \to 0 .$$

This completes the proof of (46).

Letting $A = \{1 \leqq s_N \leqq N - 1\}$, it follows from the preceding argument that as $c \to 0$,

(51) $$c^{\frac{1}{2}} E_p(N I_A) \to \frac{1}{(pq)^{\frac{1}{2}}} .$$

In the following it will be shown that as $c \to 0$,

(52) $$c^{\frac{1}{2}} E(N I_A) \to \pi .$$

To this end define

$$Y_n = E\left(\frac{1}{(pq)^{\frac{1}{2}}}\,\middle|\, x_1, \cdots, x_n\right).$$

The posterior density of p as given in (25) yields

$$(53) \qquad Y_n = \frac{(n+1)\Gamma(s_n + \frac{1}{2})\Gamma(f_n + \frac{1}{2})}{\Gamma(s_n + 1)\Gamma(f_n + 1)}.$$

For $1 \leqq s_n \leqq n - 1$, an easily proved inequality on gamma functions gives

$$(54) \qquad Y_n \geqq (n+1) \cdot \frac{1}{4(s_n f_n)^{\frac{1}{2}}}.$$

Replacing n by N and noting that for $N \geqq 4$,

$$(55) \qquad s_N(N - s_N) - N \leqq s_{N-1}(N - 1 - s_{N-1}) \leqq \frac{(N-1)N}{c(N-2)(N-3)} \leqq \frac{6}{c},$$

inequality (54) becomes

$$(56) \qquad Y_N \geqq \frac{(N+1)}{4}\left(\frac{6}{c} + N\right)^{-\frac{1}{2}}.$$

Since $N \leqq I(c) < 4/c$, for a constant $K > 0$ (56) becomes,

$$(57) \qquad K \cdot Y_N \geqq c^{\frac{1}{2}} N I_A.$$

Thus,

$$c^{\frac{1}{2}} E(N I_A) \leqq K \cdot E Y_N = K \cdot \pi,$$

and statement (52) follows by (51) and the dominated convergence theorem.

The Bayes risk is given to be

$$(58) \qquad \begin{aligned} B(N, \delta_N{}^*) &= E(H_N(s_N) + cN) \\ &= E\left\{\left(\frac{N(N+1)}{s_N f_N(N-2)} + cN\right) I_A + \left(\frac{N_0 + 1}{N_0 - 1} + cN_0\right) I_{\bar{A}}\right\}, \end{aligned}$$

where $\bar{A} = \{(s_{N_0} = 0) \cup (s_{N_0} = N_0)\}$.

Using the fact that $P(\bar{A}) = 2/(N_0 + 1)$, it follows that

$$(59) \qquad B(N, \delta_N{}^*)/c^{\frac{1}{2}} \leqq 2c^{\frac{1}{2}} E(N I_A) + \frac{2}{c^{\frac{1}{2}}(N_0 - 1)} + 2c^{\frac{1}{2}}.$$

For a *particular value* of N_0, say $N_0 = c^{-\frac{3}{4}}$, statements (52) and (59) imply that as $c \to 0$,

$$(60) \qquad \limsup B(N, \delta_N{}^*)/c^{\frac{1}{2}} \leqq 2\pi.$$

It follows that statement (60) *must hold for the Bayes procedure*. Since

$$B(N, \delta_N{}^*)/c^{\frac{1}{2}} > \frac{2}{c^{\frac{1}{2}}(N_0 + 1)},$$

statement (60) implies that $N_0 \to \infty$ as $c \to 0$.

Making use of (55) it can be seen that whenever $1 \leqq s_N \leqq N - 1$,

$$\frac{1}{H_N(s_N)} \leqq \frac{s_{N-1}(N-1-s_{N-1})(N-2)}{N(N+1)} + \frac{(N-2)}{(N+1)} \leqq \frac{1}{c(N-3)} + 1 , \tag{61}$$

which in turn implies that

$$H_N(s_N) \geqq c(N-3) - c^2(N-3)^2 . \tag{62}$$

Thus,

$$E_p H_N(s_N)/c^{\frac{1}{2}} \geqq c^{\frac{1}{2}} E_p(N-3)I_A - c^{\frac{3}{2}} E_p N^2 I_A , \tag{63}$$

and by (51) it follows that as $c \to 0$,

$$\liminf E_p H_N(s_N)/c^{\frac{1}{2}} \geqq \frac{1}{(pq)^{\frac{1}{2}}} . \tag{64}$$

By Fatou's lemma, as $c \to 0$

$$\liminf B(N, \delta_N{}^*)/c^{\frac{1}{2}} \geqq 2\pi \tag{65}$$

and hence as $c \to 0$

$$B(N, \delta_N{}^*)/c^{\frac{1}{2}} \to 2\pi , \tag{66}$$

as was to be proved.

6. Results for general beta priors. If instead of a uniform prior on p, a beta prior (23) with $a, b > 1$ is considered, then the problem exhibits the monotone property for *all* sample paths. In particular if $a, b \geqq 2$, then the Bayes estimator is given by

$$\delta_n{}^* = \frac{s_n + a - 2}{n + a + b - 4} , \qquad n \geqq 1 , \tag{67}$$

and (29) becomes

$$H_n(s_n) = \frac{(n+a+b-1)(n+a+b-2)}{(s_n+a-1)(f_n+b-1)(n+a+b-4)} , \qquad n \geqq 1 . \tag{68}$$

The optimal rule is simply

(69) $N =$ first $n \geqq 1$ such that

$$(s_n + a - 1)(f_n + b - 1) \geqq \frac{1}{c} \frac{(n+a+b-1)(n+a+b-2)}{(n+a+b-3)(n+a+b-4)} .$$

As before, for any fixed $0 < p < 1$, this rule satisfies properties (8) and (9). Further, as $c \to 0$

$$c^{\frac{1}{2}} EN \to (a + b - 1) \frac{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})}{\Gamma(a)\Gamma(b)} . \tag{70}$$

Making use of the above, it may be shown in a manner similar to the proof of (66), that, in this case, as $c \to 0$

$$\frac{B(N, \delta_N^*)}{c^{\frac{1}{2}}} \to 2(a + b - 1) \frac{\Gamma(a - \frac{1}{2})\Gamma(b - \frac{1}{2})}{\Gamma(a)\Gamma(b)} .$$

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