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ON THE JOINT DISTRIBUTION OF FRIEDMAN'S χ_r^2 STATISTICS[1]

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This study is concerned with the joint distribution of Gerig's (1969) statistics when applied to tests for shift in various marginal distributions pertaining to complete two-way multivariate data. The exact small-sample distribution can be found using conditional permutation arguments, and the limiting permutation distribution is shown to belong to a known class of multivariate chi-square distributions. A special case yields the limiting joint distribution of Friedman's (1937) χ_r^2 statistics for the one-dimensional marginal distributions. Berry-Esséen bounds are given for the rate of convergence of the joint distribution to its limiting form when the underlying distributions are identical over replications.

1. Introduction. Let $\mathbf{Y}_{ij} = [Y_{ij}^1, \cdots, Y_{ij}^m]'$ be the vector of responses to the jth of k treatments on the ith of N replications in a complete two-way classification scheme, and let $\{F_{ij}(\mathbf{z}); 1 \leqq j \leqq k, 1 \leqq i \leqq N\}$ denote their cumulative distribution functions (cdf's), assumed independent for $i = 1, 2, \cdots, N$. For $\{F_{ij}(\cdot)\}$ continuous, Gerig (1969) devised a test for the hypothesis $H_0\colon F_{i1} \equiv F_{i2} \equiv \cdots \equiv F_{ik} \equiv F_i$, $1 \leqq i \leqq N$, against the translation alternatives $H_A\colon F_{ij}(\mathbf{z}) = F_i(\mathbf{z} - \boldsymbol{\mu}_j)$, $1 \leqq j \leqq k$, $1 \leqq i \leqq N$, using permutation arguments together with a nonparametric version of the statistics of Lawley (1938) and Hotelling (1951), and using within-replication ranks in order to avoid the assumption that replication effects are additive. The small-sample distribution of Gerig's statistic can be found exactly using conditional permutation arguments, and its limiting distribution as $N \to \infty$ is chi-square (χ^2) having $m(k-1)$ degrees of freedom; further details are sketched in Section 2.4.

Under translation alternatives the hypothesis H_0 can be written as $H_0\colon \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_k$, where $\boldsymbol{\mu}_j \in \mathbb{R}^m$, $1 \leqq j \leqq k$. Several parametric tests for the latter are available under Gaussian theory, including those based on the Lawley-Hotelling statistic whose limiting ($N \to \infty$) distribution is χ^2 having $m(k-1)$ degrees of freedom.

Conclusions more specific than those supported by tests for the single hypothesis H_0 frequently are required. Here it is informative to examine the effects of the k treatments separately for each of the m responses or for natural subsets

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of them, either in terms of marginal distributions of various orders associated with $\{F_{ij}(\cdot)\}$ or, under shift alternatives, in terms of the location parameters themselves. We thus partition $\mathbf{z} = [\mathbf{z}_1', \cdots, \mathbf{z}_r']'$ and $\boldsymbol{\mu}_j = [\boldsymbol{\mu}_{j1}', \cdots, \boldsymbol{\mu}_{jr}']'$; we let $\{F_{ij}^1(\mathbf{z}_1), \cdots, F_{ij}^r(\mathbf{z}_r)\}$ be the corresponding marginal cdf's; and we consider tests for the r hypotheses

$$H_{0t}: F_{i1}^t \equiv F_{i2}^t \equiv \cdots \equiv F_{ik}^t \equiv F_i^t, \qquad 1 \leqq i \leqq N \tag{1.1}$$

simultaneously against the translation alternatives

$$H_{At}: F_{ij}^t(\mathbf{z}_t) = F_i^t(\mathbf{z}_t - \boldsymbol{\mu}_{jt}), \qquad 1 \leqq j \leqq k,\ 1 \leqq i \leqq N \tag{1.2}$$

where t ranges from 1 to r. In view of (1.2), the hypotheses (1.1) can be written equivalently as

$$H_{0t}: \boldsymbol{\mu}_{1t} = \boldsymbol{\mu}_{2t} = \cdots = \boldsymbol{\mu}_{kt}, \qquad 1 \leqq t \leqq r. \tag{1.3}$$

Although a normal-theory procedure is known (cf. Jensen (1970)) for testing $\{H_{01}, \cdots, H_{0r}\}$ using the Lawley-Hotelling statistics, we now abandon the Gaussian requirement and study a nonparametric version based on an extension of Gerig's (1969) work as summarized in Section 2.4. Specifically, we apply Gerig's permutation test to each hypothesis, the properties of this procedure thus depending on the joint distribution of several statistics of Gerig's type based on rankings within subsets of the m responses. A permutation argument in Section 3 yields the exact joint distribution of the several statistics in small samples, and their limiting joint permutation distribution is shown in Section 4 to belong to a known class of multivariate χ^2 distributions (cf. Jensen (1970)) which also includes the large-sample joint distribution of the Lawley-Hotelling statistics themselves. In Section 5 we investigate the rate of convergence of the joint distribution to its limiting form in the spirit of Berry (1941) and Esséen (1945) for the special case that $F_{ij} \equiv F_j$, $1 \leqq i \leqq N$.

2. Preliminaries. We adopt the following.

2.1. *Notation.* Here $\mathbf{x}$ is a column vector, $\mathbf{A}$ a matrix, $\mathbf{A}'$ its transpose and, where appropriate, $\mathbf{A}^{-1}$ is its inverse; dimensions sometimes are given parenthetically. Special arrays are $\mathbf{I}_n$, the identity matrix of order n; the direct product $\mathbf{A} \times \mathbf{B} = [a_{ij}\mathbf{B}]$; the diagonal matrix $\mathrm{Diag}(a_1, \cdots, a_n)$; and the vector $\mathbf{1}_n$, of order $(n \times 1)$, containing unit elements. The Kronecker indicator is δ_{ij}. Euclidean m-dimensional space is designated by $\mathbb{R}^m$ and its positive orthant by $\mathbb{R}_+^m$. Other separable metric spaces are $\mathscr{M}_{q\times m}$, the space of real matrices of order $(q \times m)$ having maximal rank; $\mathscr{M}_m^0$, the space of positive semi-definite real symmetric matrices of order $(m \times m)$; and $\mathscr{M}_m^+ \subset \mathscr{M}_m^0$, the subspace of definite matrices. Finally, $\mathscr{F}(\cdot)$ denotes the σ-field generated by one of $\mathbb{R}^m$, $\mathscr{M}_{q\times m}$ and $\mathscr{M}_m^+$, or spaces constructed from these. For example, $\mathscr{F}(\mathscr{M}_{q\times m})$ is the σ-field generated by a basis for the topology of $\mathscr{M}_{q\times m}$.

