

## ON A MEASURE OF ALIASING DUE TO FITTING AN INCOMPLETE MODEL<sup>1</sup>

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This paper proposes and develops a method for selecting a design to estimate a set of linear parametric functions in cases wherein the adequacy of the preliminary linear model is in doubt. The proposed method relies on the norm of the aliasing matrix. If the expected value of the estimator  $\hat{\phi}$  of a set of linear functions  $\phi = L_1\theta_1$  using a design  $\Gamma$ , under the true model is  $E[\hat{\phi}] = \phi + A_\Gamma\theta_2$ , then the norm  $\|A_\Gamma\| = (\text{trace } A_\Gamma'A_\Gamma)^{\frac{1}{2}}$  is presented as a measure for use in determining "alias balance" and "alias goodness." Therefore,  $\|A_\Gamma\|$  may be used in the selection of a design for experimentation, and its behaviour under various operations is discussed. Some theorems concerning aliases of rank equivalent and complementary designs in certain settings are obtained.

**0. Introduction.** Consider a  $k_1 \times k_2 \times \dots \times k_t$  factorial and let  $T$  be the set consisting of the  $N = \prod_{i=1}^t k_i$  treatment combinations. Assume the linear model, i.e.  $E[Y_T] = W_T\theta$  and  $\text{Cov}[Y_T] = \sigma^2V_T$ , where  $\theta$  is a  $k$ -vector of unknown parameters,  $W_T$  is the  $N \times k$  design matrix and  $V_T$  is an  $N \times N$  known positive definite matrix. Suppose that an experimenter assumes in advance that the  $n_2$ -vector  $\theta_2 = 0$  in the partitioned vector  $\theta' = (\theta_1'; \theta_2')$ ,  $1 \leq n_1 < k$ ,  $1 \leq n_2 < k - 1$ ,  $n_1 + n_2 = k$ . Further suppose that his interest lies in estimation of a  $p$ -vector of linear functions  $\phi = L_1\theta_1$  with a factorial arrangement or design. Let there be a class of feasible designs (i.e. a class such that each design satisfies the experimental constraints and costs the same) such that each design is capable of providing an unbiased estimator of  $L_1\theta_1$ . Let  $\hat{\phi}_\Gamma$  be the least squares estimate of  $\phi_\Gamma$  using design  $\Gamma$  under the model  $E[Y_\Gamma] = X_{1\Gamma}\theta_1$ . If the experimenter wishes to find out how well  $\hat{\phi}_\Gamma$  has done under the general model  $E[Y_\Gamma] = X_{1\Gamma}\theta_1 + X_{2\Gamma}\theta_2$ , i.e. whether the restricted model  $E[Y_\Gamma] = X_{1\Gamma}\theta_1$  was adequate or not, then he should find out how large the bias is of  $\hat{\phi}_\Gamma$ . Since  $E[\hat{\phi}_\Gamma] = L_1\theta_1 + L_1(X_{1\Gamma}'X_{1\Gamma})^{-1}X_{1\Gamma}'X_{2\Gamma}\theta_2 = \phi_\Gamma + A_\Gamma\theta_2 = \phi_\Gamma + B_\Gamma$ , and  $\text{MSE}[\hat{\phi}_\Gamma] = \text{Cov}[\hat{\phi}_\Gamma] + A_\Gamma\theta_2\theta_2'A_\Gamma' = \bar{V}(\Gamma) + \bar{B}(\Gamma)$ , it is clear that an experimenter will seek a design  $\Gamma$  which minimizes an objective function of  $\text{MSE}[\hat{\phi}_\Gamma]$ . This problem can be solved only under very special assumptions, because  $\text{MSE}[\hat{\phi}_\Gamma]$  depends on  $\sigma^2$  and  $\theta_2$ .

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One could introduce separate objective functions for  $\bar{V}(\Gamma)$  and  $\bar{B}(\Gamma)$  and find an optimal design.

This paper does not take the approach outlined above. Instead, we utilize the  $E[\hat{\phi}_\Gamma]$  to show how one design can be preferred over another, through consideration of the aliasing matrix  $A_\Gamma$ . The measure used is the norm  $\|A_\Gamma\| = (\text{trace } A_\Gamma' A_\Gamma)^{\frac{1}{2}}$ , where  $A_\Gamma$  is the alias matrix using the design  $\Gamma$ . Some properties of the matrix  $A_\Gamma$  are studied.

In Section 1 we provide the basic definitions of a factorial, factorial arrangements and fractional factorial arrangements. These definitions are then utilized in Section 2 for the discussion of selecting a design to estimate linear parametric functions. In this same section we introduce the matrix norm  $\|A_\Gamma\| = (\text{trace } A_\Gamma' A_\Gamma)^{\frac{1}{2}}$  as an objective function and point out some of its properties. The related concepts of "alias balance" and "alias goodness" as reflected through the aliasing matrix are given along with an illustrative example to show the basic calculations. The invariance of the alias measure  $\|A_\Gamma\|$  and other results are given in Section 3 under the operations of replication and level permutation. Section 4 deals with the aliases of two rank equivalent designs and Section 5 contains some results on the problem of aliases of a design and its complement in a specific setting. An illustration is provided for bringing out the notion of variance balance and orthogonality in the case of complementary designs.

