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ON INVARIANCE AND RANDOMIZATION IN FRACTIONAL REPLICATION[1]

BY J. N. SRIVASTAVA, B. L. RAKTOE AND H. PESOTAN

Colorado State University and University of Guelph

This paper generalizes the results of Paik and Federer (1970), Ehrenfeld and Zacks (1961) and Zacks (1963, 1964) regarding invariance and randomization in fractional replication. It is shown that (i) the characteristic roots of the information matrix of a design in the general factorial relative to an admissible vector of effects remain invariant under a permutation of levels; (ii) the unbiased estimation of a linear function of an admissible vector of effects can be obtained under equal probability randomization. In addition some applications of the results are indicated.

1. Introduction. Ever since the concept of fractional replication was introduced, many researchers have contributed either to the construction or the analysis of these designs. More recent writings (Ehrenfeld and Zacks (1961), Zacks (1963, 1964), Paik and Federer (1970)) have brought out some of the randomization and invariance aspects of regular and irregular fractional replicates. These papers dealt mostly with proper fractions from the symmetrical factorials.

This paper considers arbitrary fractions from factorials in which the number of levels of each factor is a positive integer. Section 2 studies the class of all row permuted matrices generated from a submatrix of a real orthogonal matrix of order s with each entry in the first column equal to $1/s^{\frac{1}{2}}$, and, establishes that any matrix in this class is related to another by post multiplication with an orthogonal matrix. Further, a property of the resulting class of orthogonal matrices is noted and these results are then utilized in Section 3. In this section the characteristic roots invariance of the information matrix of a fraction relative to an *admissible vector* of effects under a permutation of levels is established in the general factorial setting. This generalizes the result of Paik and Federer (1970) which was proved in the case of the symmetrical factorial for a parametric vector consisting of the main effects and the mean. Section 4 establishes the unbiased estimation of a linear function of an admissible vector of effects under equal probability randomization. The main result of this section broadens the randomization results of Ehrenfeld and Zacks (1961) and Zacks (1963, 1964) for regular fractions to arbitrary fractions. In the final section possible applications are indicated.

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2. Preparatory lemma. Let $C = (c_{ij})$ be a real orthogonal matrix of order s ($s \geqq 2$) with $c_{i1} = 1/s^{\frac{1}{2}}$. It follows that the sum of the entries in each column of C besides the first is zero. Delete the first column of C and call the resulting matrix C^*. Let $\{C_1^*, C_2^*, \cdots, C_{s!}^*\}$ be the set of all row permuted matrices obtained from C^* and suppose $C^* = C_1^*$. Let C_j be the matrix of order s obtained from C_j^* by adjoining a first column, each entry of which is $1/s^{\frac{1}{2}}$. The following lemma will be useful in the next section.

LEMMA 2.1. *There exists an orthogonal matrix* U_j *such that*

$$C^* U_j = C_j^* .$$

Moreover,

$$\sum_{j=1}^{s!} U_j = 0 ,$$

where 0 *is the zero matrix.*

PROOF. Observe that the matrix $C_1'C_j = C'C_j$ is an orthogonal matrix which has for its first row the $1 \times s$ vector $\mathbf{a}' = (1, 0, 0, \cdots, 0)$ and for its first column the vector $\mathbf{a}$. Let U_j be the lower right matrix of order $s - 1$ in $C_1'C_j$. The first part of the lemma can now be verified. The second part of the lemma is a consequence of the definition of U_j, and, the observation that the sum of the elements in any column of C_j besides the first is zero.

3. The invariance theorem. Paik and Federer (1970) established the characteristic root invariance of the information matrix of a fraction from the symmetric factorial with respect to the main effects and the mean under a permutation of levels. The purpose of this section is to generalize this result to the setting of the general factorial where the information matrices of fractions are taken relative to an admissible vector of effects.