2.2. *A result on weak convergence.* Let $\mathscr{A}$, $\mathscr{A}_X$ and $\mathscr{A}_Y$ be separable metric spaces and $\{X, X_N; N = 1, 2, \cdots\}$ a stochastic sequence in $\mathscr{A}_X$ having the probability measures $\{P(\cdot), P_N(\cdot); N = 1, 2, \cdots\}$. If X_N converges in probability to X, we write $X_N \to_P X$. If X_N converges in distribution to X, i.e. if $\lim_{N\to\infty} P_N(\cdot) = P(\cdot)$ at every continuity set of the latter, we write $\mathscr{L}_\infty(X_N) = \mathscr{L}(X)$. Upon combining Theorem 4.4 and Theorem 5.1 of Billingsley (1968), we have

LEMMA 1. *Let $\{(X, c), (X_N, Y_N); N = 1, 2, \cdots\}$ be a stochastic sequence in $\mathscr{A}_X \times \mathscr{A}_Y$, and $g : \mathscr{A}_X \times \mathscr{A}_Y \to \mathscr{A}$ a continuous mapping, such that*

(i) $\mathscr{L}_\infty(X_N) = \mathscr{L}(X)$

(ii) $Y_N \to_P c$, *a point in* $\mathscr{A}_Y$.

Then $\mathscr{L}_\infty[g(X_N, Y_N)] \to \mathscr{L}[g(X, c)]$.

2.3. *A multivariate χ^2 distribution.* Let $L_h(\mathbf{x}; \boldsymbol{\alpha})$ be the Laguerre polynomial on $\mathbb{R}^r$, of total order $h = h_1 + \cdots + h_r$, defined with respect to the weight function

$$\phi(\mathbf{x}; \boldsymbol{\alpha}) = \prod_{j=1}^r x_j^{\alpha_j - 1} e^{-x_j}/\Gamma(\alpha_j)\,; \qquad 0 < x_j, \alpha_j < \infty$$

and satisfying an r-dimensional Rodrigues' formula (cf. Jensen (1970)). Upon defining

$$f_h(2\mathbf{x}; 2\boldsymbol{\alpha}) = \prod_{j=1}^r \frac{h_j!\,\Gamma(\alpha_j)}{\Gamma(\alpha_j + h_j)2^r}\,\phi(\mathbf{x}; \boldsymbol{\alpha})L_h(\mathbf{x}; \boldsymbol{\alpha})\,,$$

making a simple change of scale in Theorem 2 of Jensen (1970), and abbreviating probability density function as pdf, we have

LEMMA 2. *Let $\mathbf{W} = [\mathbf{W}_{ij}]$ be a partitioned Wishart matrix having τ degrees of freedom and the matrix $\boldsymbol{\Sigma} = [\boldsymbol{\Sigma}_{ij}]$ of scale parameters such that $\mathbf{W}_{ij}$ and $\boldsymbol{\Sigma}_{ij}$ are of order $(m_i \times m_j)$, $i, j = 1, 2, \cdots, r$, and $m_1 + \cdots + m_r = m$. Define $Q_j = \operatorname{tr} \mathbf{W}_{jj}\boldsymbol{\Sigma}_{jj}^{-1}$; then the joint* pdf *of $\{Q_1, \cdots, Q_r\}$ admits the series expansion*

$$\phi(Q_1, \cdots, Q_r; \boldsymbol{\nu}) = \sum_{n=0}^\infty \frac{\Gamma(\frac{1}{2}\tau + n)}{n!\,\Gamma(\frac{1}{2}\tau)} \sum_{a_1=0}^n \cdots \sum_{a_r=0}^n A(\mathbf{a}) f_a(\mathbf{Q}; \boldsymbol{\nu}) \tag{2.1}$$

where $\mathbf{Q} = [Q_1, \cdots, Q_r]'$; $\boldsymbol{\nu} = [\nu_1, \cdots, \nu_r]'$, $\nu_j = \tau m_j$, $1 \leqq j \leqq r$; and the coefficients $A(\mathbf{a}) = A(a_1, \cdots, a_r)$ depend explicitly on the matrix $\boldsymbol{\Sigma}$ through $\boldsymbol{\Sigma}_{ii}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-\frac{1}{2}}$, $1 \leqq i, j \leqq r$.

2.4. *A resumé of Gerig's work.* Following Gerig (1969) and otherwise retaining the notation of Section 1, let R_{ij}^s designate the rank of Y_{ij}^s among the k scalars $\{Y_{i1}^s, \cdots, Y_{ik}^s\}$. These ranks are arrayed as rows of the matrix

$$\mathbf{R}_i = \begin{bmatrix} R_{i1}^1 & \cdots & R_{ik}^1 \\ \vdots & & \vdots \\ R_{i1}^m & \cdots & R_{ik}^m \end{bmatrix}, \qquad 1 \leqq i \leqq N. \tag{2.2}$$

Define

$$T_{Nj}^s = \sum_{i=1}^N R_{ij}^s/N \tag{2.3}$$

$$\mathbf{V}_N = [\sum_{i=1}^N (\sum_{j=1}^k R_{ij}^s R_{ij}^{s'} - k(k+1)^2/4)/N(k-1)] \tag{2.4}$$

and

$$\mathbf{W}_N = [\sum_{j=1}^{k} (T_{Nj}^{s} - \tfrac{1}{2}(k+1))(T_{Nj}^{s'} - \tfrac{1}{2}(k+1))] . \tag{2.5}$$