### 1. Preliminary definitions and notations.

**DEFINITION 1.1.** A factorial arrangement  $\Gamma$  with parameters  $k_1, k_2, \dots, k_t; m; n; r_1, r_2, \dots, r_N$  is defined to be a collection of  $n = \sum_{j=1}^N r_j$  treatments of  $T$  such that the  $j$ th treatment in  $T$  has multiplicity  $r_j \geq 0$ , with at least one nonzero  $r_j$ , and  $m$  is the number of nonzero  $r_j$ 's. We denote such a factorial arrangement by the symbol FA( $k_1, k_2, \dots, k_t; m; n; r_1, r_2, \dots, r_N$ ). Note that in design terminology the multiplicity  $r_j$  is the replication number of the  $j$ th treatment.

**DEFINITION 1.2.** A factorial arrangement is said to be *complete* if  $r_j > 0$  for all  $j$ .

**DEFINITION 1.3.** A complete factorial arrangement is said to be *minimal* if  $r_j = 1$  for all  $j$  and it is designated here by MFA( $k_1, k_2, \dots, k_t$ ) or simply MFA if there is no ambiguity.

**DEFINITION 1.4.** A factorial arrangement is said to be a *fractional factorial arrangement*, or simply a fractional replicate, if some but not all  $r_j > 0$ . We denote a fractional replicate by FFA( $k_1, k_2, \dots, k_t; m; n; r_1, r_2, \dots, r_N$ ) or simply FFA( $m; n; r_1, r_2, \dots, r_N$ ), whenever the underlying factorial structure is clear.

With each treatment  $g$  in  $T$  we associate a random variable  $y_g$ , which is called an observation or response or measurement, with  $E[y_g] = \theta'f(g)$ , where  $\theta$  is a  $k$ -vector of unknown parameters, also called factorial effects, and  $f$  is a  $k$ -vector of  $k$  continuous real-valued known functions on the  $g$ 's in  $T$ . In matrix notation

the linear model is written as:

$$(1.1) \quad E[Y_T] = W_T \theta, \quad \text{Cov}[Y_T] = \sigma^2 V_T,$$

where  $V_T$  is a known positive definite  $N \times N$  matrix, which with no loss of generality can be assumed to be the identity matrix of order  $N$ . The element in the  $g$ th row and the  $j$ th column of  $W_T$  is equal to  $f_j(g)$ . The  $N \times k$  matrix  $W_T$  is known as the design matrix corresponding to observation vector  $Y_T$  and the parametric vector  $\theta$ . Linear models of type (1.1) are popular in applications and a celebrated one is the polynomial model. Note that the model in (1.1) is associated with the minimal complete factorial arrangement.

Corresponding to a factorial arrangement  $\Gamma$  the model for the  $n \times 1$  observation vector  $Y_\Gamma$  induced by (1.1) is equal to.

$$(1.2) \quad E[Y_\Gamma] = X_\Gamma \theta,$$

where  $X_\Gamma$  is an  $n \times k$  matrix simply read off from  $W_T$  taking repetitions of treatments in  $\Gamma$  into account.

**2. Estimation of linear parametric functions.** In this section we introduce a general partitioning of  $\theta$  in order to estimate a set of linear functions of its components. We distinguish four distinct cases. This general framework includes the classical BLU estimation of  $\theta$ , response surface fitting, odd and even resolution problems, and biased fitting or estimation as special cases. Consider the following partitioning of the vector  $\theta$ ,

$$\theta' = (\theta_1' : \theta_2' : \theta_3'),$$

where  $\theta_1$  is an  $n_1$ -vector to be estimated,  $\theta_2$  is an  $n_2$ -vector not of interest for estimation and not assumed to be known, and  $\theta_3$  is an  $n_3$ -vector assumed to be known such that:  $1 \leq n_1 \leq k$ ,  $0 \leq n_2 \leq k - 1$ ,  $0 \leq n_3 \leq k - 1$  with  $n_1 + n_2 + n_3 = k$ . Let  $L_1$  be a  $p \times n_1$  matrix of rank  $\leq p$  and suppose our interest lies in estimating the set of  $p$  linear functions of  $\theta_1$  given by  $\psi = L_1 \theta_1$ . Explicitly we then have the following four cases:

- (i)  $n_1 = k$ ,  $n_2 = n_3 = 0$
- (ii)  $n_2 = 0$ ,  $n_3 \neq 0$
- (iii)  $n_2 \neq 0$ ,  $n_3 \neq 0$
- (iv)  $n_2 \neq 0$ ,  $n_3 = 0$ .

Case (i) with  $L_1$  equal to the identity matrix of order  $k$  gives us the BLU estimation problem of  $\theta$ . If  $L_1 = W_\Gamma$ , then under case (i) we have the response surface fitting problem. Cases (ii) and (iii) with  $L_1$  equal to the identity matrix of order  $n_1$  lead to odd and even resolution problems respectively. The biased linear estimation problems fall under either case (iii) or case (iv).

