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MINIMAL SUFFICIENT STATISTICS FOR FAMILIES OF PRODUCT MEASURES

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It is shown in this paper that the product family of countably many families of perfect probability measures defined on countably generated σ-fields admits a minimal sufficient statistic if and only if each component family admits a minimal sufficient statistic. Moreover, the minimal sufficient statistic of the product family is the "product" of the minimal sufficient statistics for the component families. Examples show that the assumptions on the component families cannot be omitted.

1. Introduction. Let $\mathfrak{P}_i \mid \mathscr{A}_i$, $i \in I$, be families of probability measures. In [3] it was shown without any assumptions that the family of all product measures $\times_{i \in I} P_i$ with $P_i \in \mathfrak{P}_i$, $i \in I$, admits a minimal sufficient σ-field if and only if each family $\mathfrak{P}_i \mid \mathscr{A}_i$, $i \in I$, admits a minimal sufficient σ-field. According to [4] the existence of a minimal sufficient σ-field neither implies nor is implied by the existence of a minimal sufficient statistic. In this paper we investigate the relations between minimal sufficient statistics for the component families and minimal sufficient statistics for the product family. Examples show that without any assumption the existence of minimal sufficient statistics for the component families does not imply the existence of a minimal sufficient statistic for the product family. However, for countably many families $\mathfrak{P}_i \mid \mathscr{A}_i$ of perfect probability measures each admitting a countably generated sufficient σ-field a minimal sufficient statistic for the product family exists if and only if minimal sufficient statistics for all component families exist. Moreover, the minimal sufficient statistic for the product family can be described in terms of the minimal sufficient statistics of the component families.

2. Notations. Let $\mathfrak{P} \mid \mathscr{A}$ be a family of probability measures (p-measures) on a basic set X. Let $\mathscr{B}$, $\mathscr{C}$ be subsystems of $\mathscr{A}$; we write $\mathscr{B} \subset \mathscr{C}(\mathfrak{P} \mid \mathscr{A})$ iff for every $B \in \mathscr{B}$ there exists $C \in \mathscr{C}$ with $P(B \triangle C) = 0$ for all $P \in \mathfrak{P}$. We write $\mathscr{B} \sim \mathscr{C}(\mathfrak{P} \mid \mathscr{A})$ iff $\mathscr{B} \subset \mathscr{C}(\mathfrak{P} \mid \mathscr{A})$ and $\mathscr{C} \subset \mathscr{B}(\mathfrak{P} \mid \mathscr{A})$. A σ-field $\mathscr{A}_0 \subset \mathscr{A}$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff for every $A \in \mathscr{A}$ there exists a conditional expectation of A, given $\mathscr{A}_0$, which is independent of $P \in \mathfrak{P}$. The σ-field $\mathscr{A}_0 \subset \mathscr{A}$ is minimal sufficient for $\mathfrak{P} \mid \mathscr{A}$ if $\mathscr{A}_0$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ and $\mathscr{A}_0 \subset \mathscr{A}_1(\mathfrak{P})$ for each sufficient σ-field $\mathscr{A}_1$ for $\mathfrak{P} \mid \mathscr{A}$.

A statistic $f \colon X \to Y$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff the σ-field

$$\mathscr{A}(f) \equiv \{f^{-1}C \colon C \subset Y\} \cap \mathscr{A}$$

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is sufficient for $\mathfrak{P}|\mathscr{A}$. A sufficient statistic $f: X \to Y$ is minimal sufficient for $\mathfrak{P}|\mathscr{A}$ iff for every sufficient statistic $g: X \to Z$ there exists a function $h: Z \to Y$ such that $f = h \circ g$ $\mathfrak{P}|\mathscr{A}$-a.e. A p-measure $P|\mathscr{A}$ is perfect iff for every real-valued $\mathscr{A}$-measurable function f and every $C \subset \mathbb{R}$ with $f^{-1}C \in \mathscr{A}$ there exists a Borel set $B \subset C$ such that $P(f^{-1}B) = P(f^{-1}C)$. It is well known that each measure defined on the Borel-field of a complete separable metric space is perfect. We remark that a measure is perfect if and only if it is quasicompact in the sense of Ryll-Nardzewski [6]. For each $i \in I$ let $\mathfrak{P}_i|\mathscr{A}_i$ be a family of p-measures on X_i. We denote by $\times_{i\in I} \mathfrak{P}_i | \times_{i\in I} \mathscr{A}_i$ the family of all product measures $\times_{i\in I} P_i$ on the product σ-field $\times_{i\in I} \mathscr{A}_i$, where $P_i \in \mathfrak{P}_i$ for each $i \in I$. If $f_i: X_i \to Y_i$, $i \in I$, we denote by $\times_{i\in I} f_i$ the function from $\times_{i\in I} X_i$ into $\times_{i\in I} Y_i$, defined by $(\times_{i\in I} f_i)(x_i)_{i\in I} = (f_i(x_i))_{i\in I}$.

