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ON UNEQUALLY SPACED TIME POINTS IN TIME SERIES

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This article discusses the sampling of stationary discrete-time stochastic processes at fixed but unequally spaced time points. The pattern of the sampling times is periodic with a cycle of p time units. One of the major problems is to determine given p the minimum number of sampling points required per cycle in order to estimate the covariances at all lags. The second problem is to find a pattern of distribution for the sampling points within the cycle which will allow the estimation of all covariances. A discussion of the references which describe the statistical properties of the estimates of covariances and spectra in this sampling situation is given.

1. Introduction. One of the authors (Van Ness) was approached by oceanographers who wanted to study the spectra of certain time-varying features of the ocean (salinity, temperature, etc.). They wished to measure these features at several different depths and for a length of time of the order of magnitude of the cruise time. However, they had only one sensing device on board and could not keep the ship out long enough to hold the device at each depth for a suitable length of time. They proposed, therefore, to lower and raise the sensor in a cyclic fashion keeping its velocity magnitude constant, thus giving a depth versus time plot as shown in Figure 1.

At a fixed depth, one has a time series $\{X(t_1), X(t_2), \cdots\}$ observed at unequally spaced time points. If $t_3 - t_2$ and $t_2 - t_1$ are both integral multiples of some number, the problem involves observations at unequally but cyclically spaced time points of a discrete time series.

We generalize this problem as follows. Given an integer $p > 2$, to be called the sampling period, let the sampling pattern, $S = \{s_1, \cdots, s_k\}$, be a subset of the integers from 0 to $p - 1$. Sample $\{X(t)\}$ only at those values of t for which $0 \leqq t \leqq T$ and $t \pmod p) \in S$. For example, if $\{Z(t)\}$ is the underlying continuous time process in the example of Figure 1 and $t_2 - t_1$ is an integral multiple of $t_3 - t_2$, one could define $X(t) = Z(t(t_3 - t_2) + t_1)$, $p = (t_3 - t_1)/(t_3 - t_2)$ and $S = \{0, (t_2 - t_1)/(t_3 - t_2)\}$. General sampling patterns are treated in this paper.

Jones (1962 and 1971) and Parzen (1963) discuss special cases of this problem. The latter gives other examples where such problems arise. Loynes (1970) reviews the state of the literature on random processes observed at unequally spaced time points. As Loynes (1969) points out, very little additional work

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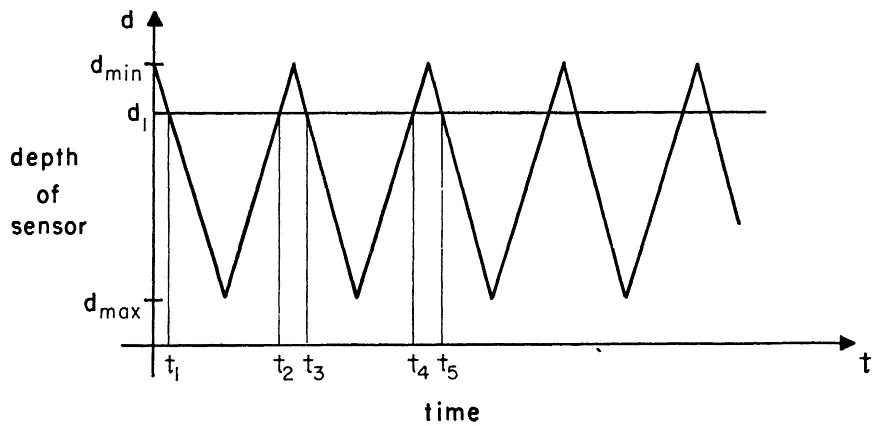

FIG. 1. Oceanographer's suggested sampling plan.

has been done on the case of missing observations at fixed (nonrandom) time points.

The major concern in this paper is to find the minimum number of observations per cycle which will allow estimation of the complete covariance sequence and spectrum in the discrete time case.

