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REPEATED GAMES WITH ABSORBING STATES

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A zero-sum two person game is repeatedly played. Some of the payoffs are "absorbing" in the sense that, once any of them is reached, all future payoffs remain unchanged. Let v_n denote the value of the n-times repeated game, and let v_∞ denote the value of the infinitely-repeated game. It is shown that $\lim v_n$ always exists. When the information structure is symmetric, v_∞ also exists and $v_\infty = \lim v_n$.

Introduction. This paper is concerned with a situation in which two players repeatedly play a zero-sum game with a payoff matrix that contains some "absorbing" entries. That is, once any such entry is reached, all payoffs in the future plays of the game must be equal to that same entry, regardless of the players' future actions. We call such games "repeated games with absorbing states."

As an example, we describe "the big match," due to Blackwell and Ferguson [1]:

Every day player II chooses a number, 0 or 1, and player I tries to predict II's choice, winning a point if he is correct. This continues as long as player I predicts 0. But if he ever predicts 1, all future choices for both players are required to be the same as that day's choices: if player I is correct on that day, he wins a point every day thereafter; if he is wrong on that day, he wins zero every day thereafter.

It is easily verified that "the big match" is indeed a repeated game with absorbing states. Its payoff matrix is

1	0
0*	1*

where the asterisks denote absorbing elements.

Our aim is to find the values and optimal strategies in repeated games with absorbing states that have a large number of stages (i.e., plays).

Now, when the number of stages is large, some normalization becomes necessary. Since the relevant notion is that of "value per play," we agree to define the payoff in the n-stage game as the sum of the payoffs in the individuals plays divided by n.

First, we consider v_n, the value of the n-stage game. In Section 1, we show that v_n always converges, and describe a method for the computation of the

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limit (Theorem 1.6). We also suggest an inductive procedure by which optimal strategies can be computed for the n-stage game (Lemma 1.2).

Next, we try to extend our knowledge to games that not only consist of a large number of stages, but in which the players do not know in advance exactly how many stages there are going to be. To approximate such games, we introduce infinitely-repeated games (Definitions 2.2 and 2.3).

In Section 2, we set out to find the values and the optimal strategies in infinitely-repeated games. The main problem encountered is that of establishing exactly how much "weight" should be attributed to an absorbing element as compared to a non-absorbing element: It is quite clear that, with n-stages to go, an absorbing element "outweighs" a non-absorbing element by a factor of n. But what if n is infinite? To illustrate the difficulty involved, note that in a "big match" with finitely many stages, player I's (unique) optimal strategy is to choose row 2 with probability $1/(n + 1)$, where n is the number of stages left to the end of play. It follows that, in an infinitely-repeated version of "the big match," for player I ever to choose row 2 with positive probability would mean "underrating" the absorbing elements. On the other hand, it makes no sense for him to choose row 1 with certainty all the time (this, in fact, would be "overrating" the absorbing elements).

This illustration shows that, in infinitely-repeated games, a player might not be able to avoid either underrating or overrating the absorbing elements. Both ways he suffers losses, so none of his strategies can be optimal (Example 2.9).

Still, the player might try to keep his losses down to ε, making his strategy ε-optimal. Our main result in Section 2 is that this is indeed possible, provided information on the opponent's past actions is available. The general idea is that, with the above information, the player can tell at each stage what proportion of his losses is still temporary (in the sense that it does not involve absorbing elements), and regulate his "weighting" of the absorbing elements accordingly.

To be more specific, we define *symmetric information* as that state of information in which each player, in addition to remembering his own actions, is informed of his opponent's past actions. We show that, assuming symmetric information, ε-optimal strategies, hence the value, always exist. We also show that this value (denoted V_∞) is equal to $\lim v_n$ (Theorem 2.1). An immediate application is that V_∞ may be computed by the method of Section 1.

It turns out that, lacking information on the opponent's past actions, a player's losses might accumulate without his being able to do anything about it. Indeed, if symmetric information is not assumed, ε-optimal strategies, hence the value, may fail to exist (Example 2.10).

The results of Section 2 are not altogether satisfactory, since the definition of V_∞ (Definition 2.3) does not exactly correspond to the notion of value in games with a large but unspecified number of stages.

To amend this, we introduce a definition of a value v_∞ (Definition 3.3) that requires of an ε-optimal strategy that it should get the player's payoff within ε

of v_∞ after a number of stages that does not depend on the opponent's actions. Section 2 is devoted to the proof that, in the case of symmetric information, v_∞ exists and $v_\infty = V_\infty = \lim v_n$ (Theorem 3.4).

1. The asymptotic behavior of v_n. This section is devoted to the proof that $\lim v_n$ exists for all repeated games with absorbing states.

We start with several lemmas.

LEMMA 1.1.

$$|v_n - v_{n-1}| \leqq |A| \frac{2}{n-1},$$

where A denotes the payoff matrix and $|A|$ is the maximal absolute value of elements in A.

PROOF. This follows from the fact that

$$(n-1)v_{n-1} - |A| \leqq n \cdot v_n \leqq (n-1)v_{n-1} + |A|$$

and that

$$|v_n| \leqq |A| .$$

We introduce the notation $A(u, t)$ for the matrix defined as follows:

$$(1.1) \qquad \begin{aligned} A(u, t)_{ij} &= a_{ij} && \text{if } a_{ij} \text{ is absorbing} \\ &= ta_{ij} + (1-t)u && \text{otherwise.} \end{aligned}$$

We denote the value of this matrix by $v(u, t)$.

Our next result is a recursive formula for the value of the n-stage game. Note that this formula may be applied to get an inductive procedure for the computation of optimal strategies.

