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REGRESSION DESIGN FOR SOME EQUIVALENCE CLASSES OF KERNELS[1]

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Earlier results on asymptotically optimal sequences of regression designs for autoregressive stochastic processes are extended (nearly) to the equivalence classes of such processes.

1. Introduction. Let $\{Y(t), t \in T\}$ be a stochastic process of the form

$$Y(t) = \theta f(t) + X(t)$$

where θ is an unknown constant, $f(t)$ is a known function on T, T is a closed bounded interval which we take to be $[0, 1]$, and $\{X(t), t \in T\}$ is a zero-mean Gaussian stochastic process with known continuous covariance kernel Q, $EX(t)X(t') = Q(t, t')$. The regression design problem is to choose an n-point subset (or "design") T_n,

$$T_n = \{t_1 < t_2 < \cdots < t_n, t_i \in T\}$$

so that the variance $\sigma^2_{T_n}$ of the Gauss–Markov estimate of θ given $\{Y(t), t \in T_n\}$ is as small as possible.

This problem has been considered by Sacks and Ylvisaker, Wahba, and Hájek and Kimeldorf [3], [11], [12], [13], [14], [16] for various special cases of Q. It is known that $\sigma^2_{T_n}$ is bounded away from 0 as $\Delta = \max_i |t_{i+1} - t_i|$ tends to 0 if and only if $f \in \mathscr{H}_Q$, where $\mathscr{H}_Q$ is the unique reproducing kernel Hilbert space (RKHS) with reproducing kernel (RK)Q, see [8]. It will be assumed that the reader is familiar with the basic properties of RKHS as given in [8], [16], see also [1].

For fixed $t \in T$, let Q_t represent the evaluation functional at t in $\mathscr{H}_Q$, that is

$$\langle Q_t, f \rangle_Q = f(t), \qquad f \in \mathscr{H}_Q$$

and

$$Q_t(t') = Q(t, t'),$$

where $\langle \cdot, \cdot \rangle_Q$ is the inner product in $\mathscr{H}_Q$.

Let P_{T_n} be the projection operator in $\mathscr{H}_Q$ onto the subspace spanned by $\{Q_t, t \in T_n\}$. It is well known that if $f \in \mathscr{H}_Q$, then $\sigma_{T_n}^{-2} = ||P_{T_n} f||_Q^2$ and $\sigma_T^{-2} = ||f||_Q^2$, where $||\cdot||_Q$ is the norm in $\mathscr{H}_Q$ and σ_T^2 is the variance of the Gauss Markov estimate of θ, given $\{Y(t), t \in T\}$. Hence $\sigma^2_{T_n}$ is minimized by minimizing $||f - P_{T_n} f||_Q^2$. From this point of view, the problem becomes one of choosing

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an optimal subspace in $\mathscr{H}_Q$ of the form span $\{Q_t, t \in T_n\}$, for the purpose of approximating the given element f. In this context, the problem has been considered by Karlin [5], [6]. The solution also has applications to the approximate solution of linear differential and integral equations, see [17], [18].

We suppose that $\{X(t), t \in T\}$ has exactly $m - 1$ quadratic mean derivatives. This entails that the functions $Q_t^{(\nu)}(\cdot)$ defined by

$$Q_t^{(\nu)}(\cdot) = \left(\frac{\partial^\nu}{\partial s^\nu}\right) Q(s, \cdot)\bigg|_{s=t}$$

are all well defined and in $\mathscr{H}_Q$, for $t \in T$ and $\nu = 1, 2, \cdots, m - 1$. Let P_{m,T_n} be the projection operator in $\mathscr{H}_Q$ onto the subspace of $\mathscr{H}_Q$ spanned by

$$\{Q_t^{(\nu)}, t \in T_n, \nu = 0, 1, \cdots, m - 1\} . \tag{1.1}$$

The optimal experimental design problem becomes tractable if we attempt to minimize $||f - P_{m,T_n} f||_Q$ rather than $||f - P_{T_n} f||_Q$, and the results are still useful, because of the relation ([16], (1.15))

$$\inf_{T_{nm}} ||f - P_{T_{mn}} f||_Q \leqq \inf_{T_n} ||f - P_{m,T_n} f||_Q \leqq \inf_{T_n} ||f - P_{T_n} f||_Q . \tag{1.2}$$

Further information about the role of derivatives may be found in Karlin [6], especially Theorem 3(i) and Theorem 4, and Sacks and Ylvisaker [13]. In particular, ([6], equation (13), [13], Theorem 4) if $m = 2$ and other conditions are satisfied, the right hand inequality in (1.2) becomes an equality.