Gerig's statistic (3.2) for testing the hypothesis H_0 now can be written in the Lawley–Hotelling form

$$X_N^2 = N \operatorname{tr} \mathbf{W}_N \mathbf{V}_N^{-1} . \tag{2.6}$$

Let $\mathbf{R}_i^*$ be the matrix obtained from $\mathbf{R}_i$ upon permuting its columns so that the integers $\{1, 2, \cdots, k\}$ appear in sequence in the first row. Gerig's permutation test for H_0 against the translation alternatives H_A rests on the uniformity of the distribution of $\mathbf{R}_i$ over the $k!$ elements of the set $S(\mathbf{R}_i^*)$ of matrices permutationally equivalent to $\mathbf{R}_i^*$, conditionally on $\mathbf{R}_i^*$ and H_0, where two matrices are permutationally equivalent if the first can be obtained through a finite number of permutations of the columns of the second. Gerig's test procedure thus is specified completely by the conditional probability law

$$\mathscr{P}_N = P\{\mathbf{R}_i = \mathbf{R}_{0i}, 1 \leqq i \leqq N \mid S(\mathbf{R}_i^*), 1 \leqq i \leqq N; H_0\} = \left(\frac{1}{k!}\right)^N \tag{2.7}$$

where $\mathbf{R}_{0i} \in S(\mathbf{R}_i^*)$. Because the statistic X_N^2 at (2.6) depends explicitly on $\{\mathbf{R}_1, \cdots, \mathbf{R}_N\}$, the exact permutation distribution of X_N^2 can be found, thereby determining a randomized test of the hypothesis H_0 which is strictly distribution-free under H_0 and which has exact size α.

In large samples, however, the computations become prohibitive. The final result of this section, due to Gerig (1969), provides the key to the asymptotic permutation distribution not only of X_N^2, as shown by Gerig, but of the joint distribution of the several statistics introduced in Section 3, details of which are supplied in Section 4.

Let $\mathbf{\Sigma}_i^* = [\sigma_{iss'}^*]$, $1 \leqq i \leqq N$, where, for $s = s'$,

$$\sigma_{iss}^* = k(k+1)/12 \tag{2.8a}$$

and, for $s \neq s'$,

$$\sigma_{iss'}^* = \sum_g \sum_h \sum_j P\{Y_{ij}^s > Y_{ig}^s, Y_{ij}^{s'} > Y_{ih}^{s'}\}/(k-1) - k(k-1)/4 , \tag{2.8b}$$

the sums all ranging from 1 to k, and define

$$\mathbf{\Sigma}_N = N^{-1} \sum_{i=1}^{N} \mathbf{\Sigma}_i^* . \tag{2.9}$$

Clearly, $\mathbf{\Sigma}_N$ depends on the underlying distributions $\{F_{ij}(\cdot); 1 \leqq j \leqq k, 1 \leqq i \leqq N\}$ and, when H_0 is true, on $\{F_i(\cdot); 1 \leqq i \leqq N\}$. In the latter case we employ the following

DEFINITION 1. Let $\mathscr{F}_0$ be the class of sequences $\{F_N\}$ of cdf's on $\mathbb{R}^m$ such that (i) $F_1, F_2, \cdots$ are continuous, and (ii) $\lim_{N\to\infty} \mathbf{\Sigma}_N = \mathbf{\Sigma}$, where (iii) $\mathbf{\Sigma}$ is positive definite.

The following result appeared as Theorem 4.3 of Gerig (1969).

LEMMA 3 (Gerig). *Let $\{F_N\} \in \mathscr{F}_0$; then as $N \to \infty$ the limiting joint permutation distribution of*

$$\{N^{\frac{1}{2}}(T^s_{Nj} - \tfrac{1}{2}(k+1));\ 1 \leqq s \leqq m,\ 1 \leqq j \leqq k\}$$

is mk-variate Gaussian having zero means and the dispersion matrix $\mathbf{\Omega} = \mathbf{\Delta} \times \mathbf{\Sigma}$, where $\mathbf{\Delta} = [(\delta_{jj'} - 1/k);\ 1 \leqq j, j' \leqq k]$.

3. A simultaneous test procedure. We outline a nonparametric procedure for testing the r hypotheses (1.1) simultaneously against the translation alternatives (1.2), adopting the type 1 probability error rate (cf. Miller (1966)) as suitable for this family of hypotheses. Our work thus extends that of Gerig (1969) and Friedman (1937) to the case of several hypotheses. The key to these extensions lies in three facts:

(i) The ranking among treatments within replications is carried out separately for each of the m responses.

(ii) The permutation argument outlined in Section 2.4 does not depend on the number of responses.

(iii) It is clear from Gerig's work that the number of test functions defined in terms of the matrices $\{\mathbf{R}_1, \cdots, \mathbf{R}_N\}$ need not be restricted to one.

Accordingly, we now partition the m responses into $r \leqq m$ subsets such that $\mathbf{Y}'_{ij} = [\mathbf{Y}^{1\prime}_{ij}, \cdots, \mathbf{Y}^{r\prime}_{ij}]$, thus inducing the block partitions

$$\mathbf{R}_i' = [\mathbf{R}_i^{1\prime}, \cdots, \mathbf{R}_i^{r\prime}]$$

$$\mathbf{V}_N = [\mathbf{V}_{Ntt'}]$$

and

$$\mathbf{W}_N = [\mathbf{W}_{Ntt'}]$$

where $1 \leqq t,\ t' \leqq r$. Corresponding to (2.6), we now construct the r statistics

$$X^2_{Nt} = N \operatorname{tr} \mathbf{W}_{Ntt} \mathbf{V}^{-1}_{Ntt}\,; \qquad 1 \leqq t \leqq r \tag{3.1}$$

of Gerig's type for testing the respective hypotheses H_{0t} at (1.1), each statistic defined in terms of one of the several subsets of the responses (compare equations (2.2)—(2.6)).

It follows directly from the arguments in Section 2.4 that our simultaneous test procedure is determined completely by the conditional probability law (2.7). Because the scalars $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ are defined explicitly in terms of $\{\mathbf{R}_1, \cdots, \mathbf{R}_N\}$, their exact joint permutation distribution can be found, thus determining a randomized simultaneous test procedure which is strictly distribution-free under the hypothesis $H_0 = \bigcap_{t=1}^r H_{0t'}$ and having a type 1 probability error rate of exact size α.