3. The main results. In this section we investigate the implications between the existence of a minimal sufficient statistic for a product family and the existence of minimal sufficient statistics for the component families $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$. For minimal sufficient σ-fields instead of minimal sufficient statistics such existence implications hold true for arbitrary I and arbitrary component families $\mathfrak{P}_i|\mathscr{A}_i$. According to Example 2 and Example 3, for minimal sufficient statistics we need, however, restrictive conditions on I and the component families $\mathfrak{P}_i|\mathscr{A}_i$.

THEOREM 1. *Let I be a countable set and $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$, be families of perfect p-measures each of them admitting a countably generated sufficient σ-field. Then there exists a minimal sufficient statistic for $\times_{i\in I} \mathfrak{P}_i | \times_{i\in I} \mathscr{A}_i$ if and only if there exists a minimal sufficient statistic for each component family $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$. Moreover, if for each $i \in I$ f_i is a minimal sufficient statistic for the family $\mathfrak{P}_i|\mathscr{A}_i$, then $\times_{i\in I} f_i$ is a minimal sufficient statistic for the product family $\times_{i\in I} \mathfrak{P}_i | \times_{i\in I} \mathscr{A}_i$.*

PROOF. At first we prove that $\times_{i\in I} f_i$ is a minimal sufficient statistic for $\times_{i\in I} \mathfrak{P}_i | \times_{i\in I} \mathscr{A}_i$ if f_i are minimal sufficient statistics for $\mathfrak{P}_i|\mathscr{A}_i$.

According to Remark 4, for each $i \in I$ there exists a real valued $\mathscr{A}_i$-measurable minimal sufficient statistic for $\mathfrak{P}_i|\mathscr{A}_i$, say g_i. Since f_i, g_i are minimal sufficient statistics for $\mathfrak{P}_i|\mathscr{A}_i$, Theorem 4 of [5] implies that both induce a minimal sufficient σ-field for $\mathfrak{P}_i|\mathscr{A}_i$ and hence $\mathscr{A}_i(g_i) \sim \mathscr{A}_i(f_i)(\mathfrak{P}_i|\mathscr{A}_i)$, $i \in I$. Let $\mathbb{B}$ be the Borel-field of the space $\mathbb{R}$ of real numbers.

According to Lemma 2 of [5] $g_i^{-1}\mathbb{B} \sim \mathscr{A}_i(g_i)(\mathfrak{P}_i|\mathscr{A}_i)$ and hence $g_i^{-1}\mathbb{B}$ is a minimal sufficient σ-field for $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$. Therefore according to Theorem 1 of [3] $\times_{i\in I} g_i^{-1}\mathbb{B} = (\times_{i\in I} g_i)^{-1}\mathbb{B}^I$ is minimal sufficient for $\mathfrak{P}|\mathscr{A} \equiv \times_{i\in I} \mathfrak{P}_i | \times_{i\in I} \mathscr{A}_i$. According to Theorem VIII of [6] and Lemma 1 of [3] $\mathfrak{P}|\mathscr{A}$ is a family of perfect p-measures admitting a countably generated sufficient σ-field, whence according to Remark 4 (ii), $\times_{i\in I} g_i$ is a minimal sufficient statistic for $\mathfrak{P}|\mathscr{A}$.

Now we shall show that $\times_{i\in I} f_i$ is a minimal sufficient statistic for $\mathfrak{P}|\mathscr{A}$. Since f_i is a minimal sufficient statistic and g_i is a sufficient statistic for $\mathfrak{P}_i|\mathscr{A}_i$ we have $f_i = \varphi_i \circ g_i$ $\mathfrak{P}_i|\mathscr{A}_i$-a.e. with an appropriate function $\varphi_i (i \in I)$. Hence

$\times_{i\in I} f_i = \varphi \circ \times_{i\in I} g_i$ $\mathfrak{P}|\mathscr{A}$-a.e. with an appropriate function φ. To prove that $\times_{i\in I} f_i$ is a minimal sufficient statistic for $\mathfrak{P}|\mathscr{A}$ it therefore suffices to prove that $\times_{i\in I} f_i$ is a sufficient statistic for $\mathfrak{P}|\mathscr{A}$. Since $\times_{i\in I} f_i = \varphi \circ \times_{i\in I} g_i$ $\mathfrak{P}|\mathscr{A}$-a.e., Lemma 3 of [5] implies

$$(*) \qquad \times_{i\in I} \mathscr{A}_i(\times_{i\in I} f_i) \subset (\times_{i\in I} g_i)^{-1}\mathbb{B}^I(\mathfrak{P}|\mathscr{A}) .$$