2. Cyclic sampling. Let $\{X_t, t = 0, \pm 1, \pm 2, \cdots\}$ be a stationary process with mean zero and finite second-order moments. Let p and $S = \{s_1, \cdots, s_k\}$ be as defined above, only for convenience assume S is ordered so that $0 \leqq s_1 < s_2 < \cdots < s_k \leqq p - 1$. Denote the covariance of lag v by $R(v) = EX(t)X(t + v)$; $v, t = 0, \pm 1, \pm 2, \cdots$.

Assume $\{X(t)\}$ has been sampled according to S over a time period $[0, T]$ giving observations $\{X_t : t \,(\mathrm{mod}\, p) \in S, 0 \leqq t \leqq T\}$. It is convenient to use the following picture: divide a circle into p equal segments and number the division points from 0 to $p - 1$. Put a dot at those points whose numbers are in S, e.g. if $p = 8$ and $S = \{0, 1, 4\}$, we have Figure 2.

Define Q to be a "translation operator" on S such that

$$\begin{aligned} Q(S) &\equiv Q(s_1, s_2, \cdots, s_k) \\ &\equiv (s_1 + 1 \,(\mathrm{mod}\, p), s_2 + 1 \,(\mathrm{mod}\, p), \cdots, s_k + 1 \,(\mathrm{mod}\, p)) \,. \end{aligned}$$

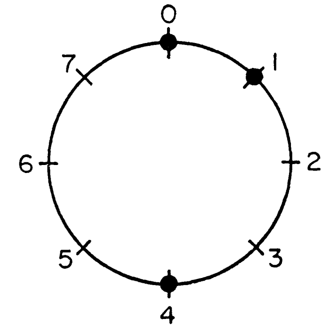

FIG. 2. Graphical representation of S.

Q shifts the dots in our picture one unit clockwise. Define $Q^2(S) \equiv Q(Q(S))$. It will also be convenient to use the notation of Parzen (1963). Let

$$(2.1) \qquad \begin{aligned} g(t) &= 0 \quad \text{if } X(t) \text{ is not observed} \\ &= 1 \quad \text{if } X(t) \text{ is observed} \end{aligned}$$

i.e. $g(t) = 1$ if and only if $t \pmod p \in S$. Define

$$(2.2) \qquad Y(t) = g(t)X(t),$$

then $\{Y(t)\}$ is a particular amplitude modulated version of $\{X(t)\}$. Note that if $g(t) = 1$ then $g(t + jp) = 1$ for all integers j.

3. Estimation of covariances. The natural estimates $R^*(v)$ of the covariances $R(\pm v)$, $v \geqq 0$ are

$$(3.1) \qquad \begin{aligned} R^*(v) = R^*(-v) &\equiv \frac{1}{M_T(v)} \sum_{t=0}^{T-v} Y(t)Y(t+v) \\ &= \frac{1}{M_T(v)} \sum_{t=0}^{T-v} g(t)X(t)g(t+v)X(t+v) \end{aligned}$$

when $M_T(v) = \sum_{t=0}^{T-v} g(t)g(t+v) > 0$. If $M_T(v) = 0$ for all T we cannot use (3.1), no matter how large the sample.

First we determine those v for which $M_T(v) > 0$ for some T. Due to symmetry we need henceforth only consider $v \geqq 0$. First note that $M_T(v) > 0$ for some T if and only if $Q^v(S) \cap S \neq \emptyset$.

A simple way of determining which $R^*(v)$ exist can be given using the pictures corresponding to Figure 2. Let card A denote the number of points in the set A. Then $k =$ card S is the number of observations per p time units. Take the circle with p segment division points and K dots distributed as described in Section 2. Draw lines connecting each dot to each other dot. With each of these $\binom{k}{2}$ lines associate two numbers giving the number of segments between its two dots counting in the two possible directions. Define V to be the set containing 0 and all the different numbers so obtained. Then one can form $R^*(v)$ if and only if $|v| \pmod p \in V$. Note that card $V \leqq 2\binom{k}{2} + 1 = k^2 - k + 1$.

Secondly we would like to know the size of $M_T(v)$. Let $[\cdot]$ denote the greatest integer function.

