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# ADMISSIBILITY OF TRANSLATION INVARIANT TOLERANCE INTERVALS IN THE LOCATION PARAMETER CASE[1]

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Given $n$ independent observations with common density $f(x-\theta)$, and a rv $z$ independent of these with density $g(x-\theta)$ ($f$, $g$ known except for $\theta$) a prediction region for $z$ is required. It is shown that the best translation invariant interval is optimal in two senses: (1) there is no other region with the same expected coverage (coverage is the probability of containing $z$) and uniformly smaller expected size (Lebesgue measure); (2) no other interval having the same confidence that the coverage exceeds $\beta$ (given) can have uniformly smaller expected length. The best invariant interval in each case is found, and the normal case is studied. The usual interval centered at $\bar{X}$ is not always optimal in the second sense if $\beta$ and/or confidence are small. A criterion involving expected coverage and the confidence of exceeding coverage $\beta$ is also examined. Again restrictions on these are needed for the usual normal interval to be optimal.

**1. Introduction.** Let $X_1, \cdots, X_n$, be real-valued, independent random variables with common density $f(x-\theta)$ where $f(x)$ is known and $\theta$ is an unknown real-valued parameter. Let $Z$ be a real-valued random variable independent of the $X$'s with density $g(z-\theta)$, known except for the value of $\theta$. On the basis of observing the $X$'s it is desired to construct a prediction region for $Z$. In the case $g(x)=f(x)$, this is the usual tolerance region, and the term tolerance region will be used here to denote the more general situation.

It will be convenient hereafter to denote the observable random variables as $X$, $Y$ where $X$ is real-valued and $Y$ is an $(n-1)$ dimensional vector (see [9] or [2] for two common transforms) having joint density $p(x-\theta, y)$. We denote the marginal density of $X$ by $p(x-\theta)$, and of $Y$ by $u(y)$, and the conditional density of $X$ given $Y$ as $p(x-\theta\,|\,y)$. Also the joint density of $(X-Z)$ and $Y$ is $\rho(v, y)$ where

$$\rho(v, y) = \int_{-\infty}^{\infty} g(x)p(x+v, y)\,dx \tag{1.1}$$

and $\rho(v\,|\,y)$ is the conditional density of $(X-Z)$ given $Y$.

A tolerance region for $Z$ can be defined by $i(z)$ where $i(z)$ denotes the probability that $z$ is included in the tolerance region. We use $T(x, y, \cdot)$ to denote the region defined by $\phi(x, y, z) = i_{x,y}(z)$ for given $x$, $y$. Translation invariant regions

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are those such that for a real number $a$ $T(x+a, y, \cdot)$ is $T(x, y, \cdot)$ shifted by the amount $a$, so that $T(x, y, \cdot) = x + T(y, \cdot)$ where $T(y, \cdot)$ is a set of real numbers. A measure of the size of a region is its Lebesgue measure, denoted $vT(x, y, \cdot)$.

The coverage of a tolerance region is $P_\theta(Z \in T(x, y, \cdot))$ and one measure of how well the region predicts $Z$ is the expected coverage $E_\theta P_\theta(Z \in T(X, Y, \cdot))$. Another measure of success in predicting $Z$ is the confidence that the coverage exceeds a given constant $\beta$ $(0 < \beta < 1)$, i.e. $P_\theta\{P_\theta[Z \in T(X, Y, \cdot) \mid X, Y] \geqq \beta\}$.

In this paper, three criteria of admissibility involving pairs of the above measures of size and coverage will be considered, and relative to each criterion certain translation invariant tolerance regions will be shown to be admissible. The results differ considerably for the different criteria.

In Section 2, a tolerance region $T_0(x, y, \cdot)$ is considered admissible with respect to expected coverage if there does not exist a region $T_1(x, y, \cdot)$ such that

$$(1.2) \qquad \begin{aligned} &\text{(i)} \quad E_\theta P_\theta[Z \in T_1(X, Y, \cdot)] \geqq E_\theta P_\theta[Z \in T_0(X, Y, \cdot)] \qquad \text{all } \theta \in R \\ &\text{(ii)} \quad E_\theta vT_1(X, Y, \cdot) \leqq E_\theta v(T_0(X, Y, \cdot)) \qquad \text{all } \theta \in R \end{aligned}$$

with inequality in either (i) or (ii) for at least one $\theta \in R$. The invariant region given by

$$(1.3) \qquad T_0(x, y, \cdot) = \{z : \rho(x - z \mid y) \geqq b\}$$

where $b > 0$ is a fixed constant will be shown to be admissible in this sense (up to an equivalence class). The need to work with equivalence classes is discussed in [8] for confidence intervals and the same considerations apply here. Optimality of the usual tolerance intervals for the normal distribution is obtained as an application of the general result.