In samples of moderate size, however, the exact permutation test becomes unwieldy. In the following section we investigate a large-sample approximation to the joint permutation distribution of the statistics $\{X^2_{N1}, \cdots, X^2_{Nr}\}$.

4. The limiting joint distribution of $\{X^2_{N1}, \cdots, X^2_{Nr}\}$. We demonstrate that:

(i) $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ are definite quadratic forms in the variables $\{N^{\frac{1}{2}}(T^s_{Nj} - \frac{1}{2}(k+1)); 1 \leqq s \leqq m, 1 \leqq j \leqq k\}$;

(ii) The limiting joint permutation distribution of the latter is Gaussian by Lemma 3;

(iii) Lemma 1 applies; and thus

(iv) The limiting joint permutation distribution of $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ belongs to the class of multivariate χ^2 distributions given in Lemma 2, a typical cdf of which is designated by $\Psi(\cdot, \cdots, \cdot; \boldsymbol{\nu})$.

Upon identifying the dimensions of $\mathbf{Y}^t_{ij}$ as $(m_t \times 1)$ such that $m_1 + \cdots + m_r = m$, we state the main result of this section as

THEOREM 1. *Let* $\{F_N\} \in \mathscr{F}_0$ *and, for each* $N = 1, 2, \cdots$, *let* $G_N(\cdot, \cdots, \cdot)$ *be the joint* cdf *of the statistics* $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ *defined at* (3.1). *The limiting joint permutation distribution of* $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ *is given by*

$$\lim_{N\to\infty} G_N(\cdot, \cdots, \cdot) = \Psi(\cdot, \cdots, \cdot; \boldsymbol{\nu}) \tag{4.1}$$

where $\boldsymbol{\nu} = [\nu_1, \cdots, \nu_r]'$ *and* $\nu_t = m_t(k-1)$, $1 \leqq t \leqq r$.

PROOF. The limiting Gaussian distribution of the variables $\{N^{\frac{1}{2}}(T^s_{Nj} - \frac{1}{2}(k+1)); 1 \leqq s \leqq m, 1 \leqq j \leqq k\}$ in Lemma 3 clearly is singular of rank $m(k-1)$. Upon applying Lemma 1 we infer that the limiting distribution of $N\mathbf{W}_N$ (compare (2.5)) is Wishart having $k-1$ degrees of freedom and the matrix $\boldsymbol{\Sigma}$ of scale parameters. Gerig (1969) demonstrated under the conditions of the theorem that $\mathbf{V}_N \to_P \boldsymbol{\Sigma}$. Applying Lemma 1 once more using the continuity of the function tr AB^{-1} together with the indicated mapping $g: \mathscr{M}_m^+ \times \mathscr{M}_m^+ \to \mathbb{R}_+^r$, we conclude that

$$\lim_{N\to\infty} G_N(z_1, \cdots, z_r) = P\{\operatorname{tr} \mathbf{W}_{ii}\boldsymbol{\Sigma}_{ii}^{-1} \leqq z_i; 1 \leqq i \leqq r\} \tag{4.2}$$

where $\mathbf{W} = [\mathbf{W}_{tt'}]$ is Wishart having the parameters $k-1$ and $\boldsymbol{\Sigma} = [\boldsymbol{\Sigma}_{tt'}]$. But the probability on the right of (4.2) is precisely the cdf corresponding to the series (2.1), and the proof is complete.

COROLLARY 1. *Let* $\{X^2_{N1}, \cdots, X^2_{Nm}\}$ *be Friedman's statistics for testing the hypotheses* (1.1) *in the case of m one-dimensional marginal distributions. Then the joint permutation distribution of* $\{X^2_{N1}, \cdots, X^2_{Nm}\}$ *has the limiting form*

$$\lim_{N\to\infty} G_N(\cdot, \cdots, \cdot) = \Psi(\cdot, \cdots, \cdot; \boldsymbol{\nu}) \tag{4.3}$$

where $\nu_t = k-1$, $1 \leqq t \leqq m$.

5. Bounds on the rate of convergence. The weak convergence of $\{X^2_{N1}, \cdots, X^2_{Nr}\}$ follows from Lemma 1 and multidimensional central limit theory requiring only that second-order moments exist. The higher moments required for establishing rates of convergence are assured in the present study as long as k is finite and thus the within-replication ranks remain bounded. Our intention here is to develop uniform bounds of the Berry–Esséen type on the difference

$$|G_N(\cdot, \cdots, \cdot) - \Psi(\cdot, \cdots, \cdot; \boldsymbol{\nu})|.$$

The presentation is simplified considerably upon accepting some loss of generality; we henceforth assume that distributions are identical over the N replications, i.e. $F_{ij}(\cdot) \equiv F_j(\cdot)$ for all $i = 1, 2, \cdots, N$ and $1 \leqq j \leqq k$.

5.1. *Some results on convexity.* We employ the following notions of convexity. A real-valued function $f(\cdot)$ is said to be convex on C (a convex set in $\mathbb{R}^n$) if, for each pair of points $\mathbf{x}, \mathbf{y} \in C$ and each $\alpha = 1 - \bar{\alpha} \in [0, 1]$, we have $f(\alpha\mathbf{x} + \bar{\alpha}\mathbf{y}) \leqq \alpha f(\mathbf{x}) + \bar{\alpha} f(\mathbf{y})$. If f is convex, the set $S(z) = \{\mathbf{x} \mid f(\mathbf{x}) \leqq z\}$ is convex in $\mathbb{R}^n$ for each $z \in R^1$ (cf. Berge (1963), Chapter 8). For later reference we introduce the

DEFINITION 2. Let $\mathscr{X}_\Gamma$ and $\mathscr{X}_0$ be the natural product spaces $\mathscr{X}_\Gamma = \mathscr{M}_{q\times m} \times (\mathscr{M}_m^+ - \mathbf{\Gamma})$ and $\mathscr{X}_0 = \mathscr{M}_{q\times m} \times \mathscr{M}_m^+$, where $\mathbf{\Gamma} \in \mathscr{M}_m^+$.

