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ON INTERVAL ESTIMATION AND SIMULTANEOUS SELECTION OF ORDERED LOCATION OR SCALE PARAMETERS[1]

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A formulation is given and a procedure is proposed for constructing a confidence interval for a certain ordered location or scale parameter and for simultaneously selecting all populations having parameters equal or larger than this ordered parameter with a preassigned minimal probability. The well-known indifference-zone formulation of the ranking problem is obtained as a special case as is the problem of interval estimation of an ordered parameter.

1. Introduction and formulation of the problem. Procedures for selection of a certain number of populations with larger parameters from a collection of several populations have been studied extensively in the past two decades; see for example, Bechhofer [3] or Barr and Rizvi [2]. Recently Saxena and Tong [6], Saxena [5], Dudewicz and Tong [4], and Alam, Saxena and Tong [1] have considered confidence intervals for the largest parameter. The present paper attempts to combine these two requirements simultaneously in a single formulation for the location and scale parameter families. The problem of interest is to construct a confidence interval for an ordered location or scale parameter and simultaneously select all populations having parameters equal or larger than this ordered parameter, with a preassigned minimal probability whenever parameters lie in a specified subspace. A procedure R is proposed to solve this problem and its performance, in terms of the probability requirement being satisfied, is evaluated.

Let $\{F(\cdot;\theta)\}$ denote a family of absolutely continuous distribution functions on the real line indexed by a parameter θ; the density corresponding to $F(\cdot,\theta)$ would be denoted by $f(\cdot,\theta)$. Let $\theta_1, \cdots, \theta_k$ be k values of θ and let $Y_1, \cdots, Y_k$ be k independent observations from k populations $\Pi_1, \cdots, \Pi_k$ with distribution functions $F(\cdot;\theta_1), \cdots, F(\cdot;\theta_k)$ respectively.

Let $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ denote the ordered values of the components of $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k) \in \Omega$. For $1 \leqq t \leqq k$, we require a procedure R that selects all Π_i $(i = 1, \cdots, k)$ *with* $\theta_i \geqq \theta_{[k-t+1]} = \theta$ (say) and simultaneously gives an interval I such that $\theta \in I$. Denote by CS the (correct) selection of all Π_j with $\theta_{[j]}$, $j = k-t+1, \cdots, k$, and by CD (correct decision) the inclusion of θ in I, and let

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$P(\boldsymbol{\theta})$ denote $\Pr\{\text{CS} \cap \text{CD} \mid R\}$. Then the procedure R, for some preassigned γ, $0 < \gamma < 1$, is more specifically required to satisfy

$$\inf_{\Omega(\psi)} P(\boldsymbol{\theta}) \geqq \gamma \,, \tag{1.1}$$

where $\Omega(\psi) = \{\boldsymbol{\theta} \in \Omega : \theta_{[k-t]} \leqq \psi(\theta_{[k-t+1]})\}$ and ψ is a given function on the real line such that $\psi(x) < x$.

We give solutions of (1.1) for location parameters and scale parameters using procedure R proposed on an ad hoc basis in Section 2. The theorems of Section 2 provide a unified treatment for both the location and scale parameters. However, the explicit solutions for the two cases are given in Section 3.

2. Main results on $P(\boldsymbol{\theta})$ for proposed R.

Proposed procedure R. Rank $Y_1, Y_2, \cdots, Y_k$, breaking ties (if any) with suitable randomization, and let $Y_{[i]}$ be the ith smallest Y_i. Consider two suitably chosen continuous increasing functions h_1 and h_2 (with inverses g_2 and g_1 respectively). Construct the random interval $I_0 = (h_1(Y_{[k-t+1]}), h_2(Y_{[k-t+1]}))$. Then assert that $\theta \in I_0$ and that the Π_j's corresponding to $Y_{[j]}$ ($j = k - t + 1, \cdots, k$) have parameters $\theta_j \geqq \theta$.

REMARK 1. It should be pointed out that in applications Y_i's are consistent estimators (preferably functions of the sufficient statistic, when it exists) of θ_i's based on samples of common size n from each population and θ_i and n appear as parameters in their distribution (n will be explicitly demonstrated in the sequel only when needed). Then (1.1) can be satisfied by a proper choice of n and the functions g_1 and g_2 (see Theorem 2).