LEMMA 1.2.

$$v_n = v\left(v_{n-1}, \frac{1}{n}\right).$$

PROOF. Suppose a_{ij} is reached at stage 1. If a_{ij} is absorbing, then the expected payoff is a_{ij}. If not, the expected payoff is $(a_{ij} + (n-1)v_{n-1})/n$. It follows that v_n is equal to the value of $A(v_{n-1}, n^{-1})$.

LEMMA 1.3. *Let $u_1 \leqq u_2$. Then for all $0 \leqq t \leqq 1$,*

$$(1.2) \qquad (v(u_1, t) - u_1) - (v(u_2, t) - u_2) \geqq t(u_2 - u_1) .$$

PROOF. It follows from the definition of $A(u, t)$ that for all i, j,

$$A(u_2, t)_{ij} - A(u_1, t)_{ij} \leqq (1-t)(u_2 - u_1) .$$

But this implies that

$$v(u_2, t) - v(u_1, t) \leqq (1-t)(u_2 - u_1) ,$$

and (1.2) follows at once.

LEMMA 1.4. *Let $\Delta(u) = \lim_{t \to 0+} t^{-1}(v(u, t) - u)$. Then $\Delta(u)$ exists for all u.*

PROOF. Clearly,

$$\begin{aligned} \Delta(u) &= +\infty && \text{if} \quad v(u,0) > u \\ &= \frac{\partial v}{\partial t_+}(u,0) && \text{if} \quad v(u,0) = u \\ &= -\infty && \text{if} \quad v(u,0) < u . \end{aligned}$$

The existence of $\partial v/\partial t_+(u, 0)$ follows from Theorem 1 of [2] where a linear-programming method for the computation of this derivative is described.

LEMMA 1.5. *$\Delta(u)$ is strictly monotone decreasing.*

PROOF. Let $u_1 < u_2$. Dividing the inequality (1.2) by t, and letting $t \to 0+$, we get

$$\Delta(u_1) - \Delta(u_2) \geqq u_2 - u_1 > 0 .$$

REMARK. The above proof was communicated to me by Jean-Francois Mertens. It is much shorter than my original proof.

At this point, we are ready to state our main result.

THEOREM 1.6. *v_n converges to the unique point u_0 for which*

$$u < u_0 \Rightarrow \Delta(u) > 0 \tag{1.3}$$

and

$$u > u_0 \Rightarrow \Delta(u) < 0 . \tag{1.4}$$

PROOF. By Lemma 1.5, a unique point u_0 with the above properties exists.

Let $\varepsilon > 0$ be given. Denote $u' = u_0 - \varepsilon/2$. Then $\Delta(u') > 0$ and a $\delta > 0$ exists such that, for all $0 \leqq t \leqq t_0$,

$$\frac{v(u', t) - u'}{t} \geqq \delta .$$

It follows from Lemmas 1.2 and 1.3 that, for $n \geqq (t_0)^{-1}$,

$$\begin{aligned} v_{n-1} \leqq u' \Rightarrow v_n - v_{n-1} &= v\left(v_{n-1}, \frac{1}{n}\right) - v_{n-1} \\ &\geqq \left(v\left(u', \frac{1}{n}\right) - u'\right) + \frac{1}{n}(u' - v_{n-1}) \geqq \delta \cdot \frac{1}{n} . \end{aligned} \tag{1.5}$$

Since $\sum_0^\infty n^{-1} = \infty$, there is an N such that $v_N \geqq u'$. Assume, w.l.o.g., that $N \geqq (t_0)^{-1}$ and that $|A| \cdot 2/N \leqq \varepsilon/2$, which, in view of Lemma 1.1, assures that $|v_n - v_{n-1}| \leqq \varepsilon/2$ for all $n > N$.

Now, if $\{n \geqq N: v_n < u_0 - \varepsilon\} \neq \phi$, denote by n_0 the smallest number in this set. Since $|v_{n_0} - v_{n_0-1}| \leqq \varepsilon/2$, it follows that $v_{n_0-1} \leqq u_0 - \varepsilon/2 = u'$, and from (1.5) we have that $v_{n_0-1} < v_{n_0}$, which contradicts the definition of n_0. (We know that $v_N \geqq u'$ so that n_0 is at least $N + 1$, and $n_0 - 1 \geqq N$.)

We conclude that $v_n \geqq u_0 - \varepsilon$ for all $n \geqq N$. In a similar way it may be shown that $v_n \leqq u_0 + \varepsilon$ for all $n \geqq N$. This completes the proof.

REMARK. If the equation

$$v(u, 0) = u$$

has a unique solution u_0, then it is clear that u_0 satisfies the requirements of Theorem 1.6 and $u_0 = \lim v_n$.

We would now like to present two examples.

EXAMPLE 1.7.

$$A = \begin{array}{|c|c|} \hline 1^* & 0 \\ \hline 0 & 1^* \\ \hline \end{array}$$

$$\begin{aligned} \Delta(u) &= +\infty && u < 1 \\ &= -\tfrac{1}{2} && u = 1 \\ &= -\infty && u > 1 \end{aligned}$$

so that $\lim v_n = 1$.

We note that this example corresponds to the remark following Theorem 1.6. We also note that here $\Delta(u)$ is never 0.

EXAMPLE 1.8. "The big match":

$$A = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0^* & 1^* \\ \hline \end{array}$$

$$\begin{aligned} \Delta(u) &= +\infty && u < 0 \\ &= 1 - 2u && 0 \leqq u \leqq 1 \\ &= -u && u > 1 \end{aligned}$$

so that $\lim v_n = \frac{1}{2}$.

At this point we would like to generalize the notion of an "absorbing element." We give $a_{ij} = \omega b^*$, $\omega > 0$, the following interpretation: whenever (i, j) is reached, a lottery is performed to decide the (i, j)th entry—with probability ω it is the absorbing element b^*, (in this case we say that "the payoff is absorbing"), and with probability $1 - \omega$ it is a non-absorbing element d.