In Section 3, attention is restricted to tolerance intervals for reasons to be discussed therein. The intervals will be taken to be closed, and denoted $T(x, y, \cdot) = [t_1(x, y), t_2(x, y)]$, invariant ones having the form $[x + t_1(y), x + t_2(y)]$ $(t_2 \geqq t_1)$. An interval $T_0(x, y, \cdot)$ will be considered admissible with respect to $\beta$-content if there is no other interval $T_1(x, y, \cdot)$ such that (1.2) (ii) and

$$(1.4) \qquad \begin{aligned} &P_\theta\{P_\theta[Z \in T_1(X, Y, \cdot) \mid X, Y] \geqq \beta\} \\ &\qquad \geqq P_\theta\{P_\theta[Z \in T_0(X, Y, \cdot) \mid X, Y] \geqq \beta\}, \qquad \text{all } \theta \in R \end{aligned}$$

with inequality in either (1.2) (ii) or (1.4) for at least one $\theta \in R$. Let $T$ represent $[t_1, t_2]$. A loss function

$$(1.5) \qquad \begin{aligned} L(T, \theta) = bvT &+ 0 \quad \text{if } P_\theta[Z \in T] \geqq \beta \\ &+ 1 \quad \text{if } P_\theta[Z \in T] < \beta \end{aligned}$$

is formed and a result of Brown and Fox [3] is used to show that the best invariant interval relative to (1.5) is admissible with respect to $\beta$-content.

For a certain class of distributions, the best invariant interval is then characterized, and as an application the usual interval for normal distributions is shown to be optimal in the sense of this section, only for certain combinations of coverage and confidence of achieving it.

In previous papers considering optimality of tolerance regions, Easterling and Weeks [4], Fraser and Guttman [6], Goodman Madansky [7] for instance, the tolerance region has been related to an appropriate hypothesis test, and the goodness criteria have been chosen correspondingly. In transferring optimality of tests to the tolerance region problem, the possible goodness criteria are limited, and in particular, it has not been possible to consider the expected length (or Lebesgue measure) of the interval, among such criteria.

Goodman and Madansky [7] mention as a possible criterion of goodness, minimizing expected coverage among procedures having a given probability that the coverage exceeds $\beta$. In particular, call $T_0(x, y, \cdot)$ admissible in this "concentration" sense if there exists no alternative procedure $T_1(x, y, \cdot)$ such that (1.4) holds and

$$E_\theta P_\theta[Z \in T_1(X, Y, \cdot)] \leqq E_\theta P_\theta[Z \in T_0(X, Y, \cdot)] \qquad \text{all } \theta \in R \tag{1.6}$$

with inequality in either (1.4) or (1.6) for at least one $\theta \in R$. In Section 4, attention is limited to normal distributions and tolerance intervals. It is shown that intervals of the form $(x + m) \pm h$ are best invariant, but $m$ will be zero, giving the classical interval centered at $X$, only under some restrictions on the content $\beta$ and the confidence $\alpha$ of achieving this content. The pathological nature of this result is similar in nature to the experience of Goodman and Madansky with this criterion. The criterion will tend to favor procedures with content close to $\beta$ when the content exceeds $\beta$, but when the content is below $\beta$, the criterion favors intervals with content far from $\beta$. A more reasonable criterion which would tend to single out procedures which concentrate their distribution of coverage near $\beta$ would involve a convex function of $|P_\theta\{Z \in T(X, Y, \cdot) \mid X, Y\} - \beta|$. We have not examined such criteria in this paper.

## 2. Admissibility with respect to expectation.

2.1 *General results.* Let the decision space $\mathscr{D} = \{i : i \text{ is measurable and } 0 \leqq i \leqq 1\}$, $i(z)$ denote the probability that $z$ is included in the tolerance region. A measurable non-randomized decision rule is then given by $\phi(x, y, z) = i_{x,y}(z)$.

