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REMARKS ON MEASURABLE SELECTIONS

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Many authors have studied the problem of obtaining an optimal plan for a dynamic programming problem or a stochastic game. In several of these works a maximal selection theorem of Dubins and Savage for u.s.c. (upper-semi-continuous) maps has played a key role. Our purpose is to relax the requirements that the maps be u.s.c. and still obtain maximal or at least nearly maximal selections.

1. Notation and preliminary remarks. The publications of Blackwell, Dubins and Savage, Maitra, and Parthasarathy, and Strauch cited in the references provide background for applications of the selection theorems. The selection theorem of Dubins and Savage can be found in [4, Chapter 2, Section 16] and in [6].

Throughout this note, X will be a separable complete metric space with distance function δ, E will be a compact metric space with distance function d, and K will denote the compact metric space of compact subsets of E with Hausdorff distance function h. The basic theory of the metric space K can be found in [5]. The Cartesian product of a set A and a set B is denoted by AB and πC denotes the projection of a subset, C, of AB on A; f will be a map of X into K, and $F = \{(x, y); y \in f(x)\}$; f is u.s.c. if for every closed subset D of E, $\{x; f(x) \cap D \neq \phi\}$ is a closed subset of X. Unless specified otherwise, u will be a nonnegative (real-valued) u.s.c. function on F: $\{u \geqq a\}$ is a closed subset of F, $a \in R$. Let $\mathscr{U}(x) = \sup \{u(x, y); y \in f(x)\}$.

Because E is compact, the projection πG of a closed subset G of XE is a closed subset of X; moreover, if $g(x) = G \cap (xE)$, $x \in \pi G$, then g is an u.s.c. map of $\pi(G)$ into K. Conversely [5, I, 175] if f is u.s.c. then F is a closed subset of XE.

The basic selection theorem of Dubins and Savage asserts that if f is u.s.c., then there is a Borel map ϕ of X into E with $\phi \subset F$ (i.e., $\phi(x) \in f(x)$) and $u(x, \phi(x)) = \mathscr{U}(x)$, for all x in X. We extend this result to Borel functions f by establishing the following.

THEOREM 1. *Let f be a Borel map of X into K and let u be a nonnegative* u.s.c. *function F. Then there is a Borel map, ϕ, of X into E such that $\phi(x) \in f(x)$ and $u(x, \phi(x)) = \mathscr{U}(x)$, $x \in X$.*

Since X is a separable complete metric space and K is a compact metric space, there is a well-known theory of maps from X to K. Thus f is a continuous map if the inverse image of every closed set is closed and f is a Borel map if the

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inverse image of every open ball $N(h, A, \varepsilon)$ is a Borel set. After the condition that f be u.s.c. was found to be sufficient for a Borel selector to exist, more general classes of maps based on the definition of u.s.c. were introduced: f is said to be B-measurable if for every closed subset D of E, $\{x; f(x) \cap D \neq \phi\}$ is a Borel set; and f is μ-measurable, where μ is the completion of a probability measure on the Borel subsets of X, if $\{x; f(x) \cap D \neq \phi\}$ is μ-measurable for every $D \in K$. The classes of Borel measurable and B-measurable maps coalesce [5, II, 47] and F is a Borel set if f is a Borel function [5, I, 167]. Theorem 1 and the latter observation verify that if f is a Borel map of X into K, then F is a Borel set that contains the graph of a Borel function ϕ. Not every Borel set contains the graph of a Borel function: see [3] and the references therein.

Before establishing Theorem 1, we transcribe another selection theorem.

THEOREM 2. *Let each of X and Y be a separable complete metric space. Let g be a map of X into the subsets of Y with $G = \{(x, y); y \in g(x)\}$ an analytic subset of XY. Let u be a nonnegative Borel function on G. Then there exists a uniformly convergent sequence ϕ_i of maps (measurable with respect to the σ-algebra generated by the analytic sets in X) of X into Y such that $\lim_i u(x, \phi_i(x)) = \mathscr{U}(x)$, uniformly in x, and for any open set V in Y, $\{x; \phi_i(x) \in V\}$ is in the σ-algebra generated by the analytic sets in X.*

The proof is a straightforward modification of the proof in Section 2: one tears G apart systematically and applies Sion's Corollary 4.4 in [8] to each part.

As the referee points out, Lemma 4 of Dubins and Savage's proof can be adapted to obtain a, perhaps, shorter proof of Theorem 1 without resorting to Sion's work [8]. However, we present an alternate proof of the Dubins–Savage selection theorem in Section 2. Our proof applies to Borel measurable maps (one merely notices that while certain of the particular types of Borel sets encountered in Section 2 need no longer be of that type, they are still Borel sets), thus yielding Theorem 1, and also produces approximations such as those in Theorem 2.