Let us denote by $\mu(A)$ the limit of the values of the n-stage games with payoff matrix A.

THEOREM 1.9. *Let A' be constructed from A by replacing an element of the form wb^* by $(wb)^*$. Then,*

$$\mu(A') \geqq 0 \Rightarrow \mu(A) \geqq \mu(A')$$
$$\mu(A') \leqq 0 \Rightarrow \mu(A) \leqq \mu(A') .$$

PROOF. Let us extend definition (1.1) to matrices with generalized absorbing elements as follows

$$\begin{aligned} A(u, t)_{ij} &= a_{ij} && \text{if } a_{ij} \text{ is absorbing} \\ &= wb + (1 - w)(td + (1 - t)u) && \text{if } a_{ij} = wb^* \\ &= ta_{ij} + (1 - t)u && \text{otherwise.} \end{aligned}$$

It is easily verified that Theorem 1.6 remains valid.

We now construct, according to the above formula, $A(u, t)$ and $A'(u, t)$ from A and A', respectively. Clearly the only difference between these two matrices is that where $wb + (1 - w)(td + (1 - t)u)$ appears in $A(u, t)$, wb appears in $A'(u, t)$. It easily follows that, regardless of d,

$$u > 0 \Rightarrow \Delta(u) \geqq \Delta'(u)$$
$$u < 0 \Rightarrow \Delta(u) \leqq \Delta'(u) ,$$

where $\Delta'(u)$ is derived from $A'(u, t)$ in the same way that $\Delta(u)$ is derived from $A(u, t)$. Applying Theorem 1.6, we obtain

$$\mu(A') > 0 \Rightarrow \mu(A) \geqq \mu(A') \tag{1.6}$$

$$\mu(A') < 0 \Rightarrow \mu(A) \leqq \mu(A') . \tag{1.7}$$

To complete the proof we must show that $\mu(A') = 0 \Rightarrow \mu(A) = 0$. Indeed, suppose $\mu(A') = 0$. Define $A'(\varepsilon)$ by adding $\varepsilon > 0$ to each payoff in A'. Clearly, $\mu(A'(\varepsilon)) = \varepsilon$. Next, construct $A(\varepsilon)$ from $A'(\varepsilon)$ by replacing $(wb + \varepsilon)^*$ by $w(b + \varepsilon/w)^*$. From (1.6), $\mu(A(\varepsilon)) \geqq \mu(A'(\varepsilon)) = \varepsilon$. Finally, construct $\hat{A}(\varepsilon)$ from $A(\varepsilon)$ by subtracting ε from each payoff in $A(\varepsilon)$; in particular, $w(b + \varepsilon/w)^*$ must be replaced by $w(b + (\varepsilon/w) - \varepsilon)^*$. Clearly, $\mu(\hat{A}(\varepsilon)) = \mu(A(\varepsilon)) - \varepsilon \geqq 0$.

The only difference between the matrices $\hat{A}(\varepsilon)$ and A is that where $w(b + (\varepsilon/w) - \varepsilon)^*$ appears in $\hat{A}(\varepsilon)$, wb^* appears in A. Moreover, since (1.6) holds regardless of the value of d, the change from $A'(\varepsilon)$ to $A(\varepsilon)$ may be carried out in such a way that the resultant d that corresponds to $w(b + (\varepsilon/w) - \varepsilon)^*$ in $\hat{A}(\varepsilon)$ be the same as the d that corresponds to wb^* in A. It follows that

$$\mu(A) \geqq \mu(\hat{A}(\varepsilon)) - \frac{1 - w}{w} \cdot \varepsilon \geqq -\frac{1 - w}{w} \cdot \varepsilon .$$

Letting $\varepsilon \to 0+$ we conclude that $\mu(A) \geqq 0$. In the same way it may be shown that $\mu(A) \leqq 0$.

REMARK. That $\mu(A)$ may differ from $\mu(A')$ may be seen by looking at the matrix

$\frac{1}{2}\cdot 2^*$	0
0	$\frac{1}{2}\cdot 2^*$

Here $\mu(A) = 1$ and $\mu(A') = 2$.

COROLLARY 1.10. *Let A be a $2 \times s$ matrix such that*

(i) *a_{1j} is non-absorbing, $1 \leqq j \leqq s$,*
(ii) *a_{2j} is generalized absorbing, $1 \leqq j \leqq s$,*
(iii) *the (minmax) value of the matrix obtained from A by erasing all the asterisks is zero.*

Then $\mu(A) = 0$.

PROOF. Let A' be constructed from A as in Theorem 1.9. It may easily be verified that, for this kind of matrix, $v_n' \equiv v_1'$. But (iii) means that $v_1' = 0$, hence $\mu(A') = 0$. Applying Theorem 1.9 we conclude that $\mu(A) = 0$.

2. The value of the infinite-stage game. This section is devoted to the proof of the following theorem:

THEOREM 2.1. *In the case of symmetric information, every infinitely-repeated game with absorbing states has an l-value* V_∞, *and* V_∞ *is equal to* $\lim v_n$.

To give the above-stated theorem a precise meaning, we must define the strategies and the l-value in infinitely-repeated games.

DEFINITION 2.2. A *strategy* for player I (resp., II) in the infinite-stage game is a sequence $\sigma = (\sigma_1, \sigma_2, \cdots)$ (resp., $\tau = (\tau_1, \tau_2, \cdots)$) where σ_k (resp., τ_k) is a function from $\prod_1^{k-1}(R \times S)$ to the set of all probability distributions on R (resp., S). Here $R = \{1, \cdots, r\}$ and $S = \{1, \cdots, s\}$ are the sets of rows and columns of the payoff matrix.