The above partitioning of the vector  $\theta$  induces a partitioning of the design matrix  $X_\Gamma$  so that the model in (1.2) can be rewritten as:

$$(2.1) \quad E[Y_\Gamma] = X_\Gamma \theta = X_{1\Gamma} \theta_1 + X_{2\Gamma} \theta_2 + X_{3\Gamma} \theta_3.$$

Since  $\theta_3$  is assumed to be known (2.1) reduces to

$$(2.2) \quad E[Y_{\Gamma}^*] = X_{1\Gamma}\theta_1 + X_{2\Gamma}\theta_2, \quad \text{where } Y_{\Gamma}^* = Y_{\Gamma} - X_{3\Gamma}\theta_3.$$

Hence without loss of generality we may hereafter assume  $\theta_3 = 0$ . Problems related to cases (i) and (ii) have received considerable attention in the published literature. Interest is currently being shown in "bias optimality". This interest stems from the fact that many researchers are becoming concerned with the correctness of their models. Many problems of "bias optimality" or "bias goodness" still remain to be resolved. Some of these problems present themselves in the following natural way. The experimenter is fairly certain that his model is:

$$(2.3) \quad E[Y_{\Gamma}] = X_{1\Gamma}\theta_1,$$

and uses the standard least squares technique to estimate  $L_1\theta_1$ . However, he wants to find out how bad his action is if indeed model (2.3) is not adequate, i.e. the true model is (2.2). This means that he wants to find out the behaviour of  $\hat{\phi}_{\Gamma}$  under model (2.2). The following are some of the properties of  $\hat{\phi}_{\Gamma}$  under model (2.2):

$$(2.4) \quad E[\hat{\phi}_{\Gamma}] = L_1\theta_1 + A_{\Gamma}\theta_2 = \phi_{\Gamma} + B_{\Gamma},$$

where  $A_{\Gamma} = L_1(X'_{1\Gamma}X_{1\Gamma})^{-}X'_{1\Gamma}X_{2\Gamma}$ . Here  $(\cdot)^{-}$  denotes a generalized inverse. We define  $A_{\Gamma}$  to be the *alias* or *contamination matrix*. It follows that any optimality measure of  $\hat{\phi}_{\Gamma}$  should not only be based on the covariance of  $\hat{\phi}_{\Gamma}$  alone but rather on its mean square error (MSE). This quantity is equal to:

$$(2.5) \quad \text{MSE}[\hat{\phi}_{\Gamma}] = \text{Cov}[\hat{\phi}_{\Gamma}] + A_{\Gamma}\theta_2\theta_2'A_{\Gamma}' = \bar{V}(\Gamma) + \bar{B}(\Gamma).$$

The problem of biased estimation as introduced above can now be formally stated.

**PROBLEM.** Let  $\Delta(L_1) = \{\Gamma_1, \Gamma_2, \dots, \Gamma_s\}$  be a set of  $s$  competing designs from the same  $\prod k_i$  factorial for the purpose of estimating  $L_1\theta_1$ . Suppose that all designs in  $\Delta(L_1)$  satisfy the constraints of the experiment and cost the same amount. Further, assume that each design in  $\Delta(L_1)$  is capable of providing an unbiased estimate of  $L_1\theta_1$  under model (2.3). (A necessary and sufficient condition for this is that for every  $\Gamma_i$  in  $\Delta(L_1)$  there exists a  $p \times n$  matrix  $K_{\Gamma_i}$  such that  $L_1 = K_{\Gamma_i}X_{1\Gamma_i}$ .) Therefore  $\Delta(L_1)$  is a class of feasible designs and the experimenter makes a choice of a design from  $\Delta(L_1)$ .

An experimenter should not choose his design in a random manner from among the designs in  $\Delta(L_1)$ , but instead he should introduce a meaningful objective function into his problem. This function which we denote by  $Q(\cdot)$  measures quantitatively a certain quality associated with each design in  $\Delta(L_1)$ . He then chooses a design in  $\Delta(L_1)$  which minimizes  $Q(\Gamma)$ ,  $\Gamma \in \Delta(L_1)$ .

Having assumed that every design in  $\Delta(L_1)$  is feasible leads us to consider those objective functions which reflect the statistical properties of  $\hat{\phi}_{\Gamma}$ . In practice, an experimenter will be unlikely to know his  $Q$  exactly. Fortunately there is usually a kind of insensitivity to using a slightly incorrect  $Q$ , i.e. if  $Q'$  is the one he

“should” use and  $Q''$  is a slightly different one, the design which minimizes  $Q''$  will not be too far from the design which minimizes  $Q'$ . This being so, in the absence of exact knowledge of this  $Q'$ , design theorists often consider a  $Q''$  which makes computation and interpretation simple.