First we investigate the infimum of $P(\boldsymbol{\theta})$ over $\Omega^*(\psi)$ for the above R, where

$$\Omega^*(\psi) = \{\theta \in \Omega : \theta_{[k-t]} \leqq \psi(\theta), \theta = \theta_{[k-t+1]} \text{ held fixed}\} \,,$$

and then determine conditions so that R satisfies (1.1). We have

$$P(\boldsymbol{\theta}) = \sum_{j=k-t+1}^{k} \int_{g_1(\theta)}^{g_2(\theta)} \prod_{r=1}^{k-t} F(y; \theta_{[r]}) \times \prod_{s=k-t+1; s\neq j}^{k} \{1 - F(y; \theta_{[s]})\} \, dF(y; \theta_{[j]}) \,. \tag{2.1}$$

Our main result is given by

THEOREM 1. *Suppose $F(y; \theta)$ is differentiable for all θ in the parameter space. Suppose that* (a) $(\partial/\partial\theta)F(y; \theta) \leqq 0$ *for all y,* (b) $[(\partial/\partial\theta)F(y; \theta)]/[(\partial/\partial\theta)F(y'; \theta)]$ *is an increasing function of θ for every $y > y'$, and* (c) $F(y; \bar{\theta}) = 0$ *for all y, where $\bar{\theta}$ is the largest possible value of the parameter ($+\infty$ included). If $F(\cdot; \theta_i)$, $i = 1, \cdots, k$ satisfy the above conditions satisfied by $F(y; \theta)$, then*

$$\inf_{\Omega^*(\psi)} P(\boldsymbol{\theta}) = \min_{r=0,1,\cdots,t-1} P(\boldsymbol{\theta}^{(r)}) \,, \tag{2.2}$$

where

$$P(\boldsymbol{\theta}^{(r)}) = (r + 1) \int_{g_1(\theta)}^{g_2(\theta)} F^{k-t}(y; \psi(\theta))[1 - F(y; \theta)]^r \, dF(y; \theta) \,, \tag{2.3}$$

and $\boldsymbol{\theta}^{(r)}$ is a vector $\boldsymbol{\theta}$ with first $(k - t)$ ordered components equal to $\psi(\theta)$, $(r + 1)$ among the last t ordered components equal to θ and the rest equal to $\bar{\theta}$.

PROOF. Let $\Omega_1(\psi) = \{\boldsymbol{\theta} \in \Omega(\psi): \theta_{[k-t+1]}(= \theta), \cdots, \theta_{[k]} \text{ held fixed}\}$. Then in view of the condition (a) the infimum of $P(\boldsymbol{\theta})$ over $\Omega_1(\psi)$ can be written as

$$\begin{aligned} \inf_{\Omega_1(\psi)} P(\boldsymbol{\theta}) = {} & F^{k-t}(g_1(\theta); \psi(\theta)) \textstyle\prod_{s=k-t+1}^{k} \{1 - F(g_1(\theta); \theta_{[s]})\} \\ & - F^{k-t}(g_2(\theta); \psi(\theta)) \textstyle\prod_{s=k-t+1}^{k} \{1 - F(g_2(\theta); \theta_{[s]})\} \\ & + (k-t) \textstyle\int_{g_1(\theta)}^{g_2(\theta)} F^{k-t-1}(y; \psi(\theta)) \\ & \times \textstyle\prod_{s=k-t+1}^{k} \{1 - F(y; \theta_{[s]})\} \, dF(y; \psi(\theta)) . \end{aligned} \tag{2.4}$$

Observe that the right side of (2.4) is a symmetric function of $\theta_{[k-t+2]}, \cdots, \theta_{[k]}$. Hence disregarding their ordering and relabeling $\theta_{[k-t+2]}, \cdots, \theta_{[k]}$ as $\eta_{k-t+2}, \cdots, \eta_k$ and letting $\boldsymbol{\eta} = (\eta_{k-t+2}, \cdots, \eta_k)$ we may symbolically write (2.4) as

$$\inf_{\Omega_1(\psi)} P(\boldsymbol{\theta}) = H(\theta, \boldsymbol{\eta}) . \tag{2.5}$$

It is clear that the infimum of $P(\boldsymbol{\theta})$ over $\Omega^*(\psi)$ is equal to the infimum of $H(\theta, \boldsymbol{\eta})$ over $\{\boldsymbol{\eta}: \theta \leqq \eta_j \leqq \bar{\theta}, j = k-t+2, \cdots, k\}$. For some j, fix $\eta_{k-t+2}, \cdots, \eta_{j-1}, \eta_{j+1}, \cdots, \eta_k$ and consider $(\partial/\partial\eta_j)H(\theta, \boldsymbol{\eta})$. Observe that the condition (b) implies that $[(\partial/\partial\eta_j)F(y; \eta_j)]/[(\partial/\partial\eta_j)F(g_2(\theta); \eta_j)]$ is a decreasing function of η_j for all $y \in [g_1(\theta), g_2(\theta)]$. Arguing in a similar manner as in Saxena [5] we conclude that $H(\theta, \boldsymbol{\eta})$ is either decreasing in η_j or first increasing and then decreasing in η_j. Consequently $\inf_{\eta_j} H(\theta, \boldsymbol{\eta})$ is either at $\eta_j = \theta$ or at $\eta_j = \bar{\theta}$. This conclusion is valid for every other j. Therefore, infimum of $P(\boldsymbol{\theta})$ over $\Omega^*(\psi)$ is achieved when a certain number r of η_j's are equal to θ and the rest equal to $\bar{\theta}$. Finally, using the condition (c) in (2.4) completes the proof.