REMARKS.

a. Note that the above definition of strategies is in accordance with the requirement that the game be of symmetric information. In particular, since "symmetric information" requires "perfect recall," we are justified to restrict our strategies to behavior strategies (see [3]).

b. In the above definition σ_k, τ_k depend only on the players' past actions but not on past lotteries (performed when generalized absorbing elements are reached). This does not restrict the generality, since there is no importance to anything the players do after a lottery that resulted with an absorbing payoff.

The probability distribution obtained from σ and τ (together with the lotteries) is denoted $\text{Prob}_{\sigma,\tau}$. Expectation with respect to $\text{Prob}_{\sigma,\tau}$ is denoted $E_{\sigma,\tau}$.

Let $\mathbf{i}_k$, $\mathbf{j}_k$, $\mathbf{g}_k$ be random variables that denote the row and the column chosen at stage k and the payoff at stage k. Let $\bar{\mathbf{g}}_n = n^{-1} \sum_1^n \mathbf{g}_k$. Since $\lim_{n\to\infty} \bar{\mathbf{g}}_n$ may fail to exist, there is no natural way to define the "payoff per play." However, it turns out that defining the payoff is not essential for defining the value.

DEFINITION 2.3. v is the *l-value* of the infinite-stage game if, for every $\varepsilon > 0$, there exist strategies σ_ε, τ_ε of players I and II, respectively, such that

$$E_{\sigma_\varepsilon,\tau}(\liminf \bar{\mathbf{g}}_n) \geqq v - \varepsilon \tag{2.1}$$

$$E_{\sigma,\tau_\varepsilon}(\limsup \bar{\mathbf{g}}_n) \leqq v + \varepsilon \tag{2.2}$$

for all strategies σ, τ of players I and II. The strategies σ_ε, τ_ε are called *ε-optimal*. A 0-optimal strategy is called *optimal*. The l-value is denoted by V_∞.

LEMMA 2.4.

$$\text{Prob}\,\{\mathbf{g}_k = d \ \text{ i.o.}\} = 0\,.$$

PROOF. Let $\omega_0 = \min\{\omega : \omega b^* \text{ is an entry of } A\}$. Now $\mathbf{g}_k = d$ only after an unsuccessful lottery so that, regardless of σ and τ:

$$\text{Prob}\,\{\mathbf{g}_k = d\} \leqq \frac{1-\omega_0}{\omega_0}\,\text{Prob}\,\{\mathbf{l} = k\}\,,$$

hence

$$\textstyle\sum_1^\infty \text{Prob}\,\{\mathbf{g}_k = d\} < \infty$$

so that

$$\text{Prob}\,\{\mathbf{g}_k = d \ \text{ i.o.}\} = 0\,.$$

LEMMA 2.5. *Let A be a $2 \times s$ matrix with value zero, and let $j_1, \cdots, j_n$ be integers, $1 \leqq j_k \leqq s$. Then*

$$\textstyle\sum_1^n a_{1j_k} < 0 \Rightarrow \sum_1^n a_{2j_k} \geqq 0\,.$$

PROOF. Denote by n_i the number of elements in the set $\{k\colon 1 \leqq k \leqq n, j_k = i\}$. Then

$$\frac{1}{n}\textstyle\sum_1^n a_{1j_k} = \sum_1^n \frac{n_i}{n}\, a_{1i}$$

and

$$\frac{1}{n}\textstyle\sum_1^n a_{2j_k} = \sum_1^n \frac{n_i}{n}\, a_{2i}\,.$$

If the lemma were false then $(n_1/n, \cdots, n_s/n)$ would be a strategy for player II in the (one-stage) game A that guarantees a payoff smaller than zero.

We are now going to prove a special case of Theorem 2.1.

LEMMA 2.6. *Let A be a $2 \times s$ matrix that satisfies* (i)—(iii) *of Corollary* 1.10. *Then the l-value V_∞ of the infinitely-repeated game with payoff matrix A exists, and $V_\infty = 0$.*

PROOF. Let y be an optimal strategy in the one-stage game that is obtained from A by erasing all the asterisks, and set $\tau_0 = (y, y, \cdots)$. It is easily verified that $E_{\sigma,\tau_0}(\limsup \bar{\mathbf{g}}_n) \leqq 0$ for all σ.

To complete the proof we must construct, for every $\varepsilon > 0$, a strategy σ_ε for which (2.2) holds with $v = 0$. Let $0 < \varepsilon < 1$ be given. Set

$$\begin{aligned} \alpha_m &= \varepsilon^2(1-\varepsilon)^m \qquad && m > 0 \\ &= \varepsilon^2 && m \leqq 0\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} m(j_1, \cdots, j_k) &= -\textstyle\sum_{i=1}^k a_{2j_i} \qquad && \text{if} \quad k \geqq 1 \\ &= 0 && \text{if} \quad k = 0 \end{aligned} \tag{2.4}$$

and define $\sigma_\varepsilon = (\sigma_1, \sigma_2, \cdots)$ (here there are only two rows so that $\sigma_k = (1 - \sigma_k^2, \sigma_k^2)$) by

$$\sigma_k^2(i_1, j_1, \cdots, i_{k-1}, j_{k-1}) = \alpha_{m(j_1,\cdots,j_{k-1})}\,. \tag{2.5}$$