Perhaps the most reasonable  $Q$  will be the one which reflects a magnitude of  $\text{MSE}(\hat{\phi}_\Gamma)$  or, in general, a convex combination of  $\bar{V}(\Gamma)$  and  $\bar{B}(\Gamma)$ . However, this is not an easy problem because  $\text{MSE}(\hat{\phi}_\Gamma)$  is a function of  $\sigma^2$  and  $\theta_2$  and thus one would have to deal with an objective function such as:

$$(2.6) \quad Q(\Gamma) = \max_{\theta \in \Lambda} \text{trace}[\bar{V}(\Gamma) + \bar{B}(\Gamma)],$$

where  $\Lambda$  is some specified set of parameter values. In certain situations where the experimenter knows something about the relative magnitudes of  $\sigma^2$  and  $\theta_2'$ , and the measure is the one given in (2.6), some progress is possible. For example, if  $\sigma^2/\theta_2'$  is “large” (the usual case in the philosophy of this section, viz., of primarily worrying about  $\theta_2 = 0$ ), then the quantity to be minimized is approximately the trace of  $\bar{V}(\Gamma)$ . On the other hand, if  $\sigma^2/\theta_2'$  is “small,” then the trace of  $\bar{B}(\Gamma)$  should be minimized. Note that these are approximate statements based on a priori knowledge concerning  $\sigma^2$  and  $\theta_2'$ . These difficulties can be partially overcome or circumvented if the experimenter limits his concern to  $\bar{V}(\Gamma)$  and to  $\bar{B}(\Gamma)$  separately. This means that two quantities should be introduced for measuring the magnitude of  $\bar{V}(\Gamma)$  and  $\bar{B}(\Gamma)$ . Quantities such as the sum or the product of the eigenvalues of  $\bar{V}(\Gamma)$  can be associated with  $\bar{V}(\Gamma)$ . The trace and similar quantities can be associated with  $\bar{B}(\Gamma)$ . For a different approach to this problem, see Anderson (1960), Box and Draper (1959, 1963), Draper and Herzberg (1971), Draper and Lawrence (1965a, 1965b, 1965c), Folks (1958), Hader, Manson and Cote (1971), Karson, Manson and Hader (1969), Kiefer (1972), and Thompson (1973).

In this paper we shall consider a somewhat different approach which utilizes the expected value of  $\hat{\phi}_\Gamma$ . Among the various measures which can be introduced, those which take into account all the entries of  $A_\Gamma$  and their magnitudes are the appealing ones. Among such well-known norms (e.g. see Bodewig (1959)) is the following:

$$(2.7) \quad Q^*(\Gamma) = \|A_\Gamma\| = (\sum_i \sum_j a_{ij}^2(\Gamma))^{\frac{1}{2}}.$$

The norm  $\|A_\Gamma\|$  enjoys some desirable properties which the other common norms do not possess, namely:

(1)  $\|A_\Gamma\|$  is orthogonally invariant, i.e.  $\|P_1 A_\Gamma\| = \|A_\Gamma P_2\| = \|A_\Gamma\|$  if  $P_1$  and  $P_2$  are orthogonal matrices.

(2)  $\|A_\Gamma\| = (\text{trace } A_\Gamma' A_\Gamma)^{\frac{1}{2}}$ , which implies that  $\|A_\Gamma\|$  is the positive square root of the sum of the eigenvalues of  $A_\Gamma' A_\Gamma$ . In particular, if  $A_\Gamma$  is a square matrix, then  $\|A_\Gamma\| = (\text{trace } A_\Gamma' A_\Gamma)^{\frac{1}{2}} = (\text{trace } A_\Gamma A_\Gamma')^{\frac{1}{2}} = (\sum_i \lambda_i^2(\Gamma))^{\frac{1}{2}}$ , where the  $\lambda_i$ 's are the eigenvalues of  $A_\Gamma$ .

Note that in fractional replication, as defined in this paper, the matrix  $A_\Gamma$  is

never equal to zero, if the number of components in  $\theta$  is  $k = N$ , which is the case in the classical setting of factorial experimentation. This implies that  $\|A_\Gamma\|$  (and indeed any other norm of  $A_\Gamma$ ) is never equal to zero in fractional replication, or, to put it in another way, fractional factorial designs can be characterized by the amount of contamination associated with their alias matrices.

Before closing this section we introduce the concepts of “alias balance” and “alias goodness.”

DEFINITION 2.1. A fractional factorial design  $\Gamma$  is said to be *contamination or alias balanced* if  $(\sum_j a_{ij}^2(\Gamma))^{\frac{1}{2}}$  is constant for all  $i$ .

Note that this definition implies that in an alias balanced design the aliasing associated with each element of  $\widehat{L_1\theta_1}$  is equal to  $\|A_\Gamma\|/c$ , where  $c$  is a positive constant. We say that the  $i$ th component of  $\widehat{L_1\theta_1}$  is estimated with less aliasing than the  $i'$ th component of  $\widehat{L_1\theta_1}$  if  $(\sum_j a_{ij}^2)^{\frac{1}{2}} < (\sum_j a_{i'j}^2)^{\frac{1}{2}}$ .

DEFINITION 2.2. Let  $\Gamma_1$  and  $\Gamma_2$  be two competing fractional factorial designs in  $\Delta(L_1)$ . Then  $\Gamma_1$  is said to be *alias better* than  $\Gamma_2$  if  $\|A_{\Gamma_1}\| - \|A_{\Gamma_2}\| < 0$ . We define  $\Gamma_1$  and  $\Gamma_2$  to be alias equivalent if  $\|A_{\Gamma_1}\| = \|A_{\Gamma_2}\|$ .