2. A proof of the basic selection theorem. To indicate the flavor of the proof, we begin by constructing an approximation, b_ε, $0 < \varepsilon < 1$, to ϕ as follows. Denote by A_k the closed set $\{u \geqq k\varepsilon\}$, where $k = 0, 1, 2, \cdots$. Since $\pi((YB) \cap A_k)$ is a closed subset of X whenever B is a closed subset of E, we can apply Theorem 4.1 of [12] to assert the existence of a Borel function b_k on $B_k = \pi A_k$ with $b_k \subset A_k$. Let $b_\varepsilon = b_k$ on $B_k - B_{k+1}$, $k = 0, 1, 2, \cdots$. Then b_ε is a Borel function and $\mathscr{U}(x) - \varepsilon \leqq u(x, b_\varepsilon(x)) \leqq \mathscr{U}(x)$. We shall choose a sequence $\varepsilon_i \to 0$ and refine the construction to obtain uniform convergence of the sequence $\phi_i = b_{\varepsilon_i}$; the limit is a Borel function ϕ with $u(x, \phi(x)) = \mathscr{U}(x)$. Thus we set $\varepsilon_i = 1/2^i$ and denote by A_{ik} the closed set $\{u \geqq k/2^i\}$. Then we recall that there is a sequence $\{S_i\}$ of finite collections $S_i = \{C_{i1}, \cdots, C_{in_i}\}$ of closures of pairwise disjoint nonempty open subsets of E, where the diameter of each C_{ij} is less than $1/2^i$ and $\bigcup_{j=1}^{n_i} C_{ij} = E$. Moreover, for each $k \leqq n_{i+1}$ there is a smallest integer $j \leqq n_i$ with $C_{(i+1)k} \subset C_{ij}$; thus, each element of S_{i+1} is identified with a unique element of S_i which contains it: we also require that $k < k_1 \Rightarrow j \leqq j_1$, where k is identified with j and k_1 is identified with j_1. Fix i for the moment and suppress it from the notation; thus we consider $S = \{C_1, \cdots, C_n\}$. We wish to make a construction for each A_{ik}. To indicate the procedure while maintaining a manageable notation, we shall temporarily identify A_{ik} with F and use the notation that is available for F. Notice that $\mathscr{U}(x) = \max \mathscr{U}_j(x)$, where $\mathscr{U}_j(x) = \sup \{u(x, y); y \in f(x) \cap (XC_j)\}$. Now fix j and suppose $\mathscr{U}(x) > \mathscr{U}_j(x)$. Then there exist two rational numbers $r < t$ such that x belongs to the closed set $T = \{z; f(z) \cap \{u \geqq t\} \neq \phi\}$ and x is also in the open set

$$\begin{aligned} R &= \{z; (f(z) \cap C_j) \subset \{u < r\}\} \\ &= \{X - \pi(F \cap (XC_j))\} \cup \{x \in \pi(F \cap (XC_j)); (f(x) \cap C_j) \subset \{u < r\}\}; \end{aligned}$$

the lattermost set is open because the restriction of u to $F \cap (XC_j)$ is u.s.c. Thus $T \cap R$ is an F_σ; taking unions over pairs of rationals $r < t$ permits us to assert that $V_j = \{x; \mathscr{U}(x) > \mathscr{U}_j(x)\}$ is an F_σ. Hence $W_j = \pi(F \cap (XC_j)) - V_j$ is a G_δ. Consequently $(W_j E) \cap F \cap (XC_j)$ is a Borel set; moreover, whenever B is closed in E, $\pi[(XB) \cap (W_j E) \cap F \cap (XC_j)] = \{\pi[(XB) \cap F \cap (XC_j)]\} \cap W_j$ is a Borel set. Therefore Sion's Theorem 4.1 in [8] implies that there exists a Borel function h_j on W_j with $h_j \subset (F \cap (XC_j))$. Let $h = h_1$ on W_1, h_2 on $W_2 - W_1, \cdots, h_j$ on $W_j - \bigcup_{i<j} W_i$, $j = 2, \cdots, n$. Now we reintroduce A_{ik}. Thus h_{ik} is a Borel function on A_{ik}. Let $\phi_i = h_{ik}$ on $A_{ik} - A_{i(k+1)}$. Then ϕ_i converges uniformly to a Borel function ϕ with $\mathscr{U}(x) = u(x, \phi(x))$.

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