Since $g_i^{-1}\mathbb{B} \sim \mathscr{A}_i(g_i) \sim \mathscr{A}_i(f_i)(\mathfrak{P}_i|\mathscr{A}_i)$, $i \in I$, we obtain

$$(\times_{i\in I} g_i)^{-1}\mathbb{B}^I = \times_{i\in I} g_i^{-1}\mathbb{B} \sim \times_{i\in I} \mathscr{A}_i(g_i) \sim \times_{i\in I} \mathscr{A}_i(f_i)(\mathfrak{P}|\mathscr{A}) .$$

Hence (*) implies

$$(\times_{i\in I} \mathscr{A}_i)(\times_{i\in I} f_i) \subset \times_{i\in I} \mathscr{A}_i(f_i) \subset \times_{i\in I} \mathscr{A}_i(\times_{i\in I} f_i)(\mathfrak{P}|\mathscr{A})$$

whence $(\times_{i\in I} \mathscr{A}_i)(\times_{i\in I} f_i) \sim \times_{i\in I} \mathscr{A}_i(f_i)(\mathfrak{P}|\mathscr{A})$. Since $\mathscr{A}_i(f_i)$ is sufficient for $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$, according to Lemma 2 of [3], $\times_{i\in I} \mathscr{A}_i(f_i)$ is sufficient for $\mathfrak{P}|\mathscr{A}$. Consequently $\times_{i\in I} f_i$ is a sufficient statistic for $\mathfrak{P}|\mathscr{A}$.

It remains to prove that each component family admits a minimal sufficient statistic if the whole product family admits a minimal sufficient statistic.

According to Theorem VIII of [6] and Lemma 1 of [3] $\mathfrak{P}|\mathscr{A}$ is a family of perfect probability measures and admits a countably generated sufficient σ-field. Hence, according to Theorem 4 of [5], the existence of a minimal sufficient statistic for $\mathfrak{P}|\mathscr{A}$ implies the existence of a minimal sufficient σ-field for $\mathfrak{P}|\mathscr{A}$. Hence according to Theorem 3 of [3] each component family $\mathfrak{P}_i|\mathscr{A}_i$, $i \in I$, admits a minimal sufficient σ-field and hence, according to Theorem 4 of [5], a minimal sufficient statistic.

We remark that the existence of a minimal sufficient statistic for a family $\mathfrak{P}|\mathscr{A}$ of p-measures neither implies nor is implied by the existence of a minimal sufficient statistic for the family of all product measures $P \times P$, $P \in \mathfrak{P}$, with identical components. This follows from Example 2 and Example 4 of [3].

The following example shows that even for two families of compact approximable, and hence perfect measures, Theorem 1 is not true any more without the assumption that *each* component family admits a countably generated sufficient σ-field; therefore the product family of two families of perfect measures, both admitting minimal sufficient statistics, need not admit a minimal sufficient statistic. Since both component families of our example admit a minimal sufficient σ-field, the product family admits a minimal sufficient σ-field (Theorem 4 of [3]). Thus we obtain a—rather simple—new example (see also Example 1 of [4] or Example 5 of [5]), answering the question of Bahadur and Lehmann (see [1] and [2]), whether the existence of a minimal sufficient σ-field implies the existence of a minimal sufficient statistic.

We remark that it remains unsolved whether or not Theorem 1 is true for not necessarily perfect measures. I guess the answer is no. But a counterexample, showing this, must necessarily be complicated; it requires—disregarding the additional aspects of the special problem—a family of p-measures on a countably generated σ-field, admitting either a minimal sufficient σ-field or a minimal sufficient statistic, but not both.

Example 2. In this example we construct a family $\mathfrak{P} \mid \mathscr{A}$ of p-measures admitting a minimal sufficient statistic and a single p-measure $Q \mid \mathscr{B}$ (the family $\{Q \mid \mathscr{B}\}$ obviously admits a minimal sufficient statistic) such that the product family $\mathfrak{P} \mid \mathscr{A} \times Q \mid \mathscr{B}$ does not admit a minimal sufficient statistic.

Let $X = Y = [0, 1]$ and $\mathscr{A} = \mathscr{B}$ be the σ-field generated by the countable subsets of $[0, 1]$. Let $\mathfrak{P} \mid \mathscr{A}$ be the family of all p-measures on $\mathscr{A}$, concentrated in a single point and $Q \mid \mathscr{B}$ be defined by $Q(B) = 0$ or 1 according as B or $\bar{B}$ is countable. Then $\mathscr{A}$ is the only sufficient σ-field for $\mathfrak{P} \mid \mathscr{A}$. Since $\mathscr{A}$ contains all one point sets, the identity is a minimal sufficient statistic for $\mathfrak{P} \mid \mathscr{A}$.