An illustrative example. Let  $t = 3$  and let the  $k_1 = k_2 = k_3 = 2$  levels of each factor be denoted by 0 and 1. Assuming model (1.1) for the following  $N = 8$  observations we have:

$$(2.8) \quad E \begin{bmatrix} y_{000} \\ y_{100} \\ y_{010} \\ y_{110} \\ y_{001} \\ y_{101} \\ y_{011} \\ y_{111} \end{bmatrix} = \begin{bmatrix} + & - & - & + & - & + & + & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + \\ + & + & + & + & - & - & - & - \\ + & - & - & + & + & - & - & + \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \\ + & + & + & + & + & + & + & + \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}.$$

Note that the parameters  $\alpha_1, \alpha_2, \dots, \alpha_8$  in (2.8) are the standard factorial effects in the order  $\mu, A, B, AB, C, AC, BC$  and  $ABC$ . Let  $\Gamma_1 = \{(000), (011), (101), (110)\}$  be a fractional factorial design, where  $(x_1x_2x_3)$  refers to a treatment combination with the  $i$ th factor being at the  $x_i$  level,  $i = 1, 2, 3$ . For this fraction  $n = 4$  and  $m = 4$ . The equation system (1.2) for  $\Gamma_1$  is:

$$E \begin{bmatrix} y_{000} \\ y_{011} \\ y_{101} \\ y_{110} \end{bmatrix} = \begin{bmatrix} + & - & - & + & - & + & + & - \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & + & + & + & - & - & - & - \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}.$$

Suppose now that the experimenter is interested in obtaining information regarding four parameters specified by  $\theta_1' = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ . Here  $n_1 = 4$  and we assume  $n_3$  to be equal to zero, so that  $n_2 = 4$ . The partitioned system for the fraction  $\Gamma_1$  is then:

$$E[Y_{\Gamma_1}] = \begin{bmatrix} + & - & - & - \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} + & + & + & - \\ - & - & + & - \\ - & + & - & - \\ + & - & - & - \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}.$$

The rank of  $X_{\Gamma_1}$  in this example is clearly equal to 4, so that separate information for each component of  $\theta_1$  is available. Therefore, if we let  $L_1$  be the identity matrix of order 4 we obtain:

$$A_{\Gamma_1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{with } \|A_{\Gamma_1}\| = 2.$$

Note that for this design  $\Gamma_1$ ,  $\sum_j a_{ij}^2(\Gamma_1) = 1$ ,  $i = 1, 2, 3, 4$  and hence  $\Gamma_1$  is alias balanced.

Next, consider the competing fraction  $\Gamma_2 = \{(000), (001), (101), (111)\}$ . Clearly  $\theta_1$  as defined earlier is estimable by the design  $\Gamma$ . The alias matrix associated with  $\Gamma_2$  is:

$$A_{\Gamma_2} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

Hence the measure of aliasing for  $\Gamma_2$  is equal to  $\|A_{\Gamma_2}\| = 2(3)^{\frac{1}{2}}$ .  $\Gamma_1$  and  $\Gamma_2$  are both alias balanced, but  $\Gamma_1$  is better than  $\Gamma_2$  from this viewpoint.

**3. Invariance of the measure of aliasing under various operations.** In this section we shall characterize some operations which leave the measure of aliasing  $\|A_{\Gamma}\|$  invariant. These operations arise naturally in many practical and theoretical settings.

*A. The replication operation.* Let  $\Gamma$  be an FFA  $(m; n; r_1, r_2, \dots, r_N)$  and let  $\Gamma(\alpha)$  be  $\alpha$  copies of  $\Gamma$ . The process of obtaining  $\Gamma(\alpha)$  from  $\Gamma$  will be denoted as the *replication operation*. Let  $\Gamma$  be an FFA and let the greatest common divisor of the nonzero  $r_j$ 's be  $d$ . Then if  $r_j' = r_j/d$ ,  $\sum r_j' = n'$ , and  $\Gamma_{d'}$  is the same FFA with the  $r_j$ 's replaced by  $r_j'$ 's and  $n$  replaced by  $n'$ ,  $\Gamma_{d'}$  is said to be the *reduced form* of  $\Gamma$ . A *minimal proper fraction*, designated as  $\Gamma_M$ , associated with  $\Gamma$  is the design obtained by deleting all duplications of treatments in  $\Gamma$ . Now with respect to the replication operation the following results are easily verified:

**THEOREM 3.1.** *The amount of alias associated with  $\Gamma$  is invariant under the replication operation, i.e.  $\|A_{\Gamma}\| = \|A_{\Gamma(\alpha)}\|$ .*

**COROLLARY 3.1.** *The amount of alias associated with the fraction  $\Gamma$  is the same as the amount of alias associated with the reduced form of  $\Gamma$ , i.e.  $\|A_\Gamma\| = \|A_{\Gamma_D}\|$ .*

**THEOREM 3.2.** *The amounts of alias of the fractions  $\Gamma$  and  $\Gamma_M$  are related through the replication matrix  $R = \text{diag}(r_1'', r_2'', \dots, r_m'')$  i.e.*

$$\|A_\Gamma\| = \|K_{\Gamma_M} X_{1\Gamma_M} (X'_{1\Gamma_M} R X_{1\Gamma_M})^{-1} X'_{1\Gamma_M} R X_{2\Gamma_M}\|,$$

and

$$\|A_{\Gamma_M}\| = \|K_{\Gamma_M} X_{1\Gamma_M} (X'_{1\Gamma_M} X_{1\Gamma_M})^{-1} X'_{1\Gamma_M} X_{2\Gamma_M}\|,$$

where  $r_1'', r_2'', \dots, r_m''$  are the nonzero  $r_i$ 's in the reduced form of  $\Gamma$ .