Following [13], a sequence T_n^*, $n = 1, 2, \cdots$ of designs is said to be asymptotically optimal (with derivatives) if

$$\lim_{n \to \infty} \frac{||f - P_{m,T_n^*} f||_Q}{\inf_{T_n} ||f - P_{m,T_n} f||_Q} = 1 .$$

In [16], asymptotically optimal designs (with derivatives) are found for the case where X is a stochastic process formally satisfying the stochastic differential equation

$$(L_m X)(t) = dW(t) , \qquad t \in [0, 1] \tag{1.3a}$$

$$X^{(\nu)}(0) = \xi_\nu , \qquad \nu = 0, 1, \cdots, m - 1 , \tag{1.3b}$$

where L_m is defined by

$$(L_m f)(t) = \textstyle\sum_{j=0}^m a_{m-j} f^{(j)}(t) ,$$

$\{W(t), t \in [0, 1]\}$ is a Wiener process and $\{\xi_\nu\}_{\nu=0}^{m-1}$ are m zero mean Gaussian random variables independent of $W(t)$, $t \in [0, 1]$. L_m (in [16]) is such that its null space is spanned by $\{\phi_\nu\}_{\nu=1}^m$ where $\{\phi_\nu\}_{\nu=1}^m$ is an extended, complete Tchebychev (ECT) system of continuity class C^{2m}. In [3], the conditions on L_m are relaxed to: $a_0 \neq 0$, $a_{m-j} \in C^j$, with $E\xi_\nu^2 = 0$. It is the purpose of this note to show that the results of [3] and [16] may be extended to "nearly all" stochastic processes equivalent to X of (1.3).

A sequence of designs may be conveniently described by a continuous positive density h on $T = [0, 1]$. Let $T_n = T_n(h) = \{t_{0n}, t_{1n}, \cdots, t_{nn}\}$ be defined by

$$\int_0^{t_{in}} h(x)\,dx = \frac{i}{n}, \qquad i = 0, 1, \cdots, n.$$

(For ease of notation we are now letting T_n contain $n + 1$ points.)

We have the following result from [16] (as a consequence of Lemma 3).

PROPOSITION 1. *Let X be as in* (1.3), *and suppose*

$$f(t) = \int_0^1 Q(t, s)\rho(s)\,ds$$

where $\rho > 0$, and ρ possess a bounded first derivative. Let $T_n = T_n(h)$. Then

$$\|f - P_{m,T_n} f\|_Q^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(2m)!\,(2m+1)!} \int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)}\,ds + o\left(\frac{1}{n^{2m}}\right) \tag{1.4}$$

where

$$\alpha(s) = \frac{1}{a_0^2(s)}.$$

Following [11], asymptotically optimal sequences of designs are found from (1.4) by using a Hölder inequality and the fact that $\int_0^1 h(s)\,ds = 1$ to show that

$$\int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)}\,ds \geqq [\int_0^1 [\rho^2(s)\alpha(s)]^{1/2m}\,ds]^{2m+1}$$

with equality iff

$$h(s) = \frac{[\rho^2(s)\alpha(s)]^{1/(2m+1)}}{\int_0^1 [\rho^2(u)\alpha(u)]^{1/(2m+1)}\,du}.$$