Consider the general $s_1 \times s_2 \times \cdots \times s_m$ factorial, $s_i \geqq 2$, where the ith factor has s_i *levels* from the set $S_i = \{0, 1, 2, \cdots, s_i - 1\}$. Let $S = S_1 \times S_2 \times \cdots \times S_m$ be the Cartesian product of the sets S_i. With each *treatment* $(i_1, i_2, \cdots, i_m)$ in S associate an *observation* and an *effect* denoted by $y(i_1, i_2, \cdots, i_m)$ and $A_1^{i_1} A_2^{i_2} \cdots A_m^{i_m}$, respectively. Let Y_0 be the set of all observations and P_0 the set of all effects. Let $X = X_1 \otimes X_2 \otimes \cdots \otimes X_m$ be the Kronecker product of real orthogonal matrices X_i of order s_i with each first column entry in X_i equal to $1/(s_i)^{\frac{1}{2}}$. Then X is a real orthogonal matrix of order $s = \pi s_i$ with each first column entry equal to $1/s^{\frac{1}{2}}$. Associate with the observation vector $\mathbf{Y}_0$ and the column vector $\mathbf{P}_0$ of single degree of freedom parameters the well-known linear model

$$E(\mathbf{Y}_0) = X\mathbf{P}_0 , \qquad \operatorname{Cov}(\mathbf{Y}_0) = \sigma^2 I_s .$$

Let $\mathbf{P}$ be a $K \times 1$ vector of single degree of freedom parameters and let $\mathbf{P}_0' = (\mathbf{P}', \mathbf{Q}')$. Then the implied linear model for a $N \times 1$ observation vector $\mathbf{Y}$ under the assumption that $\mathbf{Q} = \mathbf{0}$ is

$$E(\mathbf{Y}) = X(\mathbf{Y}, \mathbf{P})\mathbf{P} , \qquad \operatorname{Cov}(\mathbf{Y}) = \sigma^2 I_N ,$$

where $X(\mathbf{Y}, \mathbf{P})$ is the $N \times K$ submatrix of X read off from X relative to $\mathbf{Y}$ and $\mathbf{P}$.

The following concept is needed in order to formulate the main result of this section. A set of effects P will be called *admissible* iff whenever $A_1^{i_1} A_2^{i_2} \cdots A_m^{i_m}$ belongs to P and $i_j \neq 0$ $(1 \leqq j \leqq m)$ then $A_1^{i_1} A_2^{i_2} \cdots A_{j-1}^{i_{j-1}} A_j^{l} A_{j+1}^{i_{j+1}} \cdots A_m^{i_m}$ belongs to P for all $l \neq 0$ in the set S_j. An admissible set of effects can be characterized in the following way. Define formal row vectors of effects of the ith factor by

$$v_i^0 = (A_i^0) ,$$
$$v_i^1 = (A_i^1, A_i^2, \cdots, A_i^{s_i-1}) .$$

For $l_j \in \{0, 1\}$ and $1 \leqq j \leqq m$, let

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha}'(l_1, \cdots, l_m) = v_1^{l_1} \otimes v_2^{l_2} \otimes \cdots \otimes v_m^{l_m} ,$$

be the formal Kronecker product of these formal row vectors. A set of effects α will be called a *basic collection* if and only if there exists a sequence $(l_j)_{j=1}^m$ with $l_j \in \{0, 1\}$, such that α is the set of all effects formed from the entries of the row vector $\boldsymbol{\alpha}'(l_1, l_2, \cdots, l_m)$. It is clear that any basic collection of effects is admissible. Moreover, one can establish that *a set P of effects is admissible if and only if P is a disjoint union*: $P = \bigcup_{j=1}^k \alpha_j$, where α_j is a basic collection of effects. Since P_0, the set of all effects, is admissible, it follows that, *if P is admissible then the set $P_0 - P$ is admissible.*

Let Ω_i be the symmetric group of all permutations on the set S_i and let Ω be the direct product of these groups. Then $\Omega = \{\omega : \omega = (\omega_1, \cdots, \omega_m) \text{ with } \omega_i \in \Omega_i\}$. An element ω in Ω will be called a *level permutation*. If $\mathbf{Y}$ is any observation vector and ω in Ω is any level permutation then $\omega(\mathbf{Y})$ will denote the observation vector obtained from $\mathbf{Y}$ wherein each component $y(k_1, k_2, \cdots, k_m)$ in $\mathbf{Y}$ is replaced by $y(\omega_1(k_1), \omega_2(k_2), \cdots, \omega_m(k_m))$.