The loss function is given by

$$L(i, \theta) = bv(i) - c(i, \theta) \tag{2.1}$$

where $v(i) = \int_{-\infty}^{\infty} i(z)\, dz$ and $c(i, \theta) = \int_{-\infty}^{\infty} i(z) g(z - \theta)\, dz$. With $T(x, y, \cdot)$ defined by $\phi(x, y, \cdot)$, $vT(x, y, \cdot)$ is just $v(i_{x,y})$ and $P_\theta(z \in T(x, y, \cdot))$ is $c(i_{x,y}, \theta)$. As in [8] or [9] admissibility with respect to expected coverage follows from admissibility with respect to the loss $L$. Admissibility with respect to $L$ up to the equivalence given in Definition 2.1 of [9] follows from Theorem 1 below.

Two conditions will be assumed.

*Condition* 1. $X$ has finite first absolute moment, i.e.

$$\int_{-\infty}^{\infty} |x| p(x)\, dx < \infty \,. \tag{2.2}$$

*Condition* 2. Let $[\rho = b]$ denote the set $\{z : \rho(x - z \mid y) = b\}$. Then

$$\int u(y)\, dy \int_{[\rho=b]} dz = 0. \tag{2.3}$$

The left side of (2.3) is independent of $x$. Let $\phi_0(x, y, \cdot)$ be the nonrandomized procedure corresponding to (1.3) having expected Lebesgue measure $v_0$ and expected coverage $c_0$. We give first

LEMMA 1. *With loss function* (2.1), $\phi_0(x, y, \cdot)$ *is best invariant.*

PROOF. An invariant procedure has the form $i_{x,y}(z) = i_y(z - x)$ with risk

$$\begin{aligned} R(i, \theta) &= \iiint [b i_y(z - x) - i_y(z - x) g(z - \theta)] p(x - \theta \mid y) u(y)\, dz\, dx\, dy \\ &= \int \{\iint [b i_y(v) - i_y(v) g(w)] p(w - v \mid y)\, dv\, dw\} u(y)\, dy \end{aligned} \tag{2.4}$$

where $v = z - x$, $w = z - \theta$. Using (1.1) the bracketed term in (2.4) is

$$\int [b - \rho(-v \mid y)] i_y(v)\, dv \,. \tag{2.5}$$

Since $0 \leqq i \leqq 1$, this is minimized by $\phi_0$.
The main result is

THEOREM 1. *Under Conditions* 1 *and* 2, *if* $\phi_1$ *is any procedure such that*

$$E_\theta L(\phi_1(X, Y, \cdot), \theta) \leqq E_\theta L(\phi_0(X, Y, \cdot), \theta) = b v_0 - c_0 \tag{2.6}$$

*then* $\phi_1(x, y, z) = \phi_0(x, y, z)$ *for almost all* $(x, y, z)$.

PROOF. Give $\mathscr{D}$ the weak topology:

$$i_k \to i \Leftrightarrow \int_a^b i_k(z)\, dz \to \int_a^b i(z)\, dz \,, \qquad -\infty < a < b < \infty \,.$$

$\mathscr{D}$ is compact in this (non-Hausdorff) topology and $L(i, \theta)$ is lower semi-continuous. It is then straightforward to check by using Conditions 1 and 2 that Assumptions (b)—(d) and the first part of (a) of [3] are satisfied. Theorem 1 follows from Remark II following the main result of [3], and Lemma 1.

2.2 *Normal distribution*. Here, $X$ can be taken to be the average of $X_1 \cdots, X_n$ and $Y$ can be chosen to be independent of $X$. Assume each $X_i$ has variance $\omega^2$, so that $X$ has variance $\sigma^2 = (\omega^2/n)$. Let $Z$ be normal with variance $\tau^2$. It is easily seen that $(x - z \mid y)$ and $(x - z)$ have the normal distribution with mean 0 and variance $D^2 = (\sigma^2 + \tau^2)$, and the tolerance region corresponding to (1.3) is the interval $(X - T, X + T)$ where $T = D[-2 \log bD(2\pi)^{\frac{1}{2}}]^{\frac{1}{2}}$. The half length $T$ can be varied from 0 to $\infty$ by appropriate choice of $b$ in the interval $[0, 1/D(2\pi)^{\frac{1}{2}})$.