Thus if

$$\int_0^{t_{in}^*} [\rho^2(s)\alpha(s)]^{1/(2m+1)}\,ds = \frac{i}{n} \int_0^1 [\rho^2(s)\alpha(s)]^{1/(2m+1)}\,ds, \qquad i = 0, 1, \cdots, n$$

then $T_n^* = \{t_{0n}^*, t_{1n}^*, \cdots, t_{nn}^*\}$, $n = 1, 2, \cdots$, is an asymptotically optimal sequence of designs with

$$\|f - P_{m,T_n^*} f\|_Q^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(2m)!\,(2m+1)!} [\int_0^1 [\rho^2(s)\alpha(s)]^{1/(2m+1)}\,ds]^{2m+1} + o\left(\frac{1}{n^{2m}}\right).$$

The "parameter function" $\alpha(s)$, $s \in [0, 1]$, $\alpha(s) = 1/a_0^2(s)$, plays a central role in the solution. It is not hard to convince one's self (see e.g. Hájek [2]) that two stochastic processes of the form (1.3) considered in [16] are equivalent iff their "initial value" rv's (1.3 b) are equivalent and the leading coefficient $a_0(s)$ of the defining differential operator is the same for both processes.

Thus, a maximal generalization of Theorem 1 would appear to be to X's equivalent to those of the form (1.3). In the remainder of this note we show that this is, in fact, the case, modulo a regularity condition on Q which we cannot seem to get rid of.

2. Equivalence classes of kernels for X of (1.3). Let $\{X_i(t), t \in [0, 1]\}$, $i = 0$, 1 be two zero mean Gaussian stochastic processes with continuous covariances

$Q_0(s, t)$ and $Q_1(s, t)$ respectively. Now, let Q_0 and Q_1 also denote the Hilbert–Schmidt operators on $\mathscr{L}_2[0, 1]$, with Hilbert–Schmidt kernels $Q_0(s, t)$ and $Q_1(s, t)$, defined by

$$(Q_i p)(t) = \int_0^1 Q_i(t, s)p(s)\, ds\,, \qquad p \in \mathscr{L}_2[0, 1],\ i = 0, 1\,.$$

A version of the Hájek–Feldman Theorem stated in Root [10] says that the measures corresponding to X_1 and X_2 are equivalent iff

$$Q_0^{-\frac{1}{2}} Q_1 Q_0^{-\frac{1}{2}} = I - B \tag{2.1}$$

where $Q_i^{-\frac{1}{2}}$ is the symmetric square root of Q_i^{-1}, $i = 0, 1$, and B is a Hilbert–Schmidt operator with $I - B$ invertible. For simplicity we will say that Q_0 and Q_1 are equivalent if (2.1) holds.

Let

$$G_0(s, u)_+^{m-1} = \frac{(s - u)_+^{m-1}}{(m-1)!}\, c(u) \tag{2.2}$$

where

$$c(u) = 1/a_0(u)$$

and $(x)_+ = x$, $x \geqq 0$, $(x)_+ = 0$ otherwise. Let

$$Q_0(s, t) = \int_0^1 G_0(s - u)G_0(t - u)\, du\,. \tag{2.3}$$

Q_0 is the covariance of X of (1.3) with $L_m = a_0 D^m$ and $E\xi_\nu^2 = 0$, $\nu = 0, 1, \cdots, m - 1$. The Hilbert–Schmidt operator Q_0 may be written

$$Q_0 = G_0 G_0^*$$

where G_0^* is the adjoint operator to G_0, the Hilbert–Schmidt operator with kernel (2.2). Since

$$Q_0 = Q_0^{\frac{1}{2}} Q_0^{\frac{1}{2}} = G_0 G_0^*\,,$$

$Q_0^{-\frac{1}{2}} Q_1 Q_0^{-\frac{1}{2}}$ is unitarily equivalent to $G_0^{-1} Q_1 G_0^{*-1}$ and

$$Q_0^{-\frac{1}{2}} Q_1 Q_0^{-\frac{1}{2}} = I - B$$

with B Hilbert–Schmidt and $I - B$ invertible iff

$$G_0^{-1} Q_1 G_0^{*-1} = I + A$$

for A some Hilbert–Schmidt operator with $I + A$ invertible. Thus Q_0 and Q_1 are equivalent if and only if