In view of Lemma 2.4 we assume, w.l.o.g., that $\mathbf{g}_k = a_{1\mathbf{j}_k}$ whenever $\mathbf{l} > k$. Thus, if $\bar{\mathbf{g}}_n < 0$ and $\mathbf{l} > n$ then $\sum_1^n a_{1\mathbf{j}_k} < 0$, and it follows from Lemma 2.5 that $m(\mathbf{j}_1, \cdots, \mathbf{j}_k) \leqq 0$ so that $\sigma_k^2 = \varepsilon^2$. We conclude that

$$\text{Prob}\,\{\bar{\mathbf{g}}_n < 0 \wedge \mathbf{l} > n+1\} \leqq \frac{1-\omega_0\varepsilon^2}{\omega_0\varepsilon^2}\,\text{Prob}\,\{\bar{\mathbf{g}}_n < 0 \wedge \mathbf{l} = n+1\}\,,$$

hence

$$\sum_1^\infty \text{Prob}\,\{\bar{\mathbf{g}}_n < 0 \wedge \mathbf{l} = \infty\} < \infty$$

so that

$$\text{(2.6)} \qquad \text{Prob}_{\sigma_\varepsilon,\tau}\,\{\liminf \bar{\mathbf{g}}_n < 0 \wedge \mathbf{l} = \infty\} = 0 \qquad \text{for all } \tau\,.$$

To complete the proof we must show now that

$$\text{(2.7)} \qquad \text{Prob}_{\sigma_\varepsilon,\tau}\,\{\mathbf{l} < \infty\}E_{\sigma_\varepsilon,\tau}\{\liminf \bar{\mathbf{g}}_n \,|\, \mathbf{l} < \infty\} \geqq -\varepsilon \qquad \text{for all } \tau\,.$$

Of course, it suffices to prove (2.7) only for τ that are pure strategies. Here we shall deal only with pure strategies such that player II reacts in the same way to the payoff a_{1j_k} that resulted from (2, j$_k$) and an unsuccessful lottery, and to the payoff a_{1j_k} that resulted from $(1, j_k)$ (such a strategy is simply a sequence of columns $\tau = (j_1, j_2, \cdots)$). The proof for a general pure strategy goes along the same lines but requires more cumbersome notation.

We may assume, w.l.o.g., that the entries of A are rational numbers. Moreover, (i)—(iii) are not affected if A is multiplied by a positive number, so we assume that $a_{2,j}$, $1 \leqq j \leqq s$, are integers.

$$\begin{aligned}\text{Prob}\,\{\mathbf{l} < \infty\}E(\liminf \bar{\mathbf{g}}_n \,|\, \mathbf{l} < \infty) &= \textstyle\sum_{k=1}^\infty \text{Prob}\,\{\mathbf{l} = k\}b_{j_k} \\ &= \textstyle\sum_{k=1}^\infty \text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}\omega_{j_k} b_{j_k} \\ &= \textstyle\sum_{k=1}^\infty \sum_{r=1}^{|a_{2j_k}|} \text{sign}\,(a_{2j_k})\,\text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}\end{aligned}$$

where

$$\text{(2.8)} \qquad a_{2j} = \omega_j b_j^*\,.$$

Let $m(k, r) = m(j_1, \cdots, j_{k-1}) - r\,\text{sign}\,(a_{2j_k})$. By (2.5) and (2.3)

$$\text{(2.9)} \qquad \text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\} \leqq \alpha_{m(k,0)} \leqq (1 - \varepsilon)^{-|A|}\alpha_{m(k,r)}\,.$$

Set

$$T_1 = \{(k, r) : m(k, r) > \max_{(k',r') \prec (k,r)} m(k', r')\}\,,$$

where $\prec$ denotes lexicographical order. By (2.4), each time we move in T_1 from (k, r) to the next pair of indices (in lexicographical order), $m(k, r)$ is enlarged by 1, hence, by (2.9)

$$\begin{aligned}\text{(2.10)} \qquad &\left|\textstyle\sum_{(k,r)\in T_1} \text{sign}\,(a_{2j_k})\,\text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}\right| \\ &\leqq (1 - \varepsilon)^{-|A|} \textstyle\sum_{(k,r)\in T_1} \alpha_{m(k,r)} \\ &\leqq (1 - \varepsilon)^{-|A|} \textstyle\sum_0^\infty \alpha_m = (1 - \varepsilon)^{-|A|}\varepsilon\,.\end{aligned}$$

Next, denote

$$\begin{aligned}T_2 &= \{(k, r) : (k, r) \notin T_1 \text{ and } a_{2j_k} < 0\} \\ T_3 &= \{(k, r) : a_{2j_k} \geqq 0\}\,.\end{aligned}$$

There is a one-to-one correspondence from T_2 into T_3, as follows:

$(k, r) \in T_2$ is paired with the lexicographically largest $(k', r') \in T_3$ such that $(k', r') \prec (k, r)$ and $m(k', r') = m(k, r) + 1$. In this pairing

$$\text{(2.11)} \qquad k' \leqq k$$

and

$$|m(k, 0) - m(k', 0)| < 2|A| \tag{2.12}$$

so that

$$\begin{aligned}\text{Prob}\,\{\mathbf{l} \geqq k' \wedge \mathbf{i}_{k'} = 2\} &= \text{Prob}\,\{\mathbf{l} \geqq k'\}\alpha_{m(k',0)} \\ &\geqq \text{Prob}\,\{\mathbf{l} \geqq k\}\alpha_{m(k,0)}(1 - \varepsilon)^{2|A|} \\ &= \text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}(1 - \varepsilon)^{2|A|}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&|\textstyle\sum_{(k,r)\in T_2 \cup T_3} \text{sign}\,(a_{2j_k})\,\text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}| \\ &\qquad \leqq (1 - (1 - \varepsilon)^{2|A|}) \textstyle\sum_{(k,r)\in T_2} \text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\}\,.\end{aligned} \tag{2.13}$$

But

$$\begin{aligned}\textstyle\sum_{(k,r)\in T_2} \text{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\} &\leqq \frac{1}{\omega_0} \textstyle\sum_{(k,r)\in T_2} \text{Prob}\,\{\mathbf{l} = k\} \\ &\leqq \frac{|A|}{\omega_0} \textstyle\sum_{k=1}^{\infty} \text{Prob}\,\{\mathbf{l} = k\} \leqq \frac{|A|}{\omega_0}\,.\end{aligned}$$

Since $T_1 \cup T_2 \cup T_3$ is the set of all our indices (k, r), (2.7) follows from (2.10) and (2.13).