It might be noted that for the normal case, Bayes tolerance regions can be derived for loss (2.4) with normal priors for $\theta$ centered at 0 with variance $\delta^2$. The above interval is a limit of Bayes intervals as $\delta \to \infty$, and admissibility can be established by the argument used in [8].

**3. Admissibility with respect to $\beta$-content.**

3.1 *Admissibility of best invariant intervals.* Assume the following condition:

*Condition* 3. The best invariant tolerance interval relative to (1.5) is essentially uniquely defined.
As in [8], admissibility relative to (1.5) implies admissibility relative to $\beta$-content.

THEOREM 2. *Under Conditions* 1 *and* 3, *the best invariant tolerance interval relative to* (1.5) *is admissible in the class of all tolerance intervals.*

PROOF. We need only verify that (1.5) satisfies the regularity conditions of Brown and Fox [3] for admissibility of the best invariant procedure. Relation (2) is clearly satisfied and lower semi-continuity follows from the restriction to intervals and $Z$ having a density. Thus (a) is satisfied and (b) is Condition 3.

Condition (c) of [3] involves moments, and for (1.5) the integral in (c) is dominated by

$$\int |x|(bv_0(y) + 1)p(x, y)\, dx\, dy$$

which will be finite if Condition 1, (2.3), is satisfied and if $bv_0(y)$ is bounded for all $y$, where $v_0(y)$ is the length of the best invariant interval. This uniform boundedness is readily established so that Condition 1 is sufficient. Condition (d) of [3] is essentially the same as [9], equation (27), except that here attention can be limited to invariant intervals, and it can be verified as in [9] that (d) will be satisfied if Condition 1 is satisfied.

3.2 *Best invariant intervals.* A best invariant tolerance interval is given by $(x + t_1(y), x + t_2(y))$ where $(t_1(y), t_2(y))$ minimize $E_\theta L(T, x, y, \theta)$ which is independent of $\theta$. It is easily seen that for each $y$, $(t_1, t_2)$ should be chosen to minimize

$$2bh + \int_S p(x\,|\,y)\, dx \tag{3.1}$$

where

$$S = \{x : G(x + m + h) - G(x + m - h) < \beta\}\,,$$
$$G(x) = \int_{-\infty}^{x} g(u)\, du\,, \qquad m = (t_1 + t_2)/2\,, \qquad \text{and} \qquad h = (t_2 - t_1)/2\,.$$

We impose

*Condition* 4. (a) The density functions $g(x)$ and $p(x\,|\,y)$ (for each $y$) are symmetric, and (b) the families $g(x - \theta)$ and $p(x - \theta\,|\,y)$ (each $y$) have monotone likelihood ratio.

THEOREM 3. *Under Condition* 4, *the best invariant interval is given by* $m = 0$, *and if* $b \leqq b_0$, *then* $h$ *is the solution of*

$$G(k + h) - G(k - h) = \beta\,, \tag{3.2}$$

$$b = p(k\,|\,y)[g(k - h) + g(k + h)]/[g(k - h) - g(k + h)]\,, \tag{3.3}$$

*while if* $b > b_0$, $h = 0$. *The value of* $b_0$ *(which depends on* $y$*) is given by the* *solution of*

$$2(bh - P_y(k) + 1) = 1 \,, \tag{3.4}$$

*where $h$ and $k$ satisfy* (3.2) *and* (3.3), *and* $P_y(x) = \int_{-\infty}^{x} p(x \mid y)\, dz$.

PROOF. Condition 4 is used in a straightforward way to conclude that $m = 0$ and that (3.1) is minimized by the $h^*$ satisfying (3.2) and (3.3). Note that (3.1) is unity if $m = h = 0$, and it is easily seen that for $h^*$ to be best invariant, the left side of (3.4) must be less than unity when $h = h^*$. The theorem follows from the fact that the left side of (3.4) is strictly increasing in $b$, when $h = h^*$.

COROLLARY 1. *The best invariant interval has either $h = 0$ or $h > h_0$ where $h_0$ is given by setting $b = b_0$ in* (3.3).