$$Q_1 = G_0(I + A)G_0^*$$

where A is Hilbert–Schmidt and $I + A$ invertible.
We summarize these remarks as

PROPOSITION 2. *A kernel Q_1 is equivalent to Q_0 of* (2.3) iff

$$\begin{aligned} Q_1(s, t) = &\int_0^1 \frac{(s - u)_+^{m-1}}{(m-1)!} \frac{(t - u)_+^{m-1}}{(m-1)!}\, c^2(u)\, du \\ &+ \int_0^1 \int_0^1 \frac{(s - u)_+^{m-1}}{(m-1)!} \frac{(t - v)_+^{m-1}}{(m-1)!}\, c(u)A(u, v)c(v)\, du\, dv \end{aligned} \tag{2.4}$$

where

$$\int_0^1 \int_0^1 A^2(s, t)\, ds\, dt < \infty$$

and $I + A$ is invertible, A being the operator with (symmetric) Hilbert–Schmidt kernel $A(s, t)$.

Now let $\tilde{Q}_0$ be

$$\tilde{Q}_0(s, t) = \sum_{j=1}^{m} \phi_j(s)\phi_j(t) + \int_0^1 G_0(s, u)G_0(t, u)\, du \tag{2.5}$$

with G_0 given by (2.2) and $\phi_j(s) = s^{j-1}/(j-1)!$, $j = 1, \cdots, m$. A process $\{X_0(t), t \in [0, 1]\}$ with covariance (2.5) has a representation

$$X_0(t) = \sum_{\nu=0}^{m-1} X_0^{(\nu)}(0)\phi_\nu(t) + (X_0(t) - P_{m,0}X_0(t))\,, \qquad t \in [0, 1]\,,$$

where $P_{m,0}X_0(t) = E\{X_0(t) \mid X^{(\nu)}(0), \nu = 0, 1, \cdots, m-1\}$ and $\{X^{(\nu)}(0)\}_{\nu=0}^{m-1}$ are i.i.d. $\mathscr{N}(0, 1)$. The process $(X_0(t) - P_{m,0}X_0(t))$ has covariance Q_0 of (2.4). For $\tilde{Q}_1$ to be equivalent to $\tilde{Q}_0$ it is necessary and sufficient that $\{X_1^{(\nu)}(0)\}_{\nu=0}^{m-1}$ exist in q.m. and have a covariance matrix of full rank, and that the process $X_1(t) - P_{m,0}X_1(t)$ have a covariance Q_1 of the form (2.4). In this case $X_1(t)$ has a representation of the form

$$X_1(t) = \sum_{\nu=0}^{m-1} X_1^{(\nu)}(0)\psi_\nu(t) + (X_1(t) - P_{m,0}X_1(t))$$

where

$$\psi_\nu(t) = \sum_{j=0}^{m-1} \sigma^{\nu j}\eta_j(t)$$

with

$$\{\sigma^{\nu j}\} = \{\sigma_{\nu j}\}^{-1}\,, \qquad \sigma_{\nu j} = EX_1^{(\nu)}(0)X_1^{(j)}(0)\,, \qquad \nu, j = 0, 1, \cdots, m-1$$

and

$$\eta_\nu(t) = EX(t)X^{(\nu)}(0) = \frac{\partial^\nu}{\partial s^\nu}\tilde{Q}_1(t, s)\Big|_{s=0}\,, \qquad \nu = 0, 1, \cdots, m-1\,.$$

By the properties of RKHS (see [8]), the $\{\eta_\nu\}$ must all be in $\mathscr{H}_{\tilde{Q}_1}$, and if $\tilde{Q}_1$ is equivalent to $\tilde{Q}_0$, they must also be in $\mathscr{H}_{\tilde{Q}}$.
We summarize these remarks in the following