REMARK. In the proof of (2.7) we relied on a method used to prove Theorem 2 of [1].

LEMMA 2.7. *Let A be a $2 \times s$ matrix such that the first m columns of A constitute a matrix that satisfies* (i)—(iii) *of Corollary* 1.10, *and the other $s - m$ columns are all of one of two types*:

(a) *a_{ij} is non-absorbing and $a_{ij} \geqq 0$ for $i = 1, 2$.*
(b) *$a_{ij} = \omega_{ij} b^*_{ij}$, $b_{ij} \geqq 0$ for $i = 1, 2$.*

Then the repeated game with payoff matrix A has an l-value V_∞, and $V_\infty = 0$.

PROOF. This is an immediate consequence of Lemma 2.6.

LEMMA 2.8. *Let A be an $r \times s$ matrix with* (*ordinary*) *absorbing elements and let $A(u, t)$ be as in* (1.1). *Given $\eta > 0$ there exist a $t > 0$ and strategies x and $x(t)$ that are optimal for player* I *in $A(0, 0)$ and $A(0, t)$, respectively, such that*

$$\omega_j > 0 \Rightarrow \omega(t)_j > 0 \quad \textit{and} \quad |b_j - b(t)_j| < \eta \tag{2.14}$$

and

$$\omega_j = 0 \Rightarrow \omega(t)_j < \eta \quad \textit{and} \quad |e_j - e(t)_j| < \eta\,, \tag{2.15}$$

where

$$\begin{aligned}I_j &= \{i\colon 1 \leqq i \leqq r,\ a_{ij} \ \textit{is absorbing}\}\,, \\ \omega_j &= \textstyle\sum_{i\in I_j} x_i\,, \\ b_j &= \frac{1}{\omega_j} \textstyle\sum_{i\in I_j} x_i a_{ij} \qquad \textit{if} \quad \omega_j > 0\,, \\ &= 0 \qquad\qquad\qquad\quad \textit{otherwise,}\end{aligned}$$

and

$$e_j = \frac{1}{1-\omega_j} \sum_{i \in I_j} x_i a_{ij} \qquad \textit{if} \quad \omega_j < 1 ,$$
$$= 0 \qquad \textit{otherwise},$$

and where $\omega(t)_j$, $b(t)_j$ *and* $e(t)_j$ *are the corresponding averages for* $x(t)$.

PROOF. Apply the upper semi-continuity of the mapping

$$t \to \{x : x \text{ is an optimal strategy for player I in } A(0, t)\} .$$

PROOF OF THEOREM 2.1. Let A be an $r \times s$ matrix with (ordinary) absorbing elements such that $\lim v_n = u_0$. The players' situation is symmetric so that, to establish the proof, it suffices to exhibit for every $\varepsilon > 0$ a σ_ε that satisfies (2.1) with $v = u_0$.

So, let $\varepsilon > 0$ be given. Assume, w.l.o.g., that $u_0 = \varepsilon/3$. By (1.3), $\Delta(0) > 0$, so there exists a $\delta > 0$ such that, when $t \geqq 0$ is small

(2.16) $$v(0, t) \geqq \delta t .$$

Let $\eta = \min\{\varepsilon/3, \delta/2, \delta/2|A|\}$ and fix x, t and $x(t)$ such that (2.14), (2.15) and (2.16) hold.

Player I may decide that at every stage of the repeated game A he will play a probability mixture of x and $x(t)$. Restricting himself thus, player I is faced with a $2 \times s$ matrix C, where

$$c_{1j} = \sum_{i=1}^{r} x_i a_{ij}$$

and

$$c_{2j} = \sum_{i=1}^{r} x(t)_i a_{ij} .$$

Recalling our definition of a generalized absorbing element ωb^* we get

(2.17) $$c_{1j} = \omega_j b_j^* \quad \text{if} \quad \omega_j > 0$$
$$= e_j \quad \text{if} \quad \omega_j = 0$$

and

(2.18) $$c_{2j} = \omega(t)_j b(t)_j^* \quad \text{if} \quad \omega(t)_j > 0$$
$$= e(t)_j \quad \text{if} \quad \omega(t)_j = 0 .$$

Recall that $x(t)$, x are optimal in $A(0, t)$, $A(0, 0)$, respectively, so that (2.16) may be rewritten as

(2.19) $$\omega(t)_j b(t)_j + (1 - \omega(t)_j) t e(t)_j \geqq \delta \cdot t , \qquad j = 1, \cdots, s ,$$

and the special case $v(0, 0) \geqq 0$ as

(2.20) $$\omega_j b_j \geqq 0 , \qquad j = 1, \cdots, s .$$

Assume, w.l.o.g., that $\{j : \omega_j = 0 \text{ and } \omega(t)_j > 0\} = \{1, \cdots, m\}$, so the first m columns of C constitute a matrix that satisfies (i) and (ii) of Corollary 1.10.

By (2.15), $\omega_j = 0 \Rightarrow \omega(t)_j < \eta \leqq \delta/2|A|$, and it follows from (2.19) that

$$\omega(t)_j b(t)_j + t e(t)_j \geqq \delta t/2 .$$

Also from (2.15), $|e_j - e(t)_j| < \eta \leqq \delta/2$ so that $\omega(t)_j b(t)_j + te_j \geqq 0$. Dividing by $1 + t$ we get

$$\frac{1}{1+t} c_{2j} + \frac{t}{1+t} c_{1j} \geqq 0\,, \qquad j = 1, \cdots, m\,.$$

But this means that the strategy $(1/(1 + t), t/(1 + t))$ assures player I a payoff of at least zero in the one-stage game C (with asterisks erased). We may therefore assume, w.l.o.g., that C satisfies (iii) of Corollary 1.10.