PROPOSITION 1.

$$\left(\left[\frac{T-v}{p}\right]\right) \operatorname{card}(Q^v(S) \cap S) \leqq M_T(v) \leqq \left(\left[\frac{T-v}{p}\right] + 1\right) \operatorname{card}(Q^v(S) \cap S).$$

COROLLARY 2.

$$\lim_{T\to\infty} M_T(v)/T = \frac{1}{p} \operatorname{card}(Q^v(S) \cap S) \equiv m(v).$$

Following Parzen (1963) we define

$$(3.2) \qquad R_T(v) = R_T(-v) = \frac{1}{Tm(v)} \sum_{t=0}^{T-v} Y(t)Y(t+v),$$

when $m(v) > 0$, which is a simple estimate useful in spectral estimation (see Section 7).

4. Number theory results. In number theory, if $\{0, 1, 2, \cdots, p-1\} \subset V$ then S is called a difference cover for p. We wish to find how few observations we can get away with and still estimate all covariances using (3.1) or (3.2). Thus we are interested in the minimum cardinality that a difference cover for p can have. Call this number $f(p)$. As we have shown, $p \leqq f^2(p) - f(p) + 1$.

If equality holds in the previous equation then every integer $0, 1, \cdots, p-1$ is uniquely expressible as a difference. Such difference coverings are called simple difference coverings. The existence of known simple difference coverings is given by

THEOREM 3 (Singer (1938)). *A sufficient condition for the existence of* $k+1$ *integers* $v_1, \cdots, v_{k+1}$, *having the property that their* k^2+k *differences* $v_j - v_i$; $i \neq j$; $i, j = 1, \cdots, k+1$, *are congruent, modulo* $p = k^2 + k + 1$, *to* $1, 2, \cdots, k^2 + k$ *in some order is that* k *be a power of a prime.*

It has been conjectured that no simple difference set exists for $p = m^2 + m + 1$ when m is not a prime power. This conjecture has been verified for the 1321 cases in which $m \leqq 1600$, but has not been proved for all m (Mann (1965), page 86). A number of criteria for the existence of simple difference sets have been established however (see Mann (1965) or Ryser (1963) for discussion and bibliography).

A trivial covering gives the following result.

PROPOSITION 4. *If* $p \leqq 2m(n+1) + 1$ *when* m, n *positive integers*, $n \geqq 2$, *then* $f(p) \leqq m + n$.

COROLLARY 5. $1 < f^2(p)/p \leqq 2$ *for* $p \geqq 12$.

PROOF. Suppose $f^2(p) \leqq p$, then $f^2(p) \leqq f^2(p) - f(p) + 1$, giving $f(p) = 1$ which yields $p = 1$. Thus $f^2(p) > p$.

Let x be the least integer such that $x \geqq (2p+1)^{\frac{1}{2}} - 1$. Then $2p \leqq x^2 + 2x$. If x is even, $p \leqq \frac{1}{2}x^2 + x = 2(\frac{1}{2}x)(\frac{1}{2}x + 1)$ so by Proposition 4 $f(p) \leqq \frac{1}{2}x + \frac{1}{2}x = x$. If x is odd, so is $x^2 + 2x$, so $2p \leqq x^2 + 2x - 1$, from which $p \leqq (x^2 - 1)/2 + x < 2((x+1)/2)((x-1)/2 + 1)$, and again by Proposition 4, $f(p) \leqq (x+1)/2 + (x-1)/2 = x$ provided $(x-1)/2 \geqq 2$, which is the case for odd x corresponding to $p \geqq 12$. Therefore, $f(p) < (2p+1)^{\frac{1}{2}}$. Suppose $f^2(p) > 2p$. Then $2p < f^2(p) < 2p + 1$, which is impossible; hence $f^2(p) \leqq 2p$, for $p \geqq 12$. (By inspection of cases, $f^2(p) \leqq 2p$ holds for all p except $p = 4$.) □