**COROLLARY 3.2.** *The amount of alias of a fraction  $\Gamma$  is equal to the amount of alias of  $\Gamma_M$  if  $X_{1\Gamma_M}$  has full rank.*

The preceding corollary indicates that with respect to aliasing the effect of replication is of no consequence if the design matrix is nonsingular, i.e. the measure of aliasing is invariant under replication as long as the design matrix is of full rank. The practical consequence of this corollary is that the experimenter is economically better off using the proper minimal fraction in this situation. This result, by the way, also shows the unimportance of the classical notion of unbalanced (unequal numbers) designs as far as aliasing is concerned. Unequal numbers designs do affect analysis problems and variance considerations.

**B. The permutation operation.** Let  $\Gamma$  be a fractional factorial design and let  $\omega(\Gamma)$  be the corresponding permuted design obtained by applying a permutation of levels to the treatment combinations in  $\Gamma$ . In the development below we show that  $\|A_\Gamma\| = \|A_{\omega(\Gamma)}\|$ , if  $\theta_1$  is selected in a particular way, i.e. under certain conditions on the elements of  $\theta_1$  the measure of aliasing is invariant under level permutations.

Formally, let the  $k_i$  levels of the factor  $F_i$  be identified as  $\{0, 1, 2, \dots, k_i - 1\}$ ,  $i = 1, 2, \dots, t$ . Let  $\Omega$  be the set of permutations of the form  $\omega = (\omega_1, \omega_2, \dots, \omega_t)$ , where  $\omega_i$  is a permutation acting on the levels of the  $i$ th factor. A realistic choice of  $\{L_1, \theta_1, \Gamma\}$  implies that the design  $\Gamma$  should be capable of providing the desired statistical information on  $L_1\theta_1$ . However, not all realistic choices of  $\{L_1, \theta_1, \Gamma\}$  guarantee the invariance of information and amount of aliasing under a permutation  $\omega$ . An interesting and open problem is to characterize the set  $\{L_1, \theta_1\}$  such that a permutation  $\omega$  leaves the information and/or amount of aliasing invariant. A partial solution to this problem is provided by Theorem 3.3 below.

Denote an element of  $\theta$  by the symbol  $A_1^{x_1} A_2^{x_2} \dots A_t^{x_t}$ , where  $x_i \in \{0, 1, 2, \dots, k_i - 1\}$ . Note that in this notation the mean  $\mu = A_1^0 A_2^0 \dots A_t^0$  and  $\{A_1^0 A_2^0 \dots A_{i-1}^0 A_i^1 A_{i+1}^0 \dots A_t^0, A_1^0 A_2^0 \dots A_{i-1}^0 A_i^2 A_{i+1}^0 \dots A_t^0, \dots, A_1^0 A_2^0 \dots A_{i-1}^0 A_i^{k_i-1} A_{i+1}^0 \dots A_t^0\}$  represents the set of  $k_i$  normalized orthogonal parametric contrasts associated with the  $i$ th factor and the mean.

**DEFINITION 3.1.** The parametric vector  $\theta_1$  is said to be admissible if and only



if whenever  $A_1^{x_1} A_2^{x_2} \dots A_i^{x_i} \dots A_t^{x_t}$  belongs to  $\theta_1$  and  $x_i \neq 0$ , ( $1 \leq i \leq t$ ), then  $A_1^{z_1} A_2^{z_2} \dots A_i^{z_i} \dots A_t^{z_t}$  also belongs to  $\theta_1$  for all  $z \neq 0$ .

The following lemma was obtained by Srivastava, Raktoe, and Pesotan (1971).

LEMMA 3.1. *If  $\Gamma$  is an arbitrary fraction and  $\omega(\Gamma)$  is the permuted fraction obtained by the action of  $\omega \in \Omega$ , and  $\theta_1$  is admissible, then there exist orthogonal matrices  $P_{1\omega}$  and  $P_{2\omega}$  such that  $X_{1\omega(\Gamma)} = X_{1\Gamma} P_{1\omega}$  and  $X_{2\omega(\Gamma)} = X_{2\Gamma} P_{2\omega}$ .*

We next have

THEOREM 3.3. *The amount of aliasing  $\|A_\Gamma\|$  is invariant under  $\Omega$  if  $\theta_1$  is admissible.*

PROOF.