If we now show that all the columns $m + 1, \cdots, s$ are either of type (a) or (up to $\varepsilon/3$) of type (b) then we may apply Lemma 2.7 and complete the proof.

Indeed, suppose $\omega_j = 0$ and also $\omega(t)_j = 0$. By (2.19), $e(t)_j \geqq \delta$ and therefore, by (2.15), $e_j \geqq 0$. Hence, j is a column of type (a).

In all the other columns, $\omega_j > 0$. By (2.20), $b_j \geqq 0$ and it follows from (2.14) that $\omega(t)_j > 0$ and that $b(t)_j \geqq -\eta \geqq -\varepsilon/3$. Hence, j is (up to $\varepsilon/3$) of type (b) and the proof is complete.

We conclude this section with two examples. The first shows that, even if symmetric information is assumed, optimal strategies may not exist. The second example shows that Theorem 2.1 is not true if symmetric information is not assumed.

EXAMPLE 2.9.

1	0
0*	1*

with symmetric information.

It is proved in [1] that player I has no optimal strategy in the infinitely repeated game.

EXAMPLE 2.10. In [4], a certain repeated game of incomplete information is described. When closely examined, that game turns out to be equivalent to the infinitely repeated game with the same payoff matrix as in Example 2.9, but where each player only remembers his own past moves. In [4] it is proved that V_∞ does not exist. We give a sketch of the proof, using the notations of this section.

Let σ be any strategy of player I. Since player I does not know player II's past actions, σ is just a sequence of numbers $x_1, x_2, \cdots$ where $x_k = \text{Prob}\{\mathbf{i}_k = 2\}$.

If player II knows σ, he may proceed as follows:

If $\sum_1^\infty x_k = 1$, choose column 1 all the time.

If $\sum_1^\infty x_k < 1$, choose column 1 up to some n for which $\sum_n^\infty x_k \leqq \varepsilon$, and then choose column 2 all the time.

It follows that player II can hold the payoff down to ε, hence sup inf $\leqq 0$.

On the other hand, if player I knows II's strategy, then at each stage he may choose row 1 or 2 according to whether or not the probability with which II chooses column 1 at that stage is greater than $\frac{1}{2}$.

It follows that inf sup $\geqq \frac{1}{2}$, and V_∞ does not exist.

3. Other values for the infinite-stage game. In this section we show that

Theorem 2.1 is true also when a definition of the value in the infinite-stage game is adopted other than that of the l-value.

DEFINITION 3.1. v is the *weak value* of the infinite-stage game if for every $\varepsilon > 0$, there exist strategies σ_ε, τ_ε of players I and II, respectively, such that

$$\liminf E_{\sigma_\varepsilon,\tau}(\bar{\mathbf{g}}_n) \geqq v - \varepsilon \tag{3.1}$$

$$\limsup E_{\sigma,\tau_\varepsilon}(\bar{\mathbf{g}}_n) \leqq v + \varepsilon \tag{3.2}$$

for all strategies σ, τ of players I and II. The weak value is denoted by $v_\infty^\#$.

It is an immediate consequence of the Fatou–Lebesgue lemma that $v = V_\infty \Rightarrow v = v_\infty^\#$:

THEOREM 3.2. *In the case of symmetric information, every infinitely-repeated game with absorbing states has a weak value $v_\infty^\#$ and $v_\infty^\# = \lim v_n$.*

Note that (3.1) may be rewritten as

$$E_{\sigma_\varepsilon,\tau}(\bar{\mathbf{g}}_n) \geqq v - \varepsilon \qquad \text{for all} \quad n \geqq n(\tau)\,.$$

We would now like to show that Theorem 3.2 is still valid if $n(\tau)$ is required to be independent of τ, that is

DEFINITION 3.3. v is the *value* of the infinite-stage game if for every $\varepsilon > 0$, there exist strategies σ_ε, τ_ε of players I and II, respectively, such that, for all $n \geqq N(\varepsilon)$

$$E_{\sigma_\varepsilon,\tau}(\bar{\mathbf{g}}_n) \geqq v - \varepsilon \tag{3.3}$$

$$E_{\sigma,\tau_\varepsilon}(\bar{\mathbf{g}}_n) \leqq v + \varepsilon \tag{3.4}$$

for all strategies σ, τ. The value is denoted by v_∞.

THEOREM 3.4. *In the case of symmetric information, every infinitely-repeated game with absorbing states has a value v_∞ and $v_\infty = \lim v_n$.*

PROOF. The proof is carried out in complete analogy with the proof of Theorem 2.1. The main changes that are required are in (2.6) and (2.7).

First we note that if $\bar{\mathbf{g}}_n < -\varepsilon$ then $\bar{\mathbf{g}}_k < 0$ for at least $n\varepsilon/|A|$ stages prior to n. But at each such stage $\sigma_k^2 = \varepsilon^2$ hence (denote $n' = n - n\varepsilon/|A|$)

$$\begin{aligned}\text{Prob}\,\{\mathbf{g}_{n'} < 0, \cdots, \mathbf{g}_k < 0, \mathbf{g}_{k+1} < 0, \mathbf{l} > k+1\} &\\ \leqq \text{Prob}\,\{\mathbf{g}_{n'} < 0, \cdots, \mathbf{g}_k < 0, \mathbf{l} > k+1\} &\\ \leqq (1 - w_0\varepsilon^2)\,\text{Prob}\,\{\mathbf{g}_{n'} < 0, \cdots, \mathbf{g}_k < 0, \mathbf{l} > k\} &\end{aligned}$$

and it follows that

$$\text{Prob}\,\{\bar{\mathbf{g}}_n < -\varepsilon \wedge \mathbf{l} > n\} \leqq (1 - w_0\varepsilon^2)^{n\varepsilon/|A|}\,. \tag{3.5}$$