Let α be a basic collection of effects and let X_j^* be the matrix obtained by deleting the first column of X_j. Then $\boldsymbol{\alpha}' = v_1^{l_1} \otimes v_2^{l_2} \otimes \cdots \otimes v_m^{l_m}$ for a suitable sequence $l_j \in \{0, 1\}$. If $\mathbf{Y}$ is any observation vector then observe, that it follows from the Kronecker factorization of $\boldsymbol{\alpha}'$ given above, that the row of the matrix $X(\mathbf{Y}, \boldsymbol{\alpha})$ corresponding to the entry $y(i_1, \cdots, i_m)$ in $\mathbf{Y}$ is given by the Kronecker product

$$M_{i_1}^1 \otimes M_{i_2}^2 \otimes \cdots \otimes M_{i_m}^m , \tag{3.1}$$

where $M_{i_j}^j = (1/(s_j)^{\frac{1}{2}})$, the one by one matrix with entry $1/(s_j)^{\frac{1}{2}}$, if $l_j = 0$, and, $M_{i_j}^j$ is the i_jth row of X_j^*, if $l_j = 1$.

The main result of this section is the following.

THEOREM 3.1. *For each level permutation ω in Ω, a basic collection of effects α, and, any observation vector Y, there exists an orthogonal matrix $U(\boldsymbol{\alpha}, \omega)$ such that*

$$X(\mathbf{Y}, \boldsymbol{\alpha})U(\boldsymbol{\alpha}, \omega) = X(\omega(\mathbf{Y}), \boldsymbol{\alpha}) .$$

Further

$$\frac{1}{\pi(s_i!)} \sum_{\omega \in \Omega} U(\boldsymbol{\alpha}, \omega) = 0\,, \qquad \text{if} \quad \boldsymbol{\alpha}' \neq (A_1^0\, A_2^0 \cdots A_m^0)\,,$$
$$= I\,, \qquad \text{if} \quad \boldsymbol{\alpha}' = (A_1^0\, A_2^0 \cdots A_m^0)\,,$$

where 0 *is the zero matrix and* I *is the one by one identity matrix.*

PROOF. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ be a level permutation and suppose $\boldsymbol{\alpha}' = v_1^{l_1} \otimes v_2^{l_2} \otimes \cdots \otimes v_m^{l_m}$ with $l_j \in \{0, 1\}$. Let Z_j^* be the row permuted matrix obtained from X_j^* whose ith row is the $\omega_j(i)$th row of X_j^*. According to Lemma 2.1, there exists an orthogonal matrix U_j such that

$$X_j^* U_j = Z_j^*\,. \tag{3.2}$$

Define

$$U(\boldsymbol{\alpha}, \omega) = U_{l_1}(\omega_1) \otimes U_{l_2}(\omega_2) \otimes \cdots \otimes U_{l_m}(\omega_m)\,, \tag{3.3}$$

where, $U_{l_j}(\omega_j)$ is the one by one identity matrix if $l_j = 0$, and, $U_{l_j}(\omega_j) = U_j$, the matrix given in equation (3.2), if $l_j = 1\cdot$ From equations (3.1), (3.2), and (3.3), the first part of the theorem can be verified. The remaining part of the theorem follows from the second part of Lemma 2.1 and the definition of the matrix $U(\boldsymbol{\alpha}, \omega)$ given in equation (3.3).