PROPOSITION 3. *$\tilde{Q}_1$ is equivalent to $\tilde{Q}_0$ of* (2.5) *iff*

$$\tilde{Q}_1(s, t) = \sum_{j=0}^{m-1} \tilde{\psi}_j(s)\tilde{\psi}_j(t) + Q_1(s, t)\,,$$

where $\tilde{\psi}_j^{(\nu)}$ abs. cont., $\nu = 0, 1, \cdots, m-1$, $\tilde{\psi}_j^{(m)} \in \mathscr{L}_2$, the $m \times m$ matrix with ijth entry $\sigma_{\nu j}$,

$$\sigma_{\nu j} = \frac{\partial^{\nu+j}}{\partial s^\nu \partial t^j}\left(\sum_{i=0}^{m-1} \tilde{\psi}_i(s)\tilde{\psi}_i(t)\right)\Big|_{s=t=0}\,, \qquad \nu, j = 0, 1, \cdots, m-1$$

is of full rank, and $Q_1(s, t)$ is of the form (2.4).

Proposition 3 is a slight generalization of [15], Theorem 8; see also [4].

We have that $\mathscr{H}_{\tilde{Q}_1} = \mathscr{H}_{Q_1} \oplus \operatorname{span}\{\tilde{\psi}_i\}_{i=1}^{m}$, and if T_n includes the point $t = 0$, then $\|f - P_{m,T_n}f\|_{\tilde{Q}_1}^2 = \|P_{Q_1}(f - P_{m,T_n}f)\|_{\tilde{Q}_1}^2$, where P_{Q_1} is the projection operator in $\mathscr{H}_{\tilde{Q}_1}$ onto the subspace $\mathscr{H}_{Q_1}$. Thus we may without loss of generality consider Q_1 of the form (2.4). This remark holds, of course, whatever the rank of the matrix $\{\sigma_{\nu j}\}$.

3. Asymptotically optimal designs for $\tilde{Q}_1$. The purpose of this section is to prove the following

THEOREM. *Let $\tilde{Q}_1$ have a representation*

$$\begin{aligned}\tilde{Q}_1(s, t) = {} & \textstyle\sum_{i=0}^{m-1} \tilde{\phi}_i(s)\tilde{\phi}_i(t) \\ & + \int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!} \frac{(t-u)_+^{m-1}}{(m-1)!} c^2(u)\, du \\ & + \int_0^1 \int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!} \frac{(t-u)_+^{m-1}}{(m-1)!} c(u)A(u, v)c(v)\, du\, dv \end{aligned} \qquad s, t \in [0, 1]$$

where

(i) $\tilde{\phi}_i^{(\nu)}$ *abs. cont.*, $\nu = 0, 1, \cdots, m-1$, $\tilde{\phi}_i^{(m)} \in \mathscr{L}_2[0, 1]$,

(ii) $c > 0$, c' *bounded*,

(iii) $\int_0^1 \int_0^1 A^2(u, v)\, du\, dv < \infty$,

(iv) *the function γ_t given by*

$$\gamma_t(s) = \int_0^s \frac{\partial}{\partial t} \frac{1}{c(t)} A(t, \eta)c(\eta)\, d\eta$$

is well defined and is in the RKHS $\mathscr{H}_{K_1}$ *with* RK K_1 *given by*

$$K_1(s, t) = \int_0^{\min(s,t)} c^2(u)\, du + \int_0^s \int_0^t c(u)A(u, v)c(v)\, du\, dv$$

and

$$\|\gamma_t\|_{K_1} \leqq M_1 < \infty$$

where $\|\cdot\|_{K_1}$ is the norm in $\mathscr{H}_{K_1}$.

