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A NOTE ON THE STRONG CONVERGENCE OF DISTRIBUTIONS

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If a sequence of distributions converges weakly to a distribution and if the associated densities are unimodal then the sequence of density functions converges to the density of the limiting distribution. This stronger form of convergence is useful in the investigation of limiting joint distributions.

Scheffé (1947) pointed out that if a sequence of density functions $\{f_n\}$ converges pointwise to a density function $f(x)$ then $\int_B f_n$ converges to $\int_B f$ uniformly for all Borel sets B. This strong form of convergence is sometimes useful in the investigation of joint limiting distributions. Sethuraman (1961 b) has shown that if a sequence of marginal distributions converges strongly in the preceding sense, and the sequence of conditional distributions converges weakly, then the sequence of joint distributions converges weakly. He further gives examples to show that weak convergence of both conditional and marginal distributions is not sufficient. Sethuraman (1961 a), Bickel (1965) and Hettmansperger (1968) have applied these results in determining the limiting normality of estimators and test statistics.

Often it is relatively easy to establish the weak convergence for a sequence of distributions. In this note we point out sufficient conditions in order to assert the strong convergence of the sequence of distributions.

We say that a probability density function $f(x)$ is unimodal at a point a if $f(x)$ is non-decreasing on $(-\infty, a)$ and non-increasing on (a, ∞). We wish to prove the following proposition.

Proposition. *Let $\{f_n(x)\}$ be a sequence of densities which are unimodal at zero. Assume that the corresponding distribution functions $F_n(x)$ converge weakly to $F(x)$, and that $F(x)$ has a continuous density $f(x)$. Then for each $x \neq 0$, $f_n(x) \to f(x)$ as $n \to \infty$. Futhermore, convergence is uniform on any closed interval not containing zero.*

Proof. Suppose not. Then there exist an x_0 (which we may take positive) and an $\varepsilon_0 > 0$ such that $|f_n(x_0) - f(x_0)| > \varepsilon_0$ for infinitely many n. Since $f(x)$ is continuous at x_0, there is a δ such that $0 < \delta < x_0$ and $|f(x) - f(x_0)| < \varepsilon_0/2$ whenever $|x - x_0| < \delta$. If $f_{n'}(x_0) > f(x_0) + \varepsilon_0$ for some subsequence n', we then have (using weak convergence of F_n to F)

$$\delta(f(x_0) + \varepsilon_0) \leqq \lim_{n'} \int_{x_0-\delta}^{x_0} f_{n'}(t)\, dt = \int_{x_0-\delta}^{x_0} f(t)\, dt \leqq \delta(f(x_0) + \varepsilon_0/2)$$

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which is impossible. A similar contradiction is obtained if $f_n(x_0) < f(x_0) - \varepsilon_0$ for infinitely many n.

The assertion about uniform convergence follows easily since now $f(x)$ must be monotone and bounded on any closed interval not containing zero.

The following example shows that we cannot get convergence at zero. Let $g(x)$ be any continuous density function which is unimodal at zero. Let

$$\begin{aligned} h_n(x) &= n^3(x + 1/n^2) && \text{if } -1/n^2 \leqq x \leqq 0 \\ &= n^3(1/n^2 - x) && \text{if } 0 \leqq x \leqq 1/n^2 \\ &= 0 && \text{otherwise} \end{aligned}$$

and let $f_n(x) = [g(x) + h_n(x)]/[1 + n^{-1}]$. Here, $f_n(x) \to g(x)$ for $x \neq 0$, but $f_n(0) \to \infty$.

As an application of the proposition, let $X_1, \cdots, X_n$ be a random sample from a symmetric unimodal distribution with finite variance; then the sample mean also has this type of distribution by Theorem 4.5.5 in the book by Lukacs (1970). Hence the central limit theorem implies the weak convergence of the sequence of distributions of the sample mean and the preceding proposition establishes the strong convergence.

The same result holds if we drop the symmetry requirement and replace unimodal by strongly unimodal.

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