PROOF. Using (3.2) and the fact that $k$ is an increasing function of $h$, it is seen that $(k - h)$ decreases monotonely in $h$. Further, the MLR property (Condition 4 (b)) implies that both $p(x \mid y)$ and $g(x)$ are unimodal (see [1], Theorem 2, page 230). These facts are easily combined to deduce that the right side of (3.3) decreases with $h$. Hence $(dh^*/db) < 0$, and $h^* = +\infty$ at $b = 0$ while $h^* = G^{-1}((1 + \beta)/2)$ at $b = \infty$. Since $b$ must be no greater than $b_0$, this gives the result.

3.3 *Restricted admissibility.* Assume the following condition holds:

*Condition* 5. There exists a value $\hat{h}$ such that $P_\theta(Z \in T(x, y, \cdot)) < \beta$ (all $\theta$) whenever $(t_2(x, y) - t_1(x, y)) < 2\hat{h}$.

For example, under Condition 4, $\hat{h}$ is the solution of (3.2) with $k = 0$.

THEOREM 4. *Under Conditions* 1, 3 *and* 5, *the best invariant tolerance interval relative to* (1.5) *in $\hat{\mathscr{D}}$ is admissible in $\hat{\mathscr{D}}$, where $\hat{\mathscr{D}}$ is the class of tolerance interval procedures with length at least $2\hat{h}$ for each* $(x, y)$.

PROOF. The admissibility of best invariant procedures remains true as long as the decision space is closed and the conditions of [3] are satisfied, which is easily verified.

*Note* 1. From the proof of Theorem 3, it is seen that the intervals centered at $X$ with half length $h$ given by (3.2) and (3.3) for each $y$ are best invariant in $\hat{\mathscr{D}}$, under Condition 4.

*Note* 2. From the discontinuous nature of (1.5) it is seen that if a procedure $T_1$ has positive half-length less than $\hat{h}$ for some $(x, y)$ it can be improved on by one which has zero length for that $(x, y)$. Thus, a complete class of tolerance interval procedures consists of those which for each $(x, y)$ assign either a degenerate interval or one of length at least $2\hat{h}$. If one wishes to eliminate degenerate procedures it makes sense also to eliminate procedures with length under $2\hat{h}$, even though the class $\hat{\mathscr{D}}$ is not complete in the class $\mathscr{D}^+$ of nondegenerate procedures (i.e. if one wishes to try to achieve coverage $\beta$ for each

$(x, y)$ intervals of lengh between zero and $2\hat{h}$ are no better than the degenerate interval.)

3.4 *Normal distribution.* Let $X$ and $Z$ be as in Section 2.2. Since $p(x \mid y)$ is independent of $y$, the results of Sections 3.2 and 3.3 show that the constant length interval centered at $X$ is $\beta$-content admissible provided that the half length $h$ exceeds $h_0$ as described before, and that these intervals are $\beta$-content admissible for any $h$ when the class of tolerance intervals is restricted to those which for each $x$ attempt to achieve $P_\theta(Z \in T(x, \cdot)) \geqq \beta$.

We now relate the above to the classical tolerance interval which is formed by choosing a confidence coefficient $\alpha$ and desired coverage $\beta$ and using the interval $(x - h, x + h)$, with $h$ chosen so that

$$(3.5) \qquad P_0[P_0\{X - h < Z < X + h\} \geqq \beta] = \alpha .$$

THEOREM 5. *For fixed $\beta$, the tolerance interval $(X \pm h)$ is admissible in the $\beta$-content sense if and only if $\alpha \geqq \alpha_0$ where $\alpha_0$ is the solution of*

$$(3.6) \qquad -(\alpha\sigma/2h) + \varphi\left[\Phi^{-1}\left(\frac{\alpha+1}{2}\right)\right]\coth\left[(h\sigma/\tau^2)\Phi^{-1}\left(\frac{\alpha+1}{2}\right)\right] = 0$$

*where $h$ is given by* (3.5), *and $\Phi(\cdot)$, $\varphi(\cdot)$ are respectively the* cdf *and density of the standard normal distribution. For any $\alpha$, this interval is admissible in the restricted class of procedures $\hat{\mathscr{D}}$.*

PROOF. The only non-straightforward aspect is the demonstration of "only if." Let $\alpha < \alpha_0$, and $T_\alpha$ be $(X \pm h_\alpha)$ where $h_\alpha$ satisfies (3.5). Let $T_p$ be the randomized invariant procedure which with probability $p$ uses the degenerate interval, and $(1 - p)$ uses $((X \pm (2h_\alpha/1 - p))$. It is easily shown that there exists $p > 0$ such that $T_p$ is better than $T_\alpha$ in the $\beta$-content sense, completing the proof.