Let

$$f(t) = \int_0^1 \tilde{Q}_1(t, s)\rho(s)\, ds$$

with $\rho > 0$, ρ' bounded, and let $T_n = \{t_{in}\}_{i=0}^n$ with

$$\int_0^{t_{in}} h(u)\, du = \frac{i}{n}, \qquad i = 0, 1, \cdots, n$$

where

$$\int_0^1 h(u)\, du = 1, \qquad h > 0,\ h \text{ continuous.}$$

Then where $\alpha = c^2$

$$\|f - P_{m,T_n} f\|_{Q_1}^2 = \frac{1}{n^{2m}} \frac{(m!)^2}{(2m)!\,(2m+1)!} \int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)}\, ds + o\left(\frac{1}{n^{2m}}\right).$$

REMARK. The hypotheses of the Theorem do not include $I + A$ invertible. On the other hand, if $I + A$ is invertible then condition (iv) is equivalent to $\gamma_t \in \mathscr{H}_{K_0}$, the RKHS with RK

$$K_0(s, t) = \int_0^{\min(s,t)} c^2(u)\, du,$$

where $\mathscr{H}_{K_0} = \{f : f(0) = 0, f \text{ abs. cont.}, f'/c \in \mathscr{L}_2\}$. Thus if $I + A$ is invertible and (ii) holds then (iv) is equivalent to

$$\int_0^1 \left(\frac{\partial}{\partial t}\frac{1}{c(t)} A(t, \eta)\right)^2 d\eta < \tilde{M} < \infty .$$

Condition (iv) is similar to a condition used in [11]. This condition is used in the proof of Lemma 1 to follow, and we see no way to eliminate it there.

PROOF. The proof below follows closely along the lines of the proof of Theorem 1 of [13], generalized with the aid of [16].

The proof begins with Lemma 1.

LEMMA 1. *Let*

$$\begin{aligned} K_0(s, t) &= \int_0^{\min(s,t)} c^2(u)\, du \\ K_1(s, t) &= \int_0^{\min(s,t)} c^2(u)\, du + \int_0^s \int_0^t c(u)A(u, v)c(v)\, du\, dv \\ f_0(t) &= \int_0 K_0(t, u)\rho(u)\, du \\ f_1(t) &= \int_0^1 K_1(t, u)\rho(u)\, du \end{aligned}$$

where $\int_0^1 \int_0^1 A^2(u, v)\, du\, dv < \infty$, *where* $c, \rho > 0$, *continuous*, c', ρ' *bounded. Let* $\mathscr{H}_{K_i}$, $i = 0, 1$, *be the* RKHS's *with reproducing kernels* K_i, $i = 0, 1$, *and inner products* $\langle \cdot, \cdot \rangle_{K_0}$ *and* $\langle \cdot, \cdot \rangle_{K_1}$ *respectively. Suppose further that, for each* t, *the function* γ_t *defined by*

$$\gamma_t(s) = \int_0^s \frac{\partial}{\partial t}\frac{1}{c(t)} A(t, \eta)c(\eta)\, d\eta$$

satisfies

$$\gamma_t \in \mathscr{H}_{K_1}, \qquad ||\gamma_t||_{K_1} \leqq M_1 < \infty , \qquad t \in [0, 1] . \tag{3.1}$$

Then, there exists an ε *independent of* ρ *such that, for sufficiently large* n,

$$1 - \varepsilon\Delta \leqq \frac{||f_1 - P_{T_n} f_1||_{K_1}^2}{||f_0 - P_{T_n} f_0||_{K_0}^2} \leqq 1 + \varepsilon\Delta \tag{3.2}$$

where

$$\Delta = \max_i |t_{i+1} - t_i| .$$

Here, for $i = 0, 1$, $P_{T_n} f_i$ is the projection of f_i in $\mathscr{H}_{K_i}$ onto the subspace of $\mathscr{H}_{K_i}$ spanned by $\{K_{it}, t \in T_n\}$, where $K_{it}(t') = K_i(t, t')$.