Let P be an admissible set of effects. Then, as remarked earlier, $P = \bigcup_{j=1}^{l} \alpha_j$ (disjoint), where each α_j is a basic collection of effects. The column vector $\mathbf{P}$ given by $\mathbf{P}' = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \cdots, \boldsymbol{\alpha}_l')$ obtained from an admissible collection of effects will be called an *admissible vector.* If $\mathbf{Y}$ is any observation vector and $\mathbf{P}' = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \cdots, \boldsymbol{\alpha}_l')$ is an admissible vector, then let

$$X_{\mathbf{P}}(\mathbf{Y}) = [X(\mathbf{Y}, \boldsymbol{\alpha}_1)\,|X(\mathbf{Y}, \boldsymbol{\alpha}_2)|\cdots|X(\mathbf{Y}, \boldsymbol{\alpha}_l)|]\,. \tag{3.4}$$

COROLLARY 3.1. *For each level permutation* ω *in* Ω, *an admissible vector* $\mathbf{P}$, *and, any observation vector* $\mathbf{Y}$, *there exists an orthogonal matrix* $U(\mathbf{P}, \omega)$ *such that*

$$X_{\mathbf{P}}(\mathbf{Y})U(\mathbf{P}, \omega) = X_{\mathbf{P}}(\omega(\mathbf{Y}))\,.$$

Further,

$$\frac{1}{\pi(s_i!)} \sum_{\omega \in \Omega} U(\mathbf{P}, \omega) = 0\,, \qquad \text{if} \quad \mathbf{P}' \neq (A_1^0\, A_2^0 \cdots A_m^0)\,,$$
$$= I\,, \qquad \text{if} \quad \mathbf{P}' = (A_1^0\, A_2^0 \cdots A_m^0)\,,$$

where 0 *is the zero matrix and* I *is the one by one identity matrix.*

PROOF. Let $\mathbf{P}' = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \cdots, \boldsymbol{\alpha}_l')$. By Theorem 3.1, for each ω in Ω and each j ($1 \leqq j \leqq l$), there exists an orthogonal matrix $U(\boldsymbol{\alpha}_j, \omega)$ such that $X(\omega(\mathbf{Y}), \boldsymbol{\alpha}_j) = X(\mathbf{Y}, \boldsymbol{\alpha}_j)U(\boldsymbol{\alpha}_j, \omega)$. Define $U(\mathbf{P}, \omega)$ to be a diagonal block matrix with $U(\boldsymbol{\alpha}_j, \omega)$ being the (j, j)th block. The assertions in the corollary can now be easily verified using Theorem 3.1.

An immediate corollary to Corollary 3.1 is the following.

COROLLARY 3.2. *Let* $\mathbf{Y}$ *be an observation vector and* $\mathbf{P}$ *an admissible vector. Then any pair of matrices in the set of information matrices* $\{X_{\mathbf{P}}'(\omega(\mathbf{Y}))X_{\mathbf{P}}(\omega(\mathbf{Y})) : \omega \in \Omega\}$ *have the same characteristic roots.*

4. The randomization theorem. In this section the unbiased estimation of a linear function of an admissible vector is explored when a randomized observation vector is selected from among the whole class of observation vectors generated by the group of level permutations. The main result established here generalizes the results on randomization as expounded by Ehrenfeld and Zacks (1961) and Zacks (1963, 1964) for regular fractions to arbitrary fractions in the setting of the general factorial.

Let P be an admissible set of effects and let $P^c = P_0 - P$ be the set complement of P in P_0. Then, as remarked earlier, P^c is also an admissible set of effects. Let $\mathbf{Y}$ be any $N \times 1$ observation vector. Then the usual linear model under the assumption that $P^c = \mathbf{0}$ and a given distribution G is

$$E_G(\mathbf{Y}) = X_{\mathbf{P}}(\mathbf{Y})\mathbf{P}\,, \qquad \operatorname{Cov}(\mathbf{Y}) = \sigma^2 I_N\,, \tag{4.1}$$

where, $\mathbf{P}' = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \cdots, \boldsymbol{\alpha}_l')$ and $X_{\mathbf{P}}(\mathbf{Y})$ is defined by equation (3.4).