REMARK 2. The condition (a) of Theorem 1 is closely related to the stochastic ordering of a family of distributions, whereas the condition (b) is equivalent to the monotone likelihood ratio property when the family of density functions depends on a location or a scale parameter.

Note that $P(\boldsymbol{\theta}^{(r)})$ in (2.3) can be expressed as $C(r)E_r[F^{k-t}(Y; \psi(\theta))]$ where E_r denotes a certain expectation and $C(r)$ is a certain constant. Well-known results on monotonic behavior of such expectations give

COROLLARY 1. *If conditions of Theorem* 1 *hold and*

$$F(g_1(\theta); \theta) + F(g_2(\theta); \theta) \geqq 1 , \tag{2.6}$$

then

$$\begin{aligned} \inf_{\Omega^*(\psi)} P(\boldsymbol{\theta}) = P(\boldsymbol{\theta}^{(t-1)}) = {} & t \textstyle\int_{g_1(\theta)}^{g_2(\theta)} F^{k-t}(y; \psi(\theta)) \\ & \times [1 - F(y; \theta)]^{t-1} \, dF(y; \theta) . \end{aligned} \tag{2.7}$$

Now we prove that if Y_i's are consistent, it is possible to choose $g_1(\theta)$ and $g_2(\theta)$ so that the infimum of $P(\theta)$ over $\Omega^*(\psi)$ goes to unity for any fixed θ as the common sample size n tends to infinity.

Suppose Y_i's are consistent estimators. Writing $F_n(y; \theta)$ for a typical $F(y; \theta)$, $F_n(y; \theta)$ tends to unity for $y \geqq \theta$ and to zero for $y < \theta$ as n tends to infinity. Now choose functions $g_1(\theta)$, $g_2(\theta)$ and $\psi(\theta)$ such that $g_1(\theta) < \theta$, $g_2(\theta) > \theta$ and

$\psi(\theta) < \theta$. Further, choose constants c and d such that $\max(g_1(\theta), \psi(\theta)) < c < \theta < d < g_2(\theta)$. Then for a given $\varepsilon > 0$, there exists an $n_0(\varepsilon)$ such that for $n \geqq n_0(\varepsilon)$, $F_n(c; \psi(\theta)) \geqq 1 - \varepsilon$. Hence for any $r = 0, 1, \cdots, t-1$, and $n \geqq n_0(\varepsilon)$, we have from (2.3)

$$(2.8) \qquad P(\boldsymbol{\theta}^{(r)}) \geqq (1-\varepsilon)^{k-t}[\{1 - F_n(c; \theta)\}^{r+1} - \{1 - F_n(d; \theta)\}^{r+1}] .$$

Now there exist $n_1(\varepsilon)$ and $n_2(\varepsilon)$ such that for $n \geqq n_1(\varepsilon)$, $1 - F_n(c; \theta) \geqq 1 - \varepsilon$ and for $n \geqq n_2(\varepsilon)$, $1 - F_n(d; \theta) \leqq \varepsilon$. Let $n(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geqq n(\varepsilon)$, from (2.8) we have

$$(2.9) \qquad P(\boldsymbol{\theta}^{(r)}) \geqq (1-\varepsilon)^{k-t}[(1-\varepsilon)^{r+1} - \varepsilon^{r+1}] .$$

Moreover, for any given δ, $0 < \delta < 1$, it is possible to choose an $\varepsilon(0 < \varepsilon < 1)$ depending on δ and r, such that the right side of (2.9) is not less than $(1 - \delta)$. Thus, with Y_i's consistent, we have proved the following theorem.

THEOREM 2. *If $g_1(\theta) < \theta$, $g_2(\theta) > \theta$ and $\psi(\theta) < \theta$, then for any fixed θ, a given $\delta(0 < \delta < 1)$ and any $r = 0, 1, \cdots, t-1$, there exists an $n(\delta, r)$ such that $P(\boldsymbol{\theta}^{(r)}) \geqq 1 - \delta$ for $n \geqq n(\delta, r)$.*

It is possible to choose $g_1(\theta)$ and $g_2(\theta)$ so that the conditions of Theorem 2 and (2.6) are satisfied at the same time. Consequently there exists an $n(\delta)$ such that for $n \geqq n(\delta)$,

$$(2.10) \qquad \inf_{\Omega^*(\psi)} P(\boldsymbol{\theta}) = P(\boldsymbol{\theta}^{(t-1)}) \geqq 1 - \delta .$$