LEMMA 4. *Let* $\mathbf{X} \in \mathscr{M}_{q\times m}$; $\mathbf{M} \in \mathscr{M}_m^+ - \mathbf{\Gamma}$; $\mathbf{A}, \mathbf{C} \in \mathscr{M}_{s\times m}$; *and* $\mathbf{B} \in \mathscr{M}_q^0$, *where* $s \leqq m$. *Then the function*

$$g(\mathbf{X}, \mathbf{M}) = \operatorname{tr} \mathbf{C}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C}'[\mathbf{A}(\mathbf{M} + \mathbf{\Gamma})A']^{-1}$$

is convex on $\mathscr{X}_\Gamma$.

PROOF. As the problem remains invariant under suitable matrix operations, it suffices to demonstrate that $g(\mathbf{L}, \mathbf{V}) = \operatorname{tr} \mathbf{L}(\mathbf{V} + \mathbf{I})^{-1}\mathbf{L}'$ is convex. Let

$$h(\alpha) = g(\alpha\mathbf{L}_1 + \bar{\alpha}\mathbf{L}_2, \alpha\mathbf{V}_1 + \bar{\alpha}\mathbf{V}_2)\,;$$

upon differentiating once we find

$$h'(\alpha) = 2 \operatorname{tr} (\mathbf{L}_1 - \mathbf{L}_2)\mathbf{Z}^{-1}\mathbf{Y}' - \operatorname{tr} \mathbf{Y}\mathbf{Z}^{-1}(\mathbf{V}_1 - \mathbf{V}_2 + \mathbf{I})^{-1}\mathbf{Z}^{-1}\mathbf{Y}'$$

where

$$\mathbf{Z} = \alpha\mathbf{V}_1 + \bar{\alpha}\mathbf{V}_2 + \mathbf{I}\,, \qquad \mathbf{Y} = \alpha\mathbf{L}_1 + \bar{\alpha}\mathbf{L}_2\,.$$

Differentiating once more, we reduce the resulting expression to

$$h''(\alpha) = 2 \operatorname{tr} (\mathbf{T} - \mathbf{U})(\mathbf{T} - \mathbf{U})'$$

where

$$\mathbf{T} = (\mathbf{L}_1 - \mathbf{L}_2)\mathbf{Z}^{-\frac{1}{2}}\,, \qquad \mathbf{U} = \mathbf{Y}\mathbf{Z}^{-1}(\mathbf{V}_1 - \mathbf{V}_2 + \mathbf{I})^{-1}\mathbf{Z}^{-\frac{1}{2}}\,.$$

Because $h''(\alpha) \geqq 0$, $h(\alpha)$ is convex on $[0, 1]$ and thus $g(\mathbf{L}, \mathbf{V})$ is convex by Theorem 2, page 190, of Berge (1963).

COROLLARY 2. *The set*

$$S(z) = \{\mathbf{X}, \mathbf{M} \mid \operatorname{tr} \mathbf{X}'\mathbf{B}\mathbf{X}(\mathbf{M} + \mathbf{\Gamma})^{-1} \leqq z\}$$

is convex in $\mathscr{X}_\Gamma$ *for each* $z \in \mathbb{R}_+^1$.

COROLLARY 3. *Let* $\mathbf{X} \in \mathscr{M}_{q\times m}$ *and* $\mathbf{M} \in \mathscr{M}_m^+ - \mathbf{\Gamma}$. *Partition* $\mathbf{X} = [\mathbf{X}_1, \cdots, \mathbf{X}_r]$, $\mathbf{M} = [\mathbf{M}_{tt'}]$ *and* $\mathbf{\Gamma} = [\mathbf{\Gamma}_{tt'}]$ *such that* $\mathbf{\Gamma}_{tt} \in \mathscr{M}_{m_t}^+$, $\mathbf{X}_t \in \mathscr{M}_{q\times m_t}$ *and* $\mathbf{M}_{tt} \in \mathscr{M}_{m_t}^+ - \mathbf{\Gamma}_{tt}$, *where* $m_1 + \cdots + m_r = m$. *Define*

$$g_t^*(\mathbf{X}, \mathbf{M}) = \operatorname{tr} \mathbf{X}_t'\mathbf{B}\mathbf{X}_t(\mathbf{M}_{tt} + \mathbf{\Gamma}_{tt})^{-1}\,, \qquad 1 \leqq t \leqq r\,. \tag{5.1}$$

Then the set

$$S(z_1, \cdots, z_r) = \{\mathbf{X}, \mathbf{M} \mid g_1^*(\mathbf{X}, \mathbf{M}) \leqq z_1, \cdots, g_r^*(\mathbf{X}, \mathbf{M}) \leqq z_r\}$$

is convex in $\mathscr{Z}_\Gamma^r$ *for each* $z \in \mathbb{R}_+^r$.

PROOF. Let $\{S_1, \cdots, S_r\}$ be cylinder sets in $\mathscr{Z}_\Gamma^r$ defined as

$$S_t(z_t) = \{\mathbf{X}, \mathbf{M} \mid \operatorname{tr} \mathbf{X}_t'\mathbf{B}\mathbf{X}_t(\mathbf{M}_{tt} + \mathbf{\Gamma}_{tt})^{-1} \leqq z_t\}, \qquad 1 \leqq t \leqq r.$$

Each set $S_t(z_t)$ clearly is convex in cross section; applying Lemma 4 with $s = m_t$ and $\mathbf{C}$ and $\mathbf{A}$ both of the form $[\mathbf{0}, \mathbf{I}_{m_t}, \mathbf{0}]$, we find that $g_t^*(\mathbf{X}, \mathbf{M})$ is convex. The set $S(z_1, \cdots, z_r)$ is the intersection

$$S(z_1, \cdots, z_r) = \bigcap_{t=1}^r S_t(z_t)$$

of convex bodies in $\mathscr{Z}_\Gamma^r$ and thus is convex.