Let $M_{\mathbf{P}}(\mathbf{Y}) = X_{\mathbf{P}}'(\mathbf{Y})X_{\mathbf{P}}(\mathbf{Y})$ denote the information matrix of the observation vector $\mathbf{Y}$ relative to $\mathbf{P}$ and let $M_{\mathbf{P}}^-(\mathbf{Y})$ be a generalized inverse of $M_{\mathbf{P}}(\mathbf{Y})$. Further, let $\boldsymbol{\mu}$ be a column vector with the property $\boldsymbol{\mu}' = \boldsymbol{\lambda}_\omega' X_{\mathbf{P}}(\omega(\mathbf{Y}))$, for each ω in Ω, that is, $\boldsymbol{\mu}'$ is a vector which lies in the row space of $X_{\mathbf{P}}(\omega(\mathbf{Y}))$ for each ω in Ω. This formidable condition is only necessary for developing the general theory. In most practical situations of fractional factorial designs, such as Resolution III and V designs, the interest lies in the estimation of the vector $\mathbf{P}$ itself and not in linear functions of $\mathbf{P}$. The design matrix $X_{\mathbf{P}}(\mathbf{Y})$ in such situations is then of full rank, which is preserved under level permutations, so that it is not necessary to assume the row space condition.

A general solution for $\mathbf{P}$ from the system defined by equation (4.1), call it $\mathbf{P}^0$, is given by

$$\mathbf{P}^0 = M_{\mathbf{P}}^-(\mathbf{Y})X_{\mathbf{P}}'(\mathbf{Y})\mathbf{Y}\,. \tag{4.2}$$

Let $\boldsymbol{\mu}_0'\mathbf{P}$ be any linear function of $\mathbf{P}$. Then the *expected value* of $\boldsymbol{\mu}_0'\mathbf{P}^0$ using a nonrandomized design is given by

$$E_G(\boldsymbol{\mu}_0'\mathbf{P}^0 \mid \mathbf{Y}) = \boldsymbol{\mu}_0'[M_{\mathbf{P}}^-(\mathbf{Y})M_{\mathbf{P}}(\mathbf{Y})\mathbf{P} + M_{\mathbf{P}}^-(\mathbf{Y})X_{\mathbf{P}}'(\mathbf{Y})X_{\mathbf{P}^c}(\mathbf{Y})\mathbf{P}^c]\,.$$

It will be seen later that the parameters P^c play the role of nuisance parameters. Further, if ω in Ω is given and $\mathbf{P}_\omega{}^0$ is a solution to the equation $\omega(\mathbf{Y}) = X_{\mathbf{P}}(\omega(\mathbf{Y}))\mathbf{P}$ then the *expected value* of $\boldsymbol{\mu}_0'\mathbf{P}_\omega{}^0$ is defined by

$$E_\Omega(\boldsymbol{\mu}_0'\mathbf{P}_\omega{}^0) = \frac{1}{\pi(s_i!)} \sum_{\omega\in\Omega} E_G(\boldsymbol{\mu}_0'\mathbf{P}_\omega{}^0 \mid \omega(\mathbf{Y}))\,.$$

Note that in this definition the underlying randomized design is selected with equal probability.

The following lemmas will be useful in establishing the main result of this section.

LEMMA 4.1.

$$\frac{1}{\pi(s_i!)} \sum_{\omega \in \Omega} \boldsymbol{\mu}' M_{\mathbf{P}}^{-}(\omega(\mathbf{Y})) M_{\mathbf{P}}(\omega(\mathbf{Y}))\mathbf{P} = \boldsymbol{\mu}'\mathbf{P} ,$$

where $\boldsymbol{\mu}' = \boldsymbol{\lambda}_\omega' X_{\mathbf{P}}(\omega(\mathbf{Y}))$ *for each* ω *in* Ω.