$$\begin{aligned} \|A_{\omega(\Gamma)}\| &= \|K_\Gamma X_{1\Gamma} P_{1\omega} [(X_{1\Gamma} P_{1\omega})' (X_{1\Gamma} P_{1\omega})]^{-1} (X_{1\Gamma} P_{1\omega})' X_{2\Gamma} P_{2\omega}\| \\ &= \|K_\Gamma X_{1\Gamma} (X_{1\Gamma}' X_{1\Gamma})^{-1} X_{1\Gamma}' X_{2\Gamma} P_{2\omega}\|, \\ &\hspace{15em} \text{by a property of generalized inverse.} \\ &= \|A_\Gamma\|, \hspace{15em} \text{by property (1) of the norm.} \end{aligned}$$

**4. Aliases of two rank equivalent designs.** Let us explore now the behaviour of the alias measure of two rank equivalent designs. More specifically, consider two fractions  $\Gamma$  and  $\Gamma^*$  from the same factorial such that (i) they have the same cardinality, (ii) the corresponding design matrices  $X_{1\Gamma}$  and  $X_{1\Gamma^*}$  have the same rank, and (iii) the corresponding matrices  $X_{2\Gamma}$  and  $X_{2\Gamma^*}$  have the same rank. We assume that  $L_1$  and  $\theta_1$  are the same for both designs. From elementary algebra we know that:

$$X_{1\Gamma^*} = E_1 X_{1\Gamma} F_1 \quad \text{and} \quad X_{2\Gamma^*} = E_2 X_{2\Gamma} F_2,$$

where  $E_1, E_2, F_1$  and  $F_2$  are nonsingular square matrices of appropriate dimensions. This leads us to the following expression for the alias measure of design  $\Gamma^*$ .

$$\|A_{\Gamma^*}\| = \|K_\Gamma E_1 X_{1\Gamma} F_1 [(E_1 X_{1\Gamma} F_1)' (E_1 X_{1\Gamma} F_1)]^{-1} (E_1 X_{1\Gamma} F_1)' E_2 X_{2\Gamma} F_2\|.$$

If  $E_1$  and  $E_2$  are orthogonal and

$$[(X_{1\Gamma} F_1)' (X_{1\Gamma} F_1)]^{-1} = F_1^{-1} (X_{1\Gamma}' X_{1\Gamma})^{-1} F_1'^{-1}$$

then:

$$\|A_{\Gamma^*}\| = \|K_\Gamma X_{1\Gamma} (X_{1\Gamma}' X_{1\Gamma})^{-1} X_{1\Gamma}' X_{2\Gamma} F_2\|.$$

Finally, if  $F_2$  is also orthogonal then it follows that  $\|A_{\Gamma^*}\| = \|A_\Gamma\|$ . Note that Theorem 3.3 is a special case of the above setup, i.e.  $E_1$  and  $E_2$  are identity matrices, and  $F_1$  and  $F_2$  are orthogonal matrices. Further characterizations of  $E_1, E_2, F_1$  and  $F_2$  are needed so that relations can be established between  $\|A_\Gamma\|$  and  $\|A_{\Gamma^*}\|$  under various settings.

**5. Aliases of a design and its complement.** In this section we relate the aliasing of a design to the aliasing in its complementary design given that these designs satisfy certain regularity conditions.

DEFINITION 5.1. An FFA ( $m; n; r_1, r_2, \dots, r_N$ ) is said to be a (0, 1) binary fractional factorial design if  $r_i = 0$  or 1 for each  $i$ .

Note that a binary fractional factorial design is an FFA with  $m = n$ .

**DEFINITION 5.2.** Two  $(0, 1)$  binary fractions  $\Gamma$  and  $\bar{\Gamma}$  from the same  $k_1 \times k_2 \times \dots \times k_t$  factorials are said to be complementary to each other if  $\Gamma \cup \bar{\Gamma} = T$ .

It follows that if  $\Gamma$  is an FFA  $(m; n; r_1, r_2, \dots, r_N)$  then  $\bar{\Gamma}$  is an FFA  $(N - n; N - n; \bar{r}_1, \bar{r}_2, \dots, \bar{r}_N)$  such that  $r_i + \bar{r}_i = 1, i = 1, 2, \dots, N$ .

**DEFINITION 5.3.** A  $(0, 1)$  binary fraction  $\Gamma$  such that the  $n_1$ -vector  $\theta$  is estimable under the assumption that the  $n_2$ -vector  $\theta_2 = 0, n_1 + n_2 = k$ , is said to be variance balanced and orthogonal if  $X'_{1\Gamma} X_{1\Gamma} = \lambda I_{n_1}$ , where  $\lambda$  is a nonzero scalar and  $I_{n_1}$  is the identity matrix of order  $n_1$ .

Let  $\Gamma$  and  $\bar{\Gamma}$  be two  $(0, 1)$  binary complementary fractions such that  $\Gamma$  is variance balanced and orthogonal for the  $n_1$ -vector  $\theta_1$  and  $\bar{\Gamma}$  is variance balanced and orthogonal for the  $n_2$ -vector  $\theta_2$ , i.e.  $X'_{1\Gamma} X_{1\Gamma} = \lambda_1 I_{n_1}$  and  $X'_{2\bar{\Gamma}} X_{2\bar{\Gamma}} = \lambda_2 I_{n_2}$ , where  $\Gamma \cup \bar{\Gamma} = T, n_1 + n_2 = k$  and  $\lambda_1, \lambda_2$  are nonzero scalars. The following lemma relates  $\|A_\Gamma\|$  to  $\|A_{\bar{\Gamma}}\|$  in the classical orthogonal factorial setting.