An asymptotic upper bound for $f^2(p)/p$ follows from the work of Redei and Rényi (1949) on difference bases. A difference basis with respect to n is a set of integers $a_1, \cdots, a_k$ such that every positive integer $0 < v \leqq n$ can be written in the form $v = a_j - a_i$ for some $i \neq j$. Let $k(n)$ denote the minimum of k for given n. Redei and Rényi showed that $\lim_{n\to\infty} k^2(n)/n$ exists, and gave numerical bounds which were later improved by Leech (1956) and then Golay (1972). Since a difference basis for n is immediately a difference cover for $p \leqq 2n + 1$, it follows that

$$\lim_{p_0 \to \infty} \sup_{p > p_0} \frac{f^2(p)}{p} \leqq \tfrac{1}{2} \lim_{n \to \infty} \frac{k^2(n)}{n} < 1.3286 .$$

Actual calculation of $f(p)$ appears very difficult. The congruence conditions stated in the next theorem are often useful, especially for p having small divisors, but no reasonable algorithm has been developed for arbitrary p values. The following corollaries will be used later to get some further practical partial results about f.

THEOREM 6. *Let n be a divisor of p. Let $v_1, \cdots, v_b$ be a difference cover for p and let x_i be the number of elements in the cover congruent to i modulo n. Then*

$$\textstyle\sum_{i=0}^{n-1} x_i = b ;$$

$$\textstyle\sum_{i=0}^{n-1} x_i(x_i - 1) \geqq \frac{p}{n} - 1 ;$$

$$\textstyle\sum_{i=0}^{n-1} x_i x_{\mathrm{rem}\,(i+j)\bmod n} \geqq \frac{p}{n} , \quad j = 1, \cdots, n-1 ,$$

where rem (i) mod n *denotes the reduced residue representative congruent to i modulo n.*

PROOF. Consider the integers $1, \cdots, p-1$. $p/n - 1$ of them are congruent to $0 \bmod n$, and p/n are congruent to $j \bmod n$ for $j = 1, \cdots, n-1$. The inequalities merely state that sufficient differences $v_j - v_i$ are congruent to $0, 1, \cdots, n-1$ so that p may be covered by the difference cover. □

COROLLARY 7. *Let the conditions of Theorem* 6 *hold. Then* $\sum_{i=0}^{n-1} x_i^2 \leqq b^2 - ((n-1)/n)p$.

PROOF.

$$\begin{aligned} b^2 &= (\textstyle\sum_{i=0}^{n-1} x_i)^2 \\ &= \textstyle\sum_{i=0}^{n-1} x_i^2 + \sum_{j=1}^{n-1} \sum_{i=0}^{n-1} x_i x_{\mathrm{rem}\,(i+j)\bmod n} \\ &\geqq \textstyle\sum_{i=0}^{n-1} x_i^2 + \sum_{j=1}^{n-1} \frac{p}{n} . \end{aligned}$$

□

Furthermore, equality holds if $b = m + 1$ and $p = b^2 - b + 1 = m^2 + m + 1$.

COROLLARY 8. *Let the conditions of Theorem* 6 *hold with $b = m + 1$ and $p = m^2 + m + 1$, then*

$$\textstyle\sum_{i=0}^{n-1} x_i^2 = (m+1)^2 - \frac{n-1}{n}(m^2 + m + 1) = \frac{1}{n}[m^2 + (n+1)m + 1] .$$

COROLLARY 9. *Suppose $f^2(p)/p \to 1$. Let $n \geqq 2$. For each $p = kn$, $k = 1, 2, 3, \cdots$, let $V_k = \{v_1, \cdots, v_{f(p)}\}$ be a difference cover for p, and let x_{ki} be the number of elements of V_k congruent to i modulo n, $i = 0, \cdots, n-1$. Then*

$$\lim_{k \to \infty} \frac{1}{k} \textstyle\sum_{i=0}^{n-1} x_{ki}^2 = 1 .$$

PROOF. For each $k = 1, 2, \cdots, f(kn) = \sum_{i=0}^{n-1} x_{ki}$. By a well-known isoperimetric inequality, then,

$$\frac{1}{k}\sum_{i=0}^{n-1} x_{ki}^2 \geqq \frac{1}{k}\sum_{i=0}^{n-1}\left[\frac{f(kn)}{n}\right]^2$$
$$> \frac{1}{k}\sum_{i=0}^{n-1}\left[\frac{(kn)^{\frac{1}{2}}}{n}\right]^2 .$$

