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ON CONNECTEDNESS IN TWO-WAY ELIMINATION OF HETEROGENEITY DESIGNS[1]

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A necessary and sufficient condition is established for doubly-connectedness in b-row and k-column designs in which all cells are filled. A necessary condition for doubly-connectedness in a generalized two-way elimination of heterogeneity designs is provided and a property of doubly-connected designs is given.

1. Introduction. Let the experimental material be arranged in b rows and k columns and let v treatments be applied to the experimental units, the ith treatment being replicated r_i times $(i = 1, 2, \cdots, v)$. Let $u_1, u_2, \cdots, u_b$ units be treated with the treatments in the b rows and let $w_1, w_2, \cdots, w_k$ units be treated with the treatments in the k columns. If $u_1 = u_2 = \cdots = u_b = k$ and $w_1 = w_2 = \cdots = w_k = b$ we obtain the ordinary two-way elimination of heterogeneity designs; otherwise we obtain generalized two-way elimination of heterogeneity designs.

Let $N = (n_{ij})$ be a $b \times k$ matrix, where n_{ij} is the number of treated units in the ith row and jth column, $L = (l_{ij})$ be a $v \times b$ matrix, where l_{ij} is the number of treated units with the ith treatment in the jth row, and $M = (m_{ij})$ be a $v \times k$ matrix, where m_{ij} is the number of treated units with the ith treatment in the jth column. Let A^- denote a generalized inverse of A. Let $C_1, C_2, C_3, C_4, C^*, C_3^*, C_4^*$ be the well-known C matrices for the following purposes:

Matrix	*Estimating*	*Eliminating*	*Ignoring*
$C_1 = \operatorname{diag}(r_1, r_2, \cdots, r_v) - L \operatorname{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) L'$	treatment effects	row effects	column effects
$C_2 = \operatorname{diag}(r_1, r_2, \cdots, r_v) - M \operatorname{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) M'$	treatment effects	column effects	row effects
$C_3 = \operatorname{diag}(w_1, w_2, \cdots, w_k) - N' \operatorname{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N$	column effects	row effects	treatment effects

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Matrix	*Estimating*	*Eliminating*	*Ignoring*
$C_4 = \text{diag}(u_1, u_2, \cdots, u_b) - N \text{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) N'$	row effects	column effects	treatment effects
$C^* = C_1 - \left(M - L \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right) C_3^- \left(M - L \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)'$	treatment effects	row and column effects	—
$C_3^* = C_3 - \left(M - L \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)' C_1^- \left(M - L \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)$	column effects	row and treatment effects	—
$C_4^* = C_4 - \left(L - M \text{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) N'\right)' C_1^- \left(L - M \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N'\right)$	row effects	column and treatment effects	—

C^* in the case of an ordinary two-way elimination of heterogeneity designs reduces to the following:

$$C^* = \text{diag}(r_1, r_2, \cdots, r_v) - \frac{1}{k} LL' - \frac{1}{b} MM' + \frac{1}{bk} \mathbf{r}\mathbf{r}' , \tag{1.1}$$

where $\mathbf{r}' = (r_1, r_2, \cdots, r_v)$.

The design is said to be row-treatment connected if the rank of $C_1 = R(C_1) = v - 1$, column-treatment connected if $R(C_2) = v - 1$, column-row connected if $R(C_3) = k - 1$, and doubly-connected if $R(C^*) = v - 1$. It may be noted that a doubly-connected design need not necessarily ensure the estimation of all elementary contrasts of row and column effects. This fact is important in the construction and use of augmented two-way designs involving replicated treatments and unreplicated treatments such as might be used in screening experiments.

This paper is a contribution toward the theory of the doubly-connectedness property of two-way elimination of heterogeneity designs.

2. A necessary and sufficient condition for doubly-connectedness of ordinary two-way elimination of heterogeneity designs. Before we state and prove our main theorem, we state the following lemma whose proof is straightforward (e.g., see Shah (1959)):

LEMMA 2.1. *The C-matrices, C_1, C_2, C^* have rank $v - 1$ if and only if $C_1 + aJ_{v,v}$, $C_2 + aJ_{v,v}$, $C^* + aJ_{v,v}$ are nonsingular, where a is a nonzero scalar and $J_{m,n}$ is an $m \times n$ matrix with 1 everywhere.*

THEOREM 2.1. *In an ordinary two-way elimination of heterogeneity design, let $r_1 = r_2 = \cdots = r_v = r$ and let $L'M = rJ_{b,k}$. Then the design is doubly-connected if and only if it is row-treatment and column-treatment connected.*

PROOF. Under the assumptions of the theorem

$$C^* = r^{-1}C_1C_2\,, \tag{2.1}$$

$$C^* + aJ_{v,v} = r^{-1}\left(C_1 + \left(\frac{ra}{v}\right)^{\frac{1}{2}} J_{v,v}\right)\left(C_2 + \left(\frac{ra}{v}\right)^{\frac{1}{2}} J_{v,v}\right) \tag{2.2}$$

for any nonzero real a. Thus the following two-way relations establishing the theorem hold:

The design is doubly-connected

$$\begin{aligned}
&\Leftrightarrow R(C^*) = v - 1 \Leftrightarrow C^* + aJ_{v,v} \text{ is nonsingular}\\
&\Leftrightarrow C_1 + \left(\frac{ra}{v}\right)^{\frac{1}{2}} J_{v,v} \quad\text{and}\quad C_2 + \left(\frac{ra}{v}\right)^{\frac{1}{2}} J_{v,v} \quad \text{are nonsingular,}\\
&\Leftrightarrow R(C_1) = v - 1 \quad\text{and}\quad R(C_2) = v - 1\,.\\
&\Leftrightarrow \text{The design is row-treatment and column-treatment connected.}
\end{aligned}$$

Thus, we see that one need look only at row-treatment and column-treatment connectedness in searching for doubly-connected designs in the class of ordinary row-column designs. Also, note that the conditions of the theorem are certainly much stronger than what is needed, for the design

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{matrix} \tag{2.3}$$

is row-treatment, column-treatment, row-column, and doubly-connected without satisfying the assumptions of the theorem. However, some conditions are needed as the following design

$$\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} \tag{2.4}$$

does not satisfy the assumptions or the conclusions of the theorem.