5.2. *Multidimensional Berry–Esséen bounds.* Rates of convergence in multidimensional central limit theory have been studied by a number of investigators for sequences of independent and identically distributed (i.i.d.) random variables in $\mathbb{R}^s$; the following result is in a form due to Sazonov (1968), where $\mathscr{C}^s$ designates the class of all measurable convex sets in $\mathbb{R}^s$.

LEMMA 5. *Let* $\{\mathbf{y}_1, \cdots, \mathbf{y}_N\}$ *be an* i.i.d. *sequence in* $\mathbb{R}^s$ *having zero means, the nonsingular covariance matrix* $\mathbf{\Omega}$, *and the finite absolute third moments* β_{3j}, $1 \leqq j \leqq s$. *Let* $P_N(\cdot)$ *be the probability measure associated with the standardized sum* $N^{-\frac{1}{2}}(\mathbf{y}_1 + \cdots + \mathbf{y}_N)$, *and let* $P(\cdot)$ *be the limiting Gaussian measure having the parameters* $\mathbf{0}$ *and* $\mathbf{\Omega}$. *Then for each* $N = 1, 2, \cdots$,

$$\sup_{A \in \mathscr{C}^s} |P_N(A) - P(A)| \leqq c(s) \textstyle\sum_{j=1}^s \gamma_{jj}^{\frac{3}{2}} \beta_{3j}/N^{\frac{1}{2}} \tag{5.2}$$

where $\mathbf{\Gamma} = [\gamma_{ij}] = \mathbf{\Omega}^{-1}$ *and* $c(s)$ *is a finite positive constant depending only on* s.

In relation to other work, Lemma 5 also can be obtained upon modifying a proof due to Bergström (1969), who assumed the existence of third-order absolute moments, or upon specializing some findings of Bhattacharya (1968), who assumed the existence of absolute moments of order $3 + \delta$ for some positive δ. Furthermore, the work of Bergström assures that the constant $c(s)$ in (5.2) can be replaced by $c_0 \cdot s^3/d$, where c_0 is a finite positive constant not depending on any of the remaining parameters, and $d^2 = \lambda_1/\lambda_s$ in terms of the characteristic values $\{0 < \lambda_1 \leqq \cdots \leqq \lambda_s\}$ of $\mathbf{\Omega}^{-1}$.

5.3. *Bounds on* $|G_N(\cdot, \cdots, \cdot) - \Psi(\cdot, \cdots, \cdot; \boldsymbol{\nu})|$. The assumption that $F_{ij}(\cdot) \equiv F_j(\cdot)$, for each $i = 1, 2, \cdots, N$ and $1 \leqq j \leqq k$, clearly supports the conclusions:

(i) $\mathbf{\Sigma}_N$ at (2.9) is identically equal to $\mathbf{\Sigma}$, $N = 1, 2, \cdots$;

(ii) $\mathbf{V}_N$ at (2.4) converges almost surely (a.s.) to $\mathbf{\Sigma}$. The first conclusion follows from the definitions (2.8) and (2.9), and the second from one of Kolmogorov's strong laws of large numbers.

It is clear from the definition of X_{Nt}^2 at (3.1) that the limiting distribution of $\{X_{N1}^2, \cdots, X_{Nr}^2\}$ is the end result of two limiting processes which complicate our study, namely, the convergence in distribution of $N\mathbf{W}_N$ and the a.s. convergence to $\mathbf{\Sigma}$ of $\mathbf{V}_N$. In order to study these processes simultaneously, we employ a construction using bounds of the Berry–Esséen type for the joint distribution of standardized sums in the variables

$$\{[R^s_{Nj'}, R^s_{Nj}R^{s'}_{Nj}; 1 \leqq s \leqq s' \leqq m, 1 \leqq j' \leqq k-1, 1 \leqq j \leqq k]; N = 1, 2, \cdots\}.$$

Specifically, we consider for $N = 1, 2, \cdots$ the sequence

$$(5.3) \qquad [(R^s_{Nj'} - \tfrac{1}{2}(k+1)), U_{Nss'}; 1 \leqq s \leqq s' \leqq m, 1 \leqq j' \leqq k-1]$$

where

$$U_{iss'} = \textstyle\sum_{j=1}^k (R^s_{ij}R^{s'}_{ij} - k(k+1)^2/4)/(k-1)$$

and the range $1 \leqq j' \leqq k-1$ assures the nonsingularity of the joint distribution (compare Gerig (1969), page 1598). Clearly from (2.4) we have $\mathbf{V}_N = N^{-1}(\mathbf{U}_1 + \cdots + \mathbf{U}_N)$, where $\mathbf{U}_i = [U_{iss'}]$. The variables (5.3) are i.i.d. for $N = 1, 2, \cdots$, and their joint moments of all orders exist by the boundedness of $\{R^s_{ij}; 1 \leqq s \leqq m, 1 \leqq j \leqq k\}$. Details of our construction follow.

Let $\kappa' = m(k-1)$, $\kappa^* = m(m+1)/2$, and $\kappa = \kappa' + \kappa^*$. In the notation of Section 5.1 we now identity $q = k-1$ and $\mathbf{\Gamma} = N^{\frac{1}{2}}\mathbf{\Sigma}$, with the understanding that N is finite throughout. Recalling the definitions (2.2)—(2.4), let

$$(5.4) \qquad \mathbf{T}_N' = \begin{bmatrix} (T^1_{N1} - \frac{1}{2}(k+1)) & \cdots & (T^1_{N,k-1} - \frac{1}{2}(k+1)) \\ \vdots & & \vdots \\ (T^m_{N1} - \frac{1}{2}(k+1)) & & (T^m_{N,k-1} - \frac{1}{2}(k+1)) \end{bmatrix}$$

and define

$$(5.5) \qquad \mathbf{Z}_N^* = \{N^{\frac{1}{2}}\mathbf{T}_N, N^{\frac{1}{2}}(\mathbf{V}_N - \mathbf{\Sigma})\}$$

and

$$(5.6) \qquad \mathbf{Z}_N = \{N^{\frac{1}{2}}\mathbf{T}_N, N^{\frac{1}{2}}\mathbf{V}_N\}.$$