Let $\varepsilon > 0$. Let p_0 be such that $p > p_0$ implies $f^2(p)/p < 1 + \varepsilon/n$. Let k_0 be such that $k_0 n > p_0$. For $k > k_0$, then,

$$1 < \frac{1}{k}\sum_{i=0}^{n-1} x_{ki}^2$$
$$\leqq \left(\frac{f^2(kn)}{kn} - \frac{n-1}{n}\right)n$$
$$< \left(1 + \frac{\varepsilon}{n} - \frac{n-1}{n}\right)n . \qquad \square$$

The next few corollaries apply the congruence conditions modulo 2.

COROLLARY 10. *Let m and k be integers with $m \geqq 1$, $k \geqq 0$. If m is odd (even) and $f(m^2 + m - 2k) = m + 1$, then there exists an even (odd) integer y such that*

$$m - (2k + 1) \leqq y^2 \leqq m + (2k + 1) .$$

PROOF. Given a difference cover of $m + 1$ elements for $p = m^2 + m - 2k$, let x be the number of elements congruent to 0 mod 2; then $(m + 1) - x$ elements are congruent to 1 mod 2. We may assume $x \leqq (m + 1)/2$ (for if that is not so, adding 1 to each element of the cover produces a new cover in which it is true). Since $m^2 + m - 2k$ is even, Theorem 6 gives

$$x(x-1) + (m + 1 - x)(m - x) \geqq \frac{m^2 + m - 2k}{2} - 1$$

and

$$2x(m + 1 - x) \geqq \frac{m^2 + m - 2k}{2} .$$

Thus

$$x \leqq \frac{m + 1 - (m - (2k + 1))^{\frac{1}{2}}}{2}$$

and

$$\frac{m + 1 - (m + 2k + 1)^{\frac{1}{2}}}{2} \leqq x \leqq \frac{m + 1 + (m + 2k + 1)^{\frac{1}{2}}}{2}$$

giving

$$m - (2k + 1) \leqq [(m + 1) - 2x]^2 \leqq m + (2k + 1)$$

and set $y = (m + 1) - 2x$. $\square$

COROLLARY 11. *For every positive integer k_0, there exists an integer m such that*

$$f(m^2 + m - 2k) > m + 1 \qquad \text{for } k = 0, 1, \cdots, k_0 - 1 .$$

PROOF. Given k_0, let $m = (k_0 + 1)^2$. Then

$$m - (2k + 1) = k_0^2 + 2k_0 - 2k > k_0^2$$

and

$$\begin{aligned} m + (2k + 1) &= k_0^2 + 2k_0 + 2k + 2 \\ &\leqq k_0^2 + 2k_0 + 2k_0 < (k_0 + 2)^2 , \qquad k = 0, 1, \cdots, k_0 - 1 , \end{aligned}$$

so by the preceding corollary $f(m^2 + m - 2k) > m + 1$ for $k = 0, 1, \cdots, k_0 - 1$. □

The congruence conditions for p divisible by three are a little more complicated.

COROLLARY 12. *Let $p \equiv 0$ modulo 3. Consider a difference cover for p of b elements and let x_i $(i = 0, 1, 2)$ denote the number of elements in the cover congruent to i modulo 3. If $A_j = -3x_j^2 + 2bx_j + b^2 - 4p/3$, and $B_j = -3x_j^2 + 2bx_j - b^2 + 2b - 2 + 2p/3$, then*

1) $x_0 + x_1 + x_2 = b$
2) $(b - 2(b^2 - p)^{\frac{1}{2}})/3 \leqq x_i \leqq (b + 2(b^2 - p)^{\frac{1}{2}})/3$, $i = 0, 1, 2$
3) $(b - x_j - A_j^{\frac{1}{2}})/2 \leqq x_i \leqq (b - x_j + A_j^{\frac{1}{2}})/2$, $i \neq j$
4) $x_i \leqq (b - x_j - B_j^{\frac{1}{2}})/2$ *or* $x_i > (b - x_j + B_j^{\frac{1}{2}})/2$, $i \neq j$, *if* $B_j > 0$.