PROOF. For $i = 0, 1$,

$$\begin{aligned} \langle f_i - P_{T_n} f_i, f_i - P_{T_n} f_i \rangle_{K_i} &= \langle f_i, f_i - P_{T_n} f_i \rangle_{K_i} \\ &= \int_0^1 \rho(u)(f_i(u) - P_{T_n} f_i(u))\, du . \end{aligned}$$

Then

$$\begin{aligned} ||f_0 - P_{T_n} f_0||_{K_0}^2 &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)(f_0(u) - P_{T_n} f_0(u))\, du \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)\, du \int_{t_i}^{t_{i+1}} B_i(u, v)\rho(v)\, dv \end{aligned}$$

where $t_0 \equiv 0$, $t_n \equiv 1$ and, according to [16] $B_i(u, v)$ is, for $u, v \in [t_i, t_{i+1}]$, the Green's function for the differential operator $L_m{}^*L_m = g$ with boundary conditions $f(t_i) = f(t_{i+1}) = 0$,

$$(L_m{}^*L_m f)(t) = \frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}f(t) .$$

Similarly,

$$\|f_1 - P_{T_n}f_1\|_{K_1}^2 = \textstyle\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)(f_1(u) - P_{T_n}f_1(u))\, du . \tag{3.3}$$

Since $f_1(u) - P_{T_n}f_1(u) = 0$ for $t = t_0, t_1, \cdots, t_n$, and $f_1 - P_{T_n}f_1 \in \mathscr{L}_2[0, 1]$, we may write

$$\begin{aligned} &f_1(u) - P_{T_n}f_1(u) \\ &\quad = \textstyle\int_{t_i}^{t_{i+1}} B_i(u, v)\frac{d}{dv}\frac{1}{c^2(v)}\frac{d}{dv}(f_1(v) - P_{T_n}f_1(v))\, dv , \quad u \in [t_i, t_{i+1}] \end{aligned} \tag{3.4}$$

where B_i is as before. But, since

$$f_1(t) = \textstyle\int_0^1 K_0(t, u)\rho(u)\, du + \int_0^1 \rho(u)\, du \int_0^u \int_0^t c(\xi)A(\xi, \eta)c(\eta)\, d\xi\, d\eta ,$$

then

$$\begin{aligned} \frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}f_1(t) &= \rho(t) + \textstyle\int_0^1 \rho(u)\, du \int_0^u \frac{\partial}{\partial t}\frac{1}{c(t)} A(t, \eta)c(\eta)\, d\eta \\ &= \rho(t) + \textstyle\int_0^1 \rho(u)\gamma_t(u)\, du . \end{aligned} \tag{3.5}$$

By our assumption, $\gamma_t \in \mathscr{H}_{K_1}$, so that (3.5) becomes

$$\frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}f_1(t) = \rho(t) + \textstyle\int_0^1 \rho(u)\gamma_t(u) = \rho(t) + \langle\gamma_t, f_1\rangle_{K_1} . \tag{3.6}$$

Also

$$(P_{T_n}f_1)(t) = (K_1(t, t_1), K_1(t, t_2), \cdots, K_1(t, t_n))K_{1,n}^{-1}(f_1(t_1), f_1(t_2), \cdots, f_1(t_n))'$$

where $K_{1,n}$ is the $n \times n$ matrix with ijth entry $K_1(t_i, t_j)$. Now, for $t \neq t_i$,

$$\frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}K_1(t, t_i) = \frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}[\textstyle\int_0^t [\int_0^{t_i} c(\xi)A(\xi, n)c(\eta)\, d\xi\, d\eta] = \gamma_t(t_i)$$

so that, for each fixed $t \notin T_n$,

$$\begin{aligned} &\frac{d}{dt}\frac{1}{c^2(t)}\frac{d}{dt}(P_{T_n}f_1(t)) \\ &\quad = (\gamma_t(t_1), \gamma_t(t_2), \cdots, \gamma_t(t_n))K_{1,n}^{-1}(f_1(t_1), f_1(t_2), \cdots, f_1(t_n))' \\ &\quad = \langle\gamma_t, P_{T_n}f_1\rangle_{K_1} . \end{aligned} \tag{3.7}$$

Thus, by (3.3), (3.4), (3.6), (3.7),

$$\begin{aligned} \|f_1 - P_{T_n}f_1\|_{K_1}^2 &= \textstyle\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)\, du \int_{t_i}^{t_{i+1}} B_i(u, v)\rho(v)\, dv \\ &\quad + \textstyle\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)\, du \int_{t_i}^{t_{i+1}} B_i(u, v)\langle\gamma_v, f_1 - P_{T_n}f_1\rangle_{K_1}\, dv . \end{aligned}$$