From the fact that $\sum_{j=1}^k T^s_{Nj} = k(k+1)/2$ for $1 \leqq s \leqq m$ (cf. Gerig (1969), page 1601), it follows from elementary operations that $\mathbf{W}_N$ at (2.5) can be written as

$$(5.7) \qquad \mathbf{W}_N = \mathbf{T}_N'(\mathbf{I}_{k-1} + \mathbf{1}_{k-1}\mathbf{1}'_{k-1})\mathbf{T}_N.$$

In the notation of Corollary 3 and (5.11), define

$$(5.8) \qquad S_N^*(\mathbf{z}) = \{(\mathbf{X}, \mathbf{M}) \in \mathscr{X}_\Gamma \mid g_t^*(\mathbf{X}, \mathbf{M}) \leqq N^{-\frac{1}{2}}z_t; 1 \leqq t \leqq r\}$$

$$(5.9) \qquad S_N(\mathbf{z}) = \{(\mathbf{X}, \mathbf{S}) \in \mathscr{X}_0 \mid g_t(\mathbf{X}, \mathbf{S}) \leqq N^{-\frac{1}{2}}z_t; 1 \leqq t \leqq r\}$$

and

$$(5.10) \qquad A(\mathbf{z}) = \{\mathbf{X} \in \mathscr{M}_{q\times m} \mid g_t(\mathbf{X}, \mathbf{\Sigma}) \leqq z_t; 1 \leqq t \leqq r\}$$

where $\mathbf{S} = [\mathbf{S}_{tt'}] \in \mathscr{M}_m^+$ and

$$(5.11) \qquad g_t(\mathbf{X}, \mathbf{S}) = \operatorname{tr} \mathbf{X}_t'\mathbf{B}\mathbf{X}_t\mathbf{S}_{tt}^{-1}, \qquad 1 \leqq t \leqq r.$$

The sets $S_N^*(\mathbf{z})$ and $S_N(\mathbf{z})$ can be interpreted readily. Observe that $\mathbf{Z}_N^* \in \mathscr{X}_\Gamma$

and $\mathbf{Z}_N \in \mathscr{Z}_0$, and let $P_N^*(\cdot)$ and $P_N(\cdot)$ be the corresponding probability measures defined on $\mathscr{F}(\mathscr{Z}_0)$ (note that $\mathscr{F}(\mathscr{Z}_\Gamma) \equiv \mathscr{F}(\mathscr{Z}_0)$). Evidently P_N^* and P_N are identical apart from shift; more precisely, we have

$$P_N^*(S_N^*(\mathbf{z})) = P_N(S_N(\mathbf{z})) \,. \tag{5.12}$$

Moreover, upon making the identification $\mathbf{X} = N^{\frac{1}{2}} T_N$, $\mathbf{B} = (\mathbf{I}_{k-1} + \mathbf{1}_{k-1}\mathbf{1}'_{k-1})$, and $\mathbf{S} = N^{\frac{1}{2}}\mathbf{V}_N$ in (5.9) and (5.11), together with (5.7) and (3.1), we conclude that

$$\begin{aligned} P_N(S_N(\mathbf{z})) &= P_N\{N \operatorname{tr} \mathbf{W}_{Ntt}(N^{\frac{1}{2}}\mathbf{V}_{Ntt})^{-1} \leqq N^{-\frac{1}{2}} z_t,\ 1 \leqq t \leqq r\} \\ &= P_N\{X_{N1}^2 \leqq z_1, \cdots, X_{Nr}^2 \leqq z_r\} \\ &= G_N(z_1, \cdots, z_r) \end{aligned} \tag{5.13}$$

where $G_N(\cdot, \cdots, \cdot)$ is the joint cdf of $\{X_{N1}^2, \cdots, X_{Nr}^2\}$.

We finally consider Gaussian measures on $\mathscr{F}(\mathscr{Z}_0)$. So that the parameters can be identified unambiguously, we introduce the transformation $H: \mathscr{Z}_\Gamma \to \mathbb{R}^\kappa$ such that

$$H: \{\mathbf{T}_N \to \mathbf{t}_N, (\mathbf{V}_N - \mathbf{\Sigma}) \to (\mathbf{v}_N - \boldsymbol{\sigma})\} \tag{5.14}$$

where $\mathbf{t}_N' = [\mathbf{t}'_{N1}, \cdots, \mathbf{t}'_{N,k-1}]$ is defined in terms of the columns of $\mathbf{T}_N' = [\mathbf{t}_{N1}, \cdots, \mathbf{t}_{N,k-1}]$ at (5.4), and $\mathbf{V}_N \to \mathbf{v}_N$ (similarly $\mathbf{\Sigma} \to \boldsymbol{\sigma}$) column-wise using the upper triangular part of $\mathbf{V}_N$. Upon defining

$$\mathbf{z}_N' = N^{\frac{1}{2}}[\mathbf{t}_N', (\mathbf{v}_N - \boldsymbol{\sigma})'] \tag{5.15}$$

we conclude, for each $N = 1, 2, \cdots$, that the distribution of $\mathbf{z}_N$ is nonsingular with zero means and covariance matrix

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} \\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22} \end{bmatrix} \tag{5.16}$$

where $\mathbf{\Omega}_{11} = \mathbf{\Delta}^* \times \mathbf{\Sigma}$, for example (compare Gerig (1969), page 1598), and $\mathbf{\Delta}^* = [(\delta_{jj'} - 1/k);\ 1 \leqq j, j' \leqq k-1]$. With these conventions in hand let $Q_\kappa^*(\cdot)$ be the Gaussian measure on $\mathscr{F}(\mathscr{Z}_0)$ having zero means and covariance matrix $\mathbf{\Omega}$, and let $Q_{\kappa'}^*(\cdot)$ be the marginal measure of the first κ' components defined on $\mathscr{F}(\mathscr{M}_{q\times m})$, having zero means and covariance matrix $\mathbf{\Omega}_{11}$. Upon identifying $\mathbf{X} = N^{\frac{1}{2}}\mathbf{T}_N$ in (5.10), then using (5.7) and Lemma 2, we conclude that

$$Q_{\kappa'}^*(A(\mathbf{z})) = \Psi(z_1, \cdots, z_r; \boldsymbol{\nu}) \tag{5.17}$$

where $\nu_j = m_j(k-1)$, $1 \leqq j \leqq r$.