PROOF. From Theorem 6 (sums are from 0 to 2):

$$\sum x_i = b ,$$

$$\sum x_i(x_i - 1) \geqq \frac{p}{3} - 1 ,$$

$$x_0x_1 + x_1x_2 + x_2x_0 \geqq \frac{p}{3} .$$

Thus

$$x_2 = b - x_0 - x_1 ,$$

$$x_0(x_0 - 1) + x_1(x_1 - 1) + (b - x_0 - x_1)(b - x_0 - x_1 - 1) \geqq \frac{p}{3} - 1 ,$$

$$x_0x_1 + x_1(b - x_0 - x_1) + (b - x_0 - x_1)x_0 \geqq \frac{p}{3} ,$$

or

$$2x_0^2 - 2bx_0 + 2x_1^2 - 2bx_1 + 2x_0x_1 + b^2 - b + 1 - \frac{p}{3} \geqq 0 , \tag{4.1}$$

$$-x_0^2 - x_1^2 - x_0x_1 + bx_0 + bx_1 - \frac{p}{3} \geqq 0 . \tag{4.2}$$

$A_1 < 0$ gives no solution to (4.2), so

$$\frac{b - 2(b^2 - p)^{\frac{1}{2}}}{3} \leqq x_1 \leqq \frac{b + 2(b^2 - p)^{\frac{1}{2}}}{3} .$$

Then, from (4.2)

$$\frac{(b - x_1) - A_1^{\frac{1}{2}}}{2} \leqq x_0 \leqq \frac{(b - x_1) + A_1^{\frac{1}{2}}}{2}.$$

From (4.1), if $B_1 > 0$, then

$$x_0 \leqq \frac{(b - x_1) - B_1^{\frac{1}{2}}}{2} \quad \text{or} \quad x_0 \geqq \frac{(b - x_1) + B_1^{\frac{1}{2}}}{2}.$$

The symmetry of the congruence conditions permits interchanging the subscripts 0, 1, 2. □

COROLLARY 13. *Let $m \equiv 1$ modulo 3. Set $p = m^2 + m + 1$ and let x_0, x_1 and x_2 be defined as in the preceding corollary. If $b = m + 1$*

1) $(m + 1 - 2m^{\frac{1}{2}})/3 \leqq x_0, x_1, x_2 \leqq (m + 1 + 2m^{\frac{1}{2}})/3$; *and*

2) *if* $D = -3x_0^2 + 2(m + 1)x_0 - \frac{1}{3}(m - 1)^2$, *then* $x_1 = (m + 1 - x_0 \pm D^{\frac{1}{2}})/2$ *and* $x_2 = (m + 1 - x_0 \mp D^{\frac{1}{2}})/2$; *and*

3) *two of the x_i are congruent to* 1 *modulo* 3, *and the other is congruent to* 0 *modulo* 3.

PROOF. Part 1 is just the preceding corollary. In the preceding corollary, $A_0 = B_0 = D$ and this together with the fact that $x_0 + x_1 + x_2 = m + 1$, gives part 2.

$D \equiv x_0 \bmod 3$. From part 2, $D \equiv (2x_1 + x_0 - m - 1)^2$ modulo 3, from which $x_0 \equiv 0$ or $x_0 \equiv 1 \bmod 3$ since 2 is a quadratic nonresidue mod 3. If $x_0 \equiv 0$, then $2x_i \equiv m + 1$ gives $x_i \equiv 1$. If $x_0 \equiv 1$, then $2x_i \equiv 2$ or $2x_i \equiv 0$, giving $x_i \equiv 0$ or $x_i \equiv 1$; since $x_0 + x_1 + x_2 = m + 1 \equiv 2$, one of x_1, x_2, is congruent to 0 and the other to 1 mod 3. □

5. Results. The previous corollaries provide partial information about $f(p)$. For example, Corollary 10 shows that $f(20) \geqq 6$ and $f(42) \geqq 8$ while Corollary 12 shows that $f(30) \geqq 7$ and Corollary 13 shows $f(111) \geqq 12$.