3. Explicit results for location and scale parameters.

Location parameter case. Let $F_n(y; \theta_i) = F_n(y - \theta_i)$, $\psi(\theta) = \theta - \delta$, $g_1(\theta) = \theta - a$, $g_2(\theta) = \theta + b$, where $\delta \geqq 0$ and a and b with $a + b > 0$ are preassigned constants. If the density $f_n(y - \theta_i)$ has a monotone likelihood ratio in y for θ_i and a and b are chosen such that

$$(3.1) \qquad F_n(-a) + F_n(b) \geqq 1 ,$$

then for $1 \leqq t \leqq k$,

$$(3.2) \qquad \inf_{\Omega(\psi)} P(\boldsymbol{\theta}) = t \int_{-a}^{b} F_n^{k-t}(y + \delta)[1 - F_n(y)]^{t-1}\, dF_n(y) .$$

If Y_i's are consistent and both a and b are chosen to be positive constants satisfying (3.1), then the right-side expression in (3.2) for any fixed k and t goes to unity as the common sample size n tends to infinity. Thus for a preassigned γ, there exists a smallest value of n so that (1.1) is satisfied. If $F_n(\cdot)$ is symmetric about the origin, a and b can be any pair of numbers with $b \geqq a > 0$. In general, the choice of a and b may depend on the sample size n chosen.

Scale parameter case. Let $F_n(y; \theta_i) = F_n(y/\theta_i)$, $y > 0$, $\theta_i \geqq 0$, $F_n(0) = 0$, $\psi(\theta) = \zeta\theta$, $g_1(\theta) = \theta/a$, $g_2(\theta) = \theta/b$, where ζ, a, b are preassigned constants such that $0 < \zeta \leqq 1$, $0 \leqq b < a$. If $(1/\theta_i)f_n(y/\theta_i)$ has a monotone likelihood ratio in

y for θ_i and constants a and b are chosen such that

$$F_n(1/a) + F_n(1/b) \geqq 1 , \tag{3.3}$$

then for $1 \leqq t \leqq k$,

$$\inf_{\Omega(\psi)} P(\boldsymbol{\theta}) = t \int_{1/a}^{1/b} F_n^{k-t}(y/\zeta)[1 - F_n(y)]^{t-1} dF_n(y) . \tag{3.4}$$

If Y_i's are consistent and constants a and b are chosen (possibly depending on n) such that $0 \leqq b < 1 < a$ in addition to (3.3), the right-side expression in (3.4) goes to unity as n tends to infinity. Hence there exists a smallest value of n so that (1.1) is satisfied for a preassigned γ.

4. Some concluding remarks. The present formulation *includes* as a special case the indifference zone formulation of the ranking problem by taking $a = b = \infty$ in the case of location parameters and $a = \infty$ and $b = 0$ in the case of scale parameters; $\Pr\{\text{CS} \cap \text{CD} \mid R\}$ then equals $\Pr\{\text{CS} \mid R\}$ and (3.2) reduces to (7) of [2] and (3.4) reduces to (10) of [2].

The present work also *includes* the confidence interval formulation for the largest location or scale parameter (see [1]) as a special case. For $t = k$, we have $\theta = \theta_{[1]}$, $\Omega^*(\psi) = \Omega_*$, $\Omega_* = \{\boldsymbol{\theta} : \theta_{[1]}$ held fixed$\}$ and $\Pr\{\text{CS} \cap \text{CD} \mid R\}$ equals $\Pr\{\text{CD} \mid R\}$. Now the result for $\theta = \theta_{[k]}$ can be derived by considering a strictly monotone transformation of the random variables Y_i's.

If $\psi(\theta) = \theta$ (that is, no indifference zone) then the integral (2.7) can be evaluated using incomplete beta function tables and the tables of the distribution function F for any fixed value of θ.

It is well known that in the CS problem alone, a meaningful lower bound on γ of the probability requirement (1.1) is $1/\binom{k}{t}$. Such is not the case in the $\text{CS} \cap \text{CD}$ problem as indicated by (2.7).

In this formulation of interval estimation and simultaneous selection for location (scale) parameters, the upper confidence bound for $\theta_{[k-t+1]}$ can be obtained by taking $b = \infty (b = 0)$ and the lower confidence bound by taking $a = \infty (a = \infty)$. However, for $1 < t < k$, all that can then be said regarding the infimum of $P(\boldsymbol{\theta})$ over $\Omega(\psi)$ by use of Theorem 1 is that it is equal to $\min\{P(\boldsymbol{\theta}^{(r)}), r = 0, 1, \cdots, t-1\}$. In general it is not possible to find the value of r which minimizes $P(\boldsymbol{\theta}^{(r)})$. However, for $t = 1$ or $t = k$, the infimum of $P(\boldsymbol{\theta})$ can be easily obtained.

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