Finally let $H_N(z_1, \cdots, z_r)$ be the point function

$$H_N(z_1, \cdots, z_r) = Q_\kappa^*(S_N^*(z_1, \cdots, z_r)) \tag{5.18}$$

which also can be written as a definite integral using the appropriate density function. Our main result here is

THEOREM 2. *Let the array*

$$\mathbf{R}_i^0 = [(R_{ij'}^s - \tfrac{1}{2}(k+1)), U_{iss'};\ 1 \leqq s \leqq s' \leqq m,\ 1 \leqq j' \leqq k-1]'\,; \qquad i = 1, 2, \cdots$$

as in (5.3) *be an* i.i.d. *sequence in* $\mathbb{R}^\kappa$ *whose third-order absolute moments are* β_{3h}, $1 \leqq h \leqq \kappa$. *Define* $\{X_{Nt}^2; 1 \leqq t \leqq r\}$ *in terms of* $\{\mathbf{R}_i^0; 1 \leqq i \leqq N\}$ *as in* (2.3)—(2.5) *and* (3.1). *Then for each* $\mathbf{z} \in \mathbb{R}_+^r$ *and for each* $N = 1, 2, \cdots$,

$$|G_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})| \leqq c(\kappa) \textstyle\sum_{h=1}^{\kappa} \gamma_{hh}^{\frac{3}{2}} \beta_{3h}/N^{\frac{1}{2}} + |H_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})|$$

where $\Omega^{-1} = [\gamma_{ij}]$; Ω *is defined at* (5.16) *and* $H_N(\mathbf{z})$ *at* (5.18); $G_N(\mathbf{z})$ *is the joint* cdf *of* $\{X_{N1}^2, \cdots, X_{Nr}^2\}$ *and* $\Psi(\mathbf{z}; \boldsymbol{\nu})$ *its limiting form*; $\nu_t = m_t(k-1)$; *and* $c(\kappa)$ *is a finite positive constant depending only on* κ.

Proof. Applying (5.13), (5.17) and (5.12) in order, then the triangle inequality, we write

$$\begin{aligned}|G_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})| &= |P_N(S_N(\mathbf{z})) - Q_{\kappa'}^*(A(\mathbf{z}))| \\ &= |P_N^*(S_N^*(\mathbf{z})) - Q_\kappa^*(S_N^*(\mathbf{z})) + Q_\kappa^*(S_N^*(\mathbf{z})) - Q_{\kappa'}^*(A(\mathbf{z}))| \\ &\leqq |P_N^*(S_N^*(\mathbf{z})) - Q_\kappa^*(S_N^*(\mathbf{z}))| + |Q_\kappa^*(S_N^*(\mathbf{z})) - Q_{\kappa'}^*(A(\mathbf{z}))| .\end{aligned}$$

The second term following the inequality is

$$|Q_\kappa^*(S_N^*(\mathbf{z})) - Q_{\kappa'}^*(A(\mathbf{z}))| = |H_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})|$$

from (5.17) and (5.18). Under the conditions of the theorem it follows that $P_N^*(\cdot)$ converges to $Q_\kappa^*(\cdot)$, and the conditions of Lemma 5 otherwise are met. Moreover, the set $S_N^*(\mathbf{z})$ is convex in $\mathscr{X}_\Gamma$ for each $z \in \mathbb{R}_+^r$ from Corollary 3. Now applying Lemma 5, we have

$$\begin{aligned}|P_N^*(S_N^*(\mathbf{z})) - Q_\kappa^*(S_N^*(\mathbf{z}))| &\leqq \sup_{C \in \mathscr{C}^\kappa} |P_N^*(C) - Q_\kappa^*(C)| \\ &\leqq c(\kappa) \textstyle\sum_{h=1}^{\kappa} \gamma_{hh}^{\frac{3}{2}} \beta_{3h}/N^{\frac{1}{2}}\end{aligned}$$

and the proof is complete.

Corollary 4. *Let* $G_N(\cdot, \cdots, \cdot)$ *be the joint* cdf *of Friedman's* χ_r^2 *statistics for testing the hypotheses* (1.1) *in the case of m one-dimensional marginal distributions. Then, for each* $z \in \mathbb{R}_+^m$ *and each* $N = 1, 2, \cdots$,

$$|G_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})| \leqq c(\kappa) \textstyle\sum_{h=1}^{\kappa} \gamma_{hh}^{\frac{3}{2}} \beta_{3h}/N^{\frac{1}{2}} + |H_N(\mathbf{z}) - \Psi(\mathbf{z}; \boldsymbol{\nu})|$$

where $\nu_t = k - 1$, $1 \leqq t \leqq m$.

Corollary 5. *Let* $G_N(\cdot)$ *be the* cdf *of Gerig's statistic* (2.6) *for testing the hypothesis* H_0 *against the translation alternatives* H_A. *Then, for each* $z \in \mathbb{R}_+^1$ *and for each* $N = 1, 2, \cdots$,

$$|G_N(z) - \Psi_\nu(z)| \leqq c(\kappa) \textstyle\sum_{h=1}^{\kappa} \gamma_{hh}^{\frac{3}{2}} \beta_{3h}/N^{\frac{1}{2}} + |H_N(z) - \Psi_\nu(z)|$$

where $\Psi_\nu(\cdot)$ *is the* cdf *of the* χ^2 *distribution having* $\nu = m(k-1)$ *degrees of freedom and* $H_N(z) = Q_\kappa^*(S_N^*(z))$, *where*

$$S_N^*(z) = \{(\mathbf{X}, \mathbf{M}) \in \mathscr{X}_\Gamma \,|\, \mathrm{tr}\, \mathbf{X}'\mathbf{B}\mathbf{X}(\mathbf{M} + \boldsymbol{\Gamma})^{-1} \leqq z\} .$$

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