Using these corollaries, published results and calculations of difference covers in special cases we can construct Table 1. Obviously $f(p)$ is still difficult to compute.

6. Convergence. In view of Corollary 5 and the result on difference bases, it is natural to ask if the ratio $f^2(p)/p$ converges. If it does, then by Singer's theorem it must converge to 1. As of now it is not even known that f can take on a value as much as two above the bound given by $p \leqq f^2(p) - f(p) + 1$.

An interesting necessary condition for convergence is given by Corollary 9. If n divides p, then a smallest difference cover must be fairly evenly distributed among the residues mod n in order to satisfy Theorem 6, but Corollary 9 states that, if $f^2(p)/p \to 1$, then a smallest difference cover for large multiples of n must be almost perfectly distributed.

7. Conclusions. While we have made progress in determining optimal sampling

TABLE 1
f(p) for small p

p	$f(p)$	*a difference cover*								
1	1	0								
2	2	0	1							
3	2	0	1							
4	3	0	1	2						
5	3	0	1	4						
6	3	0	1	3						
7	3	0	1	3						
8	4	0	1	3	7					
9	4	0	1	3	7					
10	4	0	1	3	6					
11	4	0	1	4	6					
12	4	0	1	4	6					
13	4	0	1	3	9					
14	5	0	1	3	7	9				
15	5	0	1	3	7	10				
16	5	0	1	3	7	11				
17	5	0	1	3	7	12				
18	5	0	1	3	10	15				
19	5	0	1	3	9	15				
20	6	0	1	3	9	13	15			
21	5	0	1	4	14	16				
22	6	0	1	4	9	11	15			
23	6	0	1	4	11	13	19			
24	6	0	1	3	7	12	17			
25	6	0	1	3	7	12	15			
26	6	0	1	4	14	19	21			
27	6	0	1	4	10	12	17			
28	6	0	1	4	15	20	22			
29	?									
30	7	0	1	2	6	10	13	16		
31	6	0	1	3	8	12	18			
32	7	0	1	4	9	11	17	23		
33	7	0	1	2	3	8	13	17		
34	7	0	1	2	3	8	13	17		
35	7	0	1	2	3	8	13	17		
36	7	0	1	11	16	19	23	25		
37	7	0	1	6	10	17	23	35		
38–41	?									
42	8	0	1	6	10	19	26	37	40	
43	8	0	8	18	19	22	24	31	39	
44	8	0	8	18	19	22	24	31	39	
45	8	0	8	18	19	22	24	31	39	
46	8	0	8	18	19	22	24	31	39	
47	8	0	8	18	19	22	24	31	39	
48	8	0	8	18	19	22	24	31	39	
49	8	0	8	18	19	22	24	31	39	
50–55	$\leqq 9$									
56	9	0	1	10	19	21	37	43	49	52
57	8	0	1	3	13	32	36	43	52	

(*Recall Mann* (1965) *has results for* $p = m^2 + m + 1$, $m \leqq 1600$.)

patterns, the general answer seems extremely difficult. If one has a sample which does allow the estimation of all covariances, questions concerning the estimates and their properties are discussed in Parzen (1963).

Note the strange behavior of $f(p)$ shown by Table 1. For example, 6 points per cycle are required for $p = 20$ but only 5 for $p = 21$. This disturbance in the order of $f(p)$ tends to occur around the points $p = m^2 + m + 1$, $m = 4, 5, 6, \cdots$.

Interesting open questions besides the complete determination of $f(p)$ remain. One, mentioned above, is whether $f(p)$ is ever more than one above the bound given by $p \leqq f^2(p) - f(p) + 1$. A second is how many different difference covers for p exist. Of course, one could try to add information to Table 1. We have, for example, written an APL program which conducts a trial and error search for difference covers. It was this program that found the smallest difference covers for $p = 36, 37, 42, 56$ and found difference covers of size 9 for $p = 50, \cdots, 55$.

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