**LEMMA 5.1.** *If  $\Gamma$  and  $\bar{\Gamma}$  are as described above and  $n_1 + n_2 = N, W_T' W_T = I_N$ , then (i)  $\|A_\Gamma\| = [(N - n)(1 - \lambda_2)/\lambda_1]^{\frac{1}{2}}$  and (ii)  $\|A_{\bar{\Gamma}}\| = [n(1 - \lambda_1)/\lambda_2]^{\frac{1}{2}}$ .*

**PROOF.** We give a proof for (i); an analogous proof can be obtained for (ii). Since the alias matrix  $A_\Gamma = (X'_{1\Gamma} X_{1\Gamma})^{-1} X_{1\Gamma} X_{2\Gamma} = \lambda_1^{-1} X'_{1\Gamma} X_{2\Gamma}$  and  $A_\Gamma' A_\Gamma = \lambda_1^{-2} X'_{2\Gamma} X_{1\Gamma} X'_{1\Gamma} X_{2\Gamma} = \lambda_1^{-1} X'_{2\Gamma} X_{2\Gamma} = \lambda_1^{-1} (I_{N-n} - \lambda_2 I_{N-n}) = \lambda_1^{-1} (1 - \lambda_2) I_{N-n}$  it follows that  $\|A_\Gamma\| = [\text{trace}(A_\Gamma' A_\Gamma)]^{\frac{1}{2}} = [(N - n)(1 - \lambda_2)/\lambda_1]^{\frac{1}{2}}$ .

Note that in the above setup  $\|A_\Gamma\| \rightarrow 0$  as  $n \rightarrow N$  as intuitively expected. Proceeding under the same setting we observe that  $|\det X_{1\Gamma}| = \lambda_1^{n/2}$  and  $|\det X_{2\bar{\Gamma}}| = \lambda_2^{(N-n)/2}$ . Also from a theorem in Muir and Metzler (1933) we know that  $|\det X_{1\Gamma}| = |\det X_{2\bar{\Gamma}}|$ , so that  $\lambda_1^n = \lambda_2^{N-n}$ . Hence the following result holds:

**THEOREM 5.1.** *Let  $\Gamma$  and  $\bar{\Gamma}$  be as in Lemma 5.1, then knowledge of  $\|A_\Gamma\|$  implies knowledge of  $\|A_{\bar{\Gamma}}\|$ . Further, if  $n = N/2$  then  $\|A_\Gamma\| = \|A_{\bar{\Gamma}}\|$ .*

**COROLLARY 5.1.** *Let  $\Gamma^*$  be an  $\alpha$  fold  $(0, 1)$  binary variance balanced and orthogonal fraction in the classical fractional factorial setup, then  $\Gamma^*$  is alias balanced.*

**PROOF.** The proof follows from Corollary 3.1 and the fact that by Lemma 4.1,  $A'_{\Gamma^*} A_{\Gamma^*}$  is a multiple of an identity matrix.

Further exploration of relations between the aliases of  $\Gamma$  and  $\bar{\Gamma}$  are desirable in less restrictive settings.

*An illustrative example.* Consider the  $2 \times 2 \times 2$  factorial with the usual set of treatment combinations  $T$  and the usual parametric vector  $\theta$ . (See equation 2.8 of Section 2.) Let  $\Gamma = \{(000), (011), (101), (110)\}$  to estimate  $\theta_1' = (\mu, A, B, C)$ . The corresponding complementary design is  $\bar{\Gamma} = \{(100), (010), (001), (111)\}$  to estimate  $\theta_2' = (AB, AC, BC, ABC)$ . The matrices required to obtain  $\|A_\Gamma\|$  and  $\|A_{\bar{\Gamma}}\|$  are:

$$\begin{array}{l}
 X_{1\Gamma} = \begin{bmatrix} + & - & - & - \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix} \\
 X_{2\Gamma} = \begin{bmatrix} + & + & + & - \\ - & - & + & - \\ - & + & - & - \\ + & - & - & - \end{bmatrix}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{l}
 X_{1\bar{\Gamma}} = \begin{bmatrix} + & + & - & - \\ + & - & + & - \\ + & - & - & + \\ + & + & + & + \end{bmatrix} \\
 X_{2\bar{\Gamma}} = \begin{bmatrix} - & - & + & + \\ - & + & - & + \\ + & - & - & + \\ + & + & + & + \end{bmatrix}.
 \end{array}$$

The reader can easily verify that  $\Gamma$  and  $\bar{\Gamma}$  are  $(0, 1)$  binary variance balanced and orthogonal fractions which are also alias balanced. Moreover,  $\Gamma$  and  $\bar{\Gamma}$  carry the same amount of aliasing, i.e.  $\|A_{\Gamma}\| = \|A_{\bar{\Gamma}}\|$ .

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