Assume now that there exists a minimal sufficient statistic for the product family $\mathfrak{P} \mid \mathscr{A} \times Q \mid \mathscr{B}$, say f. Since $g: X \times Y \to X$ defined by $g(x, y) = x$ is a sufficient statistic for $\mathfrak{P} \mid \mathscr{A} \times Q \mid \mathscr{B}$, there exists a $\mathfrak{P} \times Q$-null set $N_1 \in \mathscr{A} \times \mathscr{B}$ such that $(x, y_1), (x, y_2) \in \bar{N}_1$ implies $f(x, y_1) = f(x, y_2) \equiv c_x$. Since f is a sufficient statistic and $\mathscr{A} \times Y$ is a minimal sufficient σ-field for $\mathfrak{P} \mid \mathscr{A} \times Q \mid \mathscr{B}$, we obtain $\mathscr{A} \times Y \subset (\mathscr{A} \times \mathscr{B})(f)(\mathfrak{P} \times Q)$. Consequently $c_{x_1} \neq c_{x_2}$ for $x_1 \neq x_2$. Let h: $X \times Y \to \mathbb{R}$ be the function defined by

$$\begin{aligned} h(x, y) &\equiv x \qquad \text{if} \quad x \neq y \\ &\equiv -1 \qquad \text{if} \quad x = y . \end{aligned}$$

A little reflection shows that $(\mathscr{A} \times \mathscr{B})(h) \sim \mathscr{A} \times Y(\mathfrak{P} \times Q)$. Hence h is a sufficient statistic for $\mathfrak{P} \mid \mathscr{A} \times Q \mid \mathscr{B}$. As f is a minimal sufficient statistic, consequently there exists a $\mathfrak{P} \times Q$-null set $N_2 \in \mathscr{A} \times \mathscr{B}$ such that $(x_1, y_1), (x_2, y_2) \in \bar{N}_2$ and $h(x_1, y_1) = h(x_2, y_2)$ imply $f(x_1, y_1) = f(x_2, y_2)$. Then $N \equiv N_1 \cup N_2 \in \mathscr{A} \times \mathscr{B}$ is a $\mathfrak{P} \times Q$-null set and hence $N_x \equiv \{y \in Y: (x, y) \in N\}$ is countable for all $x \in X$. Let $D \equiv \{(x, x): x \in [0, 1]\}$. It is easy to see that $C \cap D$ is countable for each $C \in \mathscr{A} \times \mathscr{B}$ with C_x uncountable for at most countably many $x \in X$. Consequently $N \cap D$ is countable whence $\bar{N} \cap D$ is uncountable. As f is injective on $\bar{N} \cap D$ and h is constant on $\bar{N} \cap D$ we obtain a contradiction.

Obviously, Q and P, $P \in \mathfrak{P}$, are compact approximable and hence perfect measures. Furthermore, $\mathfrak{P} \mid \mathscr{A}$ does not admit a countably generated sufficient σ-field, since $\mathscr{A}$ is not countably generated.

The following example shows that Theorem 1 does not hold true for uncountably many component families of perfect measures defined on a countably generated σ-field.

Example 3. Let $I \equiv \mathbb{R}$ and $X_i \equiv \{0, 1\}$, $\mathscr{A}_i$ the power set of X_i for each $i \in I$. Let $\mathfrak{P}_i \mid \mathscr{A}_i$, $i \in I$, consist of the two p-measures $P_1 \mid \mathscr{A}_i$ and $P_2 \mid \mathscr{A}_i$, where $P_1\{0\} = \frac{1}{2}$, $P_2\{0\} = \frac{1}{4}$. Obviously $\mathfrak{P}_i \mid \mathscr{A}_i$ admits a minimal sufficient statistic for each $i \in I$. With similar methods as in Example 1 of [4] it can be shown that the product family $\times_{i \in I} \mathfrak{P}_i \mid \times_{i \in I} \mathscr{A}_i$ does not admit a minimal sufficient statistic. We remind the reader that according to Theorem 5 of [3] the product family does admit a minimal sufficient σ-field.

For the sake of completeness we cite the following result which is contained in the proof of Theorem 4 of [5].

REMARK 4. Let $\mathfrak{P} \mid \mathscr{A}$ be a family of perfect p-measures admitting a countably generated sufficient σ-field.

(i) If $\mathfrak{P} \mid \mathscr{A}$ admits a minimal sufficient statistic, then $\mathfrak{P} \mid \mathscr{A}$ admits a real-valued $\mathscr{A}$-measurable minimal sufficient statistic, too.

(ii) Let Y be a complete separable metric space with Borel-field $\mathscr{B}$. If f: $X \rightarrow Y$ is an $\mathscr{A}$, $\mathscr{B}$ measurable function such that $f^{-1}\mathscr{B}$ is a minimal sufficient σ-field for $\mathfrak{P} \mid \mathscr{A}$, then f is a minimal sufficient statistic for $\mathfrak{P} \mid \mathscr{A}$.

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