PROOF. By a property of generalized inverse it follows that

$$X_{\mathbf{P}}(\omega(\mathbf{Y})) M_{\mathbf{P}}^{-}(\omega(\mathbf{Y})) M_{\mathbf{P}}(\omega(\mathbf{Y})) = X_{\mathbf{P}}(\omega(\mathbf{Y})) . \tag{4.3}$$

Hence, substituting the definition of $\boldsymbol{\mu}'$ in the expression given in the lemma and using equation (4.3) the lemma can be verified.

LEMMA 4.2. *If* $\mathbf{P}$ *is any admissible vector, then*

$$\sum_{\omega \in \Omega} M_{\mathbf{P}}^{-}(\omega(\mathbf{Y})) X_{\mathbf{P}}'(\omega(\mathbf{Y})) X_{\mathbf{P}^c}(\omega(\mathbf{Y})) = 0 ,$$

where 0 *is the zero matrix.*

PROOF. Let $\mathbf{P}' = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \cdots, \boldsymbol{\alpha}_l')$ and $(\mathbf{P}^c)' = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \cdots, \boldsymbol{\beta}_k')$ where α_i, β_j are basic collections of effects. By Corollary 3.1, there exist orthogonal matrices $U(\mathbf{P}, \omega)$ and $U(\mathbf{P}^c, \omega)$, each being diagonal block matrices of l and k blocks respectively, with the respective (j, j)th blocks being $U(\boldsymbol{\alpha}_j, \omega)$ and $U(\boldsymbol{\beta}_j, \omega)$, such that $X_{\mathbf{P}}(\omega(\mathbf{Y})) = X_{\mathbf{P}}(\mathbf{Y}) U(\mathbf{P}, \omega)$ and $X_{\mathbf{P}^c}(\omega(\mathbf{Y})) = X_{\mathbf{P}^c}(\mathbf{Y}) U(\mathbf{P}^c, \omega)$. Substituting these equations in the expression given in the lemma it follows that

$$\begin{aligned} &\sum_{\omega \in \Omega} M_{\mathbf{P}}^{-}(\omega(\mathbf{Y})) X_{\mathbf{P}}'(\omega(\mathbf{Y})) X_{\mathbf{P}^c}(\omega(\mathbf{Y})) \\ &\qquad = \sum_{\omega \in \Omega} U'(\mathbf{P}, \omega) M_{\mathbf{P}}^{-}(\mathbf{Y}) X_{\mathbf{P}}'(\mathbf{Y}) X_{\mathbf{P}^c}(\mathbf{Y}) U(\mathbf{P}^c, \omega) . \end{aligned}$$

Let $L = M_{\mathbf{P}}^{-}(\mathbf{Y}) X_{\mathbf{P}}'(\mathbf{Y}) X_{\mathbf{P}^c}(\mathbf{Y})$. Then L is a fixed matrix independent of ω in Ω. Partition L into lk matrices K_{ij}, $1 \leqq i \leqq l$ and $1 \leqq j \leqq k$, where each K_{ij} is a matrix conformable with $U'(\boldsymbol{\alpha}_i, \omega)$ and $U(\boldsymbol{\beta}_j, \omega)$. Then $U'(\mathbf{P}, \omega) L U(\mathbf{P}^c, \omega)$ is a block matrix whose (i, j)th block is $U'(\boldsymbol{\alpha}_i, \omega) K_{ij} U(\boldsymbol{\beta}_j, \omega)$. Thus to establish the lemma it is enough to show that $\sum_{\omega \in \Omega} U'(\boldsymbol{\alpha}_i, \omega) K_{ij} U(\boldsymbol{\beta}_j, \omega) = 0$. Now suppose $\boldsymbol{\alpha}_i' = v_1^{l_1} \otimes v_2^{l_2} \otimes \cdots \otimes v_m^{l_m}$ and $\boldsymbol{\beta}_j' = v_1^{k_1} \otimes v_2^{k_2} \otimes \cdots \otimes v_m^{k_m}$. Then, since α_i and β_j are disjoint sets there exists an r with $l_r = 0$ and $k_r = 1$ or $l_r = 1$ and $k_r = 0$. Suppose that $l_r = 0$ and $k_r = 1$. Then, from equation (3.3), one has that $U_{l_r}(\omega)$ is the one by one identity matrix for each ω in Ω_r. Select permutations ω_i in Ω_i $(i \neq r)$ and fix them. Set $F = \{(\omega_1, \omega_2, \cdots, \omega_{r-1}, \omega, \omega_{r+1}, \cdots, \omega_m) : \omega \in \Omega_r\}$. Then Ω is a disjoint union of such sets F. It now follows, from equation (3.3), that for any δ_1, δ_2 in F, $U(\boldsymbol{\alpha}_i, \delta_1) = U(\boldsymbol{\alpha}_i, \delta_2)$. Hence, $U(\boldsymbol{\alpha}_i, \omega) K_{ij}$ is independent of ω in F, and, $\sum_{\omega \in F} U'(\boldsymbol{\alpha}_i, \omega) K_{ij} U(\boldsymbol{\beta}_j, \omega) = (U'(\boldsymbol{\alpha}_i, \omega) K_{ij}) \sum_{\omega \in F} U(\boldsymbol{\beta}_j, \omega)$. Moreover, from Theorem 3.1, it follows that