We have thus obtained the following analogue of (2.6):

$$\text{Prob}_{\sigma_\varepsilon,\tau}\,\{\bar{\mathbf{g}}_n < -\varepsilon \wedge \mathbf{l} > n\} \leqq \varepsilon \qquad \text{for all} \quad \tau \quad \text{and}\ n \geqq N_1 \tag{3.6}$$

(and this implies that $\text{Prob}\,\{\mathbf{l} > n\}E\{\mathbf{g}_n \,|\, \mathbf{l} > n) \geqq -|A| \cdot \varepsilon - \varepsilon$).

As for (2.7), the corresponding analogue is

$$(3.7)\qquad \mathrm{Prob}_{\sigma_\varepsilon,\tau}\{\mathbf{l} \leqq n\}E_{\sigma_\varepsilon,\tau}\{\bar{\mathbf{g}}_n \mid \mathbf{l} \leqq n\} \geqq -\varepsilon \qquad \text{for all} \quad \tau \quad \text{and} \quad n \geqq N_2 .$$

We sketch its proof. Let $q = (1 - w_0\varepsilon^2)^{\varepsilon/|A|}$ and assume, w.l.o.g., that $\sum_{N_1}^{\infty} q^n < \varepsilon/|A|$. Let $N_2 = N_1/\varepsilon$. Now

$$\begin{aligned}\mathrm{Prob}\,&\{\mathbf{l} \leqq n\}E\{\bar{\mathbf{g}}_n \mid \mathbf{l} \leqq n\} \\ &= \textstyle\sum_{k=1}^{n} \mathrm{Prob}\,\{\mathbf{l} = k\}\left(\frac{k-1}{n} E(\bar{\mathbf{g}}_k \mid \mathbf{l} = k) + \frac{n-k-1}{n} b_{j_k}\right).\end{aligned}$$

But, by (3.5)

$$\begin{aligned}\mathrm{Prob}\,\{\mathbf{l} = k\}E(\bar{\mathbf{g}}_k \mid \mathbf{l} = k) &\geqq -|A| \cdot \mathrm{Prob}\,\{\mathbf{g}_k < -\varepsilon \wedge \mathbf{l} = k\} - \varepsilon\,\mathrm{Prob}\,\{\mathbf{l} = k\} \\ &\geqq -|A|q^{k-1} - \varepsilon\,\mathrm{Prob}\,\{\mathbf{l} = k\} .\end{aligned}$$

Hence

$$\textstyle\sum_{k=N_1+1}^{n} \mathrm{Prob}\,\{\mathbf{l} = k\} \frac{k-1}{n} E(\bar{\mathbf{g}}_k \mid \mathbf{l} = k) \geqq -|A| \sum_{N_1+1}^{\infty} q^{k-1} - \varepsilon = -2\varepsilon .$$

On the other hand, if $n \geqq N_2$ and $k \leqq N_1$ then $(k-1)/n \leqq \varepsilon$; hence

$$\textstyle\sum_{k=0}^{N_1} \mathrm{Prob}\,\{\mathbf{l} = k\} \frac{k-1}{n} E(\bar{\mathbf{g}}_k \mid \mathbf{l} = k+1) \leqq \varepsilon|A| .$$

It follows that, in order to establish (3.7), it suffices to prove that

$$(3.8)\qquad \textstyle\sum_{k=0}^{\infty} \mathrm{Prob}\,\{\mathbf{l} = k\} \frac{n-k-1}{n} b_{j_k} \geqq -\varepsilon \qquad \text{for all} \quad \tau \quad \text{and} \quad n \geqq N_2 .$$

But this may be done exactly in the same way as in the proof of (2.7), where we showed that

$$\textstyle\sum_{k=1}^{\infty} \mathrm{Prob}\,\{\mathbf{l} = k\} b_{j_k} \geqq -\varepsilon .$$

Indeed, all we have to do is to establish the analogues of (2.10) and (2.13). The analogue of (2.10) follows at once, since all the summands have the same sign so that multiplication of the kth summand by $(n - k - 1)/n$ (which is $\leqq 1$) cannot enlarge the absolute value of the sum.

As for the analogue of (2.13), this follows easily once we notice that (by (2.11)) $(n - k' - 1)/n \geqq (n - k - 1)/n$. Hence

$$\begin{aligned}\mathrm{Prob}\,&\{\mathbf{l} \geqq k' \wedge \mathbf{i}_{k'} = 2\} \cdot \frac{n-k'-1}{n} \\ &\geqq \mathrm{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\} \frac{n-k-1}{n} \cdot (1-\varepsilon)^{2|A|}\end{aligned}$$

so that

$$\begin{aligned}&\left|\textstyle\sum_{(k,r)\in T_2 \cup T_3} \mathrm{sign}\,(a_{2j_k})\,\mathrm{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\} \frac{n-k-1}{n}\right| \\ &\qquad \leqq (1 - (1-\varepsilon)^{2|A|}) \textstyle\sum_{(k,r)\in T_2} \mathrm{Prob}\,\{\mathbf{l} \geqq k \wedge \mathbf{i}_k = 2\} \frac{n-k-1}{n} \\ &\qquad \leqq (1 - (1-\varepsilon)^{2|A|}) \textstyle\sum_{(k,r)\in T_2} \mathrm{Prob}\,\{\mathbf{l} = k \wedge \mathbf{i}_k = 2\} .\end{aligned}$$

This completes the proof.

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