For example, when $\beta = 0.90$ and $\alpha = 0.90$ the resulting interval is admissible, but when $\alpha = 0.60$ it is not. It is admissible for $\alpha = 0.60$ among the restricted procedures.

As in Section 2.2, the results for the normal case can be obtained by taking limits of Bayes tolerance intervals both in the restricted and unrestricted decision spaces.

It is relatively easy to characterize Bayes or best invariant intervals and quite difficult to characterize such regions in general. This is the reason the results are given only for intervals.

**4. Admissibility with respect to concentration.** This section considers the question of minimizing expected coverage subject to a given confidence of covering at least a fraction $\beta$ of the population, for normal distributions, with attention limited to interval procedures.

By forming a linear combination loss function, Bayes solutions relative to the priors of Section 2.2 are readily found relative to the class of procedures which have positive probability that the coverage exceeds $\beta$.

Consider tolerance intervals of the form $(X + m - \tau h, X + m + \tau h)$ where $m$ and $h$ are chosen to mimimize expected coverage among intervals having a specified confidence $\alpha$ that the coverage exceeds $\beta$. This means that $m$ and $h$ must be given as solutions of the following equations:

$$\Phi(h + k) - \Phi(k - h) = \beta \tag{4.1}$$

$$\Phi((m + \tau k)/\sigma) - \Phi((m - \tau k)/\sigma) = \alpha \tag{4.2}$$

and

$$\tanh(\tau mk/\sigma^2) - \coth(hk)\tanh(\tau mh/D^2) = 0\,, \tag{4.3}$$

where $D$ is as in Section 2.2. These intervals can be shown to be limits of Bayes intervals. The interesting question is whether equations (4.1), (4.2) and (4.3) give $m = 0$ as a minimizing solution. Since all translation invariant intervals have this form, the question is really whether the classical intervals $(m = 0)$ are best invariant relative to the concentration criterion. We present without proof,

THEOREM 6. *If*

$$\tanh(hk) - (h\sigma^2/KD^2) > 0 \tag{4.4}$$

*when* $k = (\sigma/\tau)\Phi^{-1}(\frac{1}{2}(\alpha + 1))$ *and $h$ is given by* (4.1), *then the optimal* $m = 0$. *Otherwise the optimal $m$ is the unique positive solution of* (4.1), (4.2) *and*

$$\tanh(hk) - \tanh(\tau mh/D^2)\coth(\tau mk/\sigma^2) = 0\,. \tag{4.5}$$

By taking limits as $\sigma \to 0$ (i.e. as $n \to \infty$ since $\sigma^2 = (\delta^2/n)$), it is found that (4.4) is positive in the limit if and only if

$$\Phi^{-1}\left(\frac{\alpha + 1}{2}\right) > 1 \qquad \text{or} \qquad \alpha > 0.6826\,,$$

so that even for large $n$ the $\alpha$ values which yield $m = 0$ are restricted. When $\sigma = \tau = 1$, and $\beta = 0.90$, (4.4) is positive for $\alpha = 0.90$ so that the optimal $m = 0$. For $\alpha = 0.70$, (4.4) is negative and the optimal value of $m = 0.70$ (by symmetry $-0.70$ is also optimal). The $h$ corresponding to $m = 0.70$ is $h = 2.57$. The classical interval $(m = 0)$ giving the desired confidence is $(X - 2.32, X + 2.32)$ and it has expected coverage 0.897. The optimal interval of the form $(X - 1.87, X + 3.27)$ or $(X - 3.27, X + 1.87)$ has expected coverage 0.896.

The Brown–Fox [3] results can be used to establish admissibility of the best invariant procedure when $m = 0$. If $m \neq 0$, then the best invariant procedure is not unique (nor essentially unique) since $\pm m$ are equally good. It is then possible that the best invariant procedure is inadmissible, and that a proof could be constructed along the lines of Farrell [5].

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