An example of a doubly-disconnected design which is equi-replicated and is row-treatment and column-treatment connected was given by Shah and Khatri (1973).

3. A necessary condition for doubly-connectedness in generalized two-way elimination of heterogeneity designs. Contrary to the belief that a doubly-connected design is pairwise connected, the following result holds:

THEOREM 3.1. *A doubly-connected generalized two-way elimination of heterogeneity design is row-treatment and column-treatment connected.*

PROOF. Let, if possible, a doubly-connected design be row-treatment disconnected. Then there exists a $v \times 1$ column vector ξ orthogonal to $J_{v,1}$ such that $C_1\xi = 0_{v,1}$. Then, we have

$$\xi' C^* \xi = -\left\{\left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)\xi\right\}' C_3^- \times \left\{\left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)\xi\right\}, \tag{3.1}$$

which is a contradiction, as the left-hand side is a positive quantity and the right-hand side is a nonpositive quantity in view of C^* and C_3^- being positive semi-definite matrices. Thus a doubly-connected design is row-treatment connected. Analogously, one may show it to be column-treatment connected.

The above result can also be obtained from the model of such designs and estimability theory. A doubly-connected design need not necessarily be row-column connected. The following design where X's indicate blanks is doubly-connected and hence row-treatment and column-treatment connected, but is row-column disconnected:

1	2	X	X
2	1	X	X
X	X	1	2
X	X	2	1

In fact, the class of doubly-connected designs $I\bar{\mathrm{X}}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ is row-column disconnected, where I is the identity matrix and $\bar{\mathrm{X}}$ is the symbol for the Kronecker product of matrices.

4. A property of doubly-connected designs. One may wonder whether a doubly-connected design can be used to estimate every elementary contrast of row and column effects. The answer is provided by the following theorem:

THEOREM 4.1. *A doubly-connected design, which is also row-column connected, provides estimates for every elementary contrast of row and column effects.*

PROOF. For a doubly-connected design which is row-column connected, the following hold:

$$\begin{aligned} R(C^*) &= R(C_1) = R(C_2) = v - 1 \\ R(C_3) &= k - 1 \\ R(C_4) &= b - 1\,, \end{aligned} \tag{4.1}$$

and thus

$$
\begin{aligned}
&R\begin{bmatrix} \operatorname{diag}(u_1, u_2, \cdots, u_b) & N & L' \\ N' & \operatorname{diag}(w_1, w_2, \cdots, w_k) & M' \\ L & M & \operatorname{diag}(r_1, r_2, \cdots, r_v) \end{bmatrix} \\
(4.2)\qquad &= R\begin{bmatrix} \operatorname{diag}(u_1, u_2, \cdots, u_b) & N & L' \\ 0 & C_3 & * \\ 0 & 0 & C^* \end{bmatrix} \\
&= b + k - 1 + v - 1 = v + b + k - 2 ,
\end{aligned}
$$

where * denotes a matrix obtained in the sweeping-out process. Again

$$
\begin{aligned}
&v + b + k - 2 \\
(4.3)\qquad &= R\begin{bmatrix} \operatorname{diag}(u_1, u_2, \cdots, u_b) & N & L' \\ N' & \operatorname{diag}(w_1, w_2, \cdots, w_k) & M' \\ L & M & \operatorname{diag}(r_1, r_2, \cdots, r_v) \end{bmatrix} \\
&= R\begin{bmatrix} C_4^* & 0 & 0 \\ N' & \operatorname{diag}(w_1, w_2, \cdots, w_k) & M' \\ * & 0 & C_2 \end{bmatrix} \\
&= R(C_4^*) + k + v - 1 ,
\end{aligned}
$$

from which it follows that

$$(4.4)\qquad R(C_4^*) = b - 1 ,$$

and

$$
\begin{aligned}
&v + b + k - 2 \\
(4.5)\qquad &= R\begin{bmatrix} \operatorname{diag}(u_1, u_2, \cdots, u_b) & N & L' \\ N' & \operatorname{diag}(w_1, w_2, \cdots, w_k) & M' \\ L & M & \operatorname{diag}(r_1, r_2, \cdots, r_v) \end{bmatrix} \\
&= R\begin{bmatrix} \operatorname{diag}(u_1, u_2, \cdots, u_b) & N & L' \\ 0 & C_3^* & 0 \\ 0 & * & C_1 \end{bmatrix} \\
&= R(C_3^*) + b + v - 1 ,
\end{aligned}
$$

from which it follows that

$$(4.6)\qquad R(C_3^*) = k - 1 .$$

Thus all elementary contrasts of row and column effects are estimable, establishing the theorem.

The above result can also be proved by the estimability conditions of elementary contrasts.

The result is symmetrical in row, column, and treatments and we have the following:

THEOREM 4.2. *In a generalized two-way elimination of the heterogeneity model, if the row effects are estimable and if it is column-treatment connected, then it is* *row-column and row-treatment connected. Similarly, if the column effects are estimable and if it is row-treatment connected, then it is row-column and column-treatment connected.*

Concluding remarks. Though the results in this paper are obtained for generalized two-way elimination of heterogeneity designs, they can be translated into the terminology of any three-factor experiment without any loss of generality, in an obvious way.

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