Now ρ and B_i are nonnegative, so we may write

$$|\int_{t_i}^{t_{i+1}} \rho(u)\,du \int_{t_i}^{t_{i+1}} B_i(u, v)\langle \gamma_v, f_1 - P_{T_n} f_1\rangle_{K_1}\,dv| \\ \leqq \int_{t_i}^{t_{i+1}} \rho(u)\,du \int_{t_i}^{t_{i+1}} B_i(u, v)\,dv \times M_1\|f_1 - P_{T_n} f_1\|_{K_1},$$

where M_1 is defined in (3.1).

Now, letting

$$\xi_0(t) = \int_0^1 K_0(t, u)\,du$$

it may be shown that[2]

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \rho(u)\,du \int_{t_i}^{t_{i+1}} B_i(u, v)\,dv = \langle f_0 - P_{T_n} f_0, \xi_0 - P_{T_n}\xi_0\rangle_{K_0}. \tag{3.8}$$

By (1.4),

$$\|\xi_0 - P_{T_n}\xi_0\|_{K_0} = M_2\left(\frac{1}{n}\left(1 + o\left(\frac{1}{n}\right)\right)\right)$$

for appropriately chosen M_2.

Thus

$$\|f_1 - P_{T_n} f_1\|_{K_1}^2 = \|f_0 - P_{T_n} f_0\|_{K_0}^2 + \theta\frac{M_3}{n}\|f_1 - P_{T_n} f_1\|_{K_1}\|f_0 - P_{T_n} f_0\|_{K_0}$$

for some θ with $|\theta| < 1$ and $M_3 = M_1 M_2$, and so

$$\frac{\|f_1 - P_{T_n} f_1\|_{K_1}^2}{\|f_0 - P_{T_n} f_0\|_{K_0}^2} = 1 + \theta\frac{M_3}{n}\left(1 + o\left(\frac{1}{n}\right)\right).$$

Since $1/n \leqq \Delta$, the Lemma is proved.

LEMMA 2. *For $m \geqq 2$ let*

$$Q_i(s, t) = \int_0^s \int_0^t \frac{(s-u)_+^{m-2}}{(m-2)!}\frac{(t-u)_+^{m-2}}{(m-2)!} K_i(u, v)\,du\,dv \qquad i = 1, 2$$

where K_i, $i = 0, 1$ are as Lemma 1. *Let*

$$f_i(t) = \int_0^1 Q_i(t, u)\rho(u)\,du, \qquad i = 0, 1.$$

Then, there exists an ε independent of ρ such that, for sufficiently large n,

$$1 - \varepsilon\Delta \leqq \frac{\|f_1 - P_{m,T_n} f_1\|_{Q_1}^2}{\|f_0 - P_{m,T_n} f_0\|_{Q_0}^2} \leqq 1 + \varepsilon\Delta. \tag{3.9}$$

Here $P_{m,T_n} f_i$ is the projection of f_i in $\mathscr{H}_{Q_i}$, $i = 0, 1$ onto the subspace of (1.1) with $Q = Q_i$.

The proof of this Lemma is contained within the proof of Theorem 1 of [13], page 2065 Equations (2.28)—(2.31), where it is shown that (3.2) implies (3.9).

The Theorem now follows by using the proof of Lemma 3 of [16] (where only condition ii on c is needed for $Q = Q_0$) to show that

$$\|f_0 - P_{m,T_n} f_0\|_{Q_0}^2 = \frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!\,(2m+1)!}\int_0^1 \frac{\rho^2(s)\alpha(s)}{h^{2m}(s)}\,ds + o\left(\frac{1}{n^{2m}}\right).$$

[2] Equation (3.8) may be checked by following the argument of Lemma 1 of [16]; see equations (3.4), (3.5) and (3.22). Equation (3.4a) there should read $f(t) = EX(t)\int_0^1 X(u)\rho(u)\,du$.

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