$$\sum_{\omega \in F} U(\boldsymbol{\beta}_j, \omega) = U_{k_1}(\omega_1) \otimes \cdots \otimes \sum_{\omega \in \Omega_r} U_{k_r}(\omega) \otimes \cdots \otimes U_{k_m}(\omega_m) = 0 .$$

Hence, $\sum_{\omega \in \Omega} U'(\boldsymbol{\alpha}_i, \omega) K_{ij} U(\boldsymbol{\beta}_j, \omega) = 0$ as well. The case $l_r = 1$ and $k_r = 0$ follows similarly, completing the proof.

The main result of this section is the following.

THEOREM 4.1. *Let Y be a given observation vector and consider the class of observation vectors $\omega(\mathbf{Y})$ generated by ω in Ω. Select a permutation η in Ω with probability $(\prod_{i=1}^{m}(s_i!))^{-1}$. If $\boldsymbol{\mu}$ is a column vector such that $\boldsymbol{\mu}' = \boldsymbol{\lambda}_\omega' X_{\mathbf{P}}(\omega(\mathbf{Y}))$ for each ω in Ω, where $\mathbf{P}$ is an admissible vector, then $E_\Omega(\boldsymbol{\mu}'\mathbf{P}_\eta^0) = \boldsymbol{\mu}'\mathbf{P}$, where $\mathbf{P}_\eta^0$ is a solution to the equation $\eta(\mathbf{Y}) = X_{\mathbf{P}}(\eta(\mathbf{Y}))\mathbf{P}$.*

PROOF. Now

$$\begin{aligned} E_\Omega(\boldsymbol{\mu}'\mathbf{P}_\eta^0) &= \frac{1}{\pi(s_i!)} \sum_{\eta\in\Omega} E_G(\boldsymbol{\mu}'\mathbf{P}_\eta^0 \mid \eta(\mathbf{Y})) \\ &= \frac{1}{\pi(s_i!)} \sum_{\eta\in\Omega} \boldsymbol{\mu}'[M_{\mathbf{P}}^-(\eta(\mathbf{Y}))M_{\mathbf{P}}(\eta(\mathbf{Y}))\mathbf{P} \\ &\qquad + M_{\mathbf{P}}^-(\eta(\mathbf{Y}))X_{\mathbf{P}}'(\eta(\mathbf{Y}))X_{\mathbf{P}^c}(\eta(\mathbf{Y}))\mathbf{P}^c] . \end{aligned}$$

By Lemma 4.2 the sum of the second term in the above expression reduces to the zero matrix and by Lemma 4.1 the sum of the first term reduces to $\boldsymbol{\mu}'\mathbf{P}$, completing the proof.

5. Some applications. The results obtained in this paper have wide applicability in most practical settings of factorial experimentation. First of all, admissible vectors $\mathbf{P}$ admit such celebrated vectors as "main effects" and "main effects plus two-factor interactions", which typically appear in resolution III and V settings respectively. The interest in such cases lies in estimation of all the elements of the admissible vector $\mathbf{P}$. It is well known that a necessary and sufficient condition for estimability of $\mathbf{P}$ is that the underlying design matrix of the given fractional replicate is of full rank. If this is the case then any permutation $\omega \in \Omega$ will lead to an observation vector $\omega(\mathbf{Y})$ such that $\mathbf{P}$ is estimable. As indicated earlier in most practical settings it is not necessary to check the row space condition for estimating $\mathbf{P}$.

Secondly, the invariance theorem relates that when the amount of information is measured by a functional on the spectrum of the information matrix then each design in the class of designs generated by the action of Ω on a fixed design will contain the same amount of information, since the underlying information matrices are spectrum invariant. Hence, if no additional criteria are specified then one should pick a design randomly for the results to be unbiased as shown in the randomization theorem.

Finally, if additional meaningful economic and/or physical criteria can be introduced then random selection is not appropriate. To illustrate this realistically consider the 2^4 factorial with the objective to estimate the main effects and the mean with five treatment combinations under the assumption that all interactions are zero. It is known that a determinant-optimal saturated main effect plan is given by the treatment combinations $D_1 = \{(0000), (0111), (1011), (1101), (1110)\}$. Using the invariance theorem one gets 16 equi-information designs by applying Ω. The first one is already listed and the remaining 15 are:

$D_2 = \{(1000), (1111), (0011), (0101), (0110)\}$, $D_3 = \{(0100), (0011), (1111), (1001), (1010)\}$, $D_4 = \{(0010), (0101), (1001), (1111), (1100)\}$, $D_5 = \{(0001), (0110), (1010), (1100), (1111)\}$, $D_6 = \{(1100), (1011), (0111), (0001), (0010)\}$, $D_7 = \{(1010), (1101), (0001), (0111), (0100)\}$, $D_8 = \{(1001), (1110), (0010), (0100), (0111)\}$, $D_9 = \{(0110), (0001), (1101), (1011), (1000)\}$, $D_{10} = \{(0101), (0010), (1110), (1000), (1011)\}$, $D_{11} = \{(0011), (0100), (1000), (1110), (1101)\}$, $D_{12} = \{(1110), (1001), (0101), (0011), (0000)\}$, $D_{13} = \{(1101), (1010), (0110), (0000), (0011)\}$, $D_{14} = \{(1011), (1100), (0000), (0110), (0101)\}$, $D_{15} = \{(0111), (0000), (1100), (1010), (1001)\}$, $D_{16} = \{(1111), (1000), (0100), (0010), (0001)\}$. Let $w(D) = \mathbf{1}'D\mathbf{1}$ (= the number of 1's in D) be the weight function of a design. The weight function can be interpreted as the total cost of a design if the low level 0 of any factor is assumed to cost 0 units and the high level 1 of any factor is assumed to cost 1 unit. Evaluating the weight function for the sixteen saturated main effect plans above it is seen that D_{16} is the unique cost-optimal and determinant-optimal design, since $w(D_{16}) = 8$ and this is lower than that for the other fifteen designs.

If a situation arises where the design in the determinant-optimal (or for any other optimality criterion) class is not unique then further physical and other criteria can be introduced by the experimenter in order to select a design. If no other criteria are introduced in such a situation then one may choose a design randomly from the determinant-optimal (or for any optimality criterion) class.

6. Unsolved problem. An interesting question to consider in the context of Corollary 3.2 is the following: For what other collections P (besides admissible collections) are the matrices $X_P(\omega(\mathbf{Y}))$ generated by Ω orthogonally related?

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DEPARTMENT OF STATISTICS
COLORADO STATE UNIVERSITY
FORT COLLINS, COLORADO 80521

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF GUELPH
GUELPH, ONTARIO N1G 2W1