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THE ASYMPTOTIC MINIMAX CHARACTER OF SEQUENTIAL BINOMIAL AND SIGN TESTS

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Let p be the probability of any event in repeated independent trials. Sequential tests of the composite hypothesis $p \leqq p_0$ against the composite hypothesis $p > p_0$ are proposed, which asymtotically minimize the maximum risk when the cost of experimentation tends to zero, if the loss depends only on p and satisfies some natural regularity conditions. Asymptotic power, expected sample size and risk of the tests are also given.

1. Introduction and summary. In this paper is studied the problem of sequentially testing the hypothesis $p \leqq p_0$ against $p > p_0$ for the probability p of an event in an infinite series of independent trials of the same kind. A decision-theoretic approach is considered where the cost of experimentation is a constant c times the number of trials performed and the losses $L(p; a_0)$ and $L(p; a_1)$ of making the two possible actions a_0 and a_1 satisfy the conditions

(A) $L(p; a_0)$ and $L(p; a_1)$ are nonnegative continuous functions of p for $0 \leqq p \leqq 1$,

(B) $L(p; a_0) = 0$ for $p \leqq p_0$ and $L(p; a_1) = 0$ for $p \geqq p_0$,

(C) $\lim_{p \downarrow p_0} L(p; a_0)/(p - p_0) = \lim_{p \uparrow p_0} L(p; a_1)/(p_0 - p) = k$

where $0 < k < \infty$. When the loss depends only on p and $a = a_0$ or a_1, and satisfies the conditions (A), (B) and (C), we say that the decision problem has *local linear loss structure in p*. (Compare to Raiffa and Schlaifer [6] page 96.)

Let P denote the probability measure of the infinite series of trials, and let the risk of a (sequential) decision rule δ when the cost of experimentation is c units per trial be denoted by $R_c(P; \delta)$. A set $\{\delta_0^c : 0 < c < \infty\}$ of decision rules δ_0^c is said to be *asymptotically minimax* in a set $\mathscr{D}$ of decision rules and a set $\mathscr{P}$ of probability measures if

$$\delta_0^c \in \mathscr{D} \qquad \text{for} \quad 0 < c < \infty$$

and

$$\lim_{c \to 0} \frac{\sup_{P \in \mathscr{P}} R_c(P; \delta_0^c)}{\inf_{\delta \in \mathscr{D}} \sup_{P \in \mathscr{P}} R_c(P; \delta)} = 1 .$$

This definition seems to be mentioned first by Chung [1]. It is a minimax analogy to the better known definition of asymptotically Bayes decision rule. See Zacks [8] page 311 and page 483, where references are given to papers treating this subject.

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The main theorem (Theorem 4, Section 4) of the present paper states that a certain set of SPR-tests is asymptotically minimax for the Bernoulli problem with local linear loss structure, in the set $\mathscr{D}$ of all decision rules, and it is followed by an extension to a more general nonparametric problem. There are also given asymptotic formulas for the power, expected sample size and risk of these tests (Theorems 1, 2 and 3, Section 3). The minimax problem of testing positive drift against negative drift in Brownian motion with linear loss structure was solved by De Groot [2]. Since this problem serves as a limiting case of ours, we rely much on his results.

2. Preliminaries. In this section we give some notation and state some lemmas which will be used to prove the theorems in Sections 3 and 4.

Let $X_1, X_2, \cdots$ be independent random variables taking the values 0 and 1 with probabilities $1 - p$ and p. In the tests of the hypothesis $p \leqq p_0$ against $p > p_0$, which we are going to study, the stopping rule and the final decision rule are both based on the outcomes of the sequence of random variables $Z_n = \sum_{k=1}^{n} (X_k - p_0)$ for $n = 1, 2, \cdots$. For $0 < c < \infty$ we define δ_0^c as the sequential test such that experimentation stops as soon as $|Z_n| \geqq \rho_c$, action a_0 is chosen if $Z_n \leqq -\rho_c$ and action a_1 is chosen if $Z_n \geqq \rho_c$ where ρ_c is a constant determined by the equations

$$\rho_c = \tfrac{1}{2}k^{\frac{1}{3}}p_0^{\frac{2}{3}}(1 - p_0)^{\frac{2}{3}}\eta c^{-\frac{1}{3}} ,$$

$$\eta = (v^3/(v + \sinh v))^{\frac{1}{3}} ,$$

$$(1 + e^{-v})(v + \sinh v) = 2v \sinh v .$$

The uniqueness of the solution of the third equation follows as a special case of the motivations done by De Groot [2].

The first four lemmas are not proved here, since they are fairly standard and straightforward.

LEMMA 1. *Let X be a random variable, taking the values 0 and 1 with probabilities $1 - p$ and p, and let $\Phi_p(t) = E[e^{(X-p_0)t}]$ be the moment generating function of $X - p_0$, where $0 < p_0 < 1$.*

A. *Then the equation $\Phi_p(t) = 1$ has*

(i) *one and only one nonzero root if $p \neq p_0$,*

(ii) *no nonzero root if $p = p_0$.*

B. *If $h(p)$ is the unique nonzero root of $\Phi_p(t) = 1$ for $p \neq p_0$ and $h(p_0) = 0$, then h is a continuous decreasing function and*

$$h(p) = -\frac{2(p - p_0)}{p_0(p - p_0)} + o(p - p_0) .$$

The proof of Lemma 1 follows immediately from Lemma 3.4 in Ghosh [3] page 102 and an elementary study of the inverse of $h(p)$. The next two Lemmas (2 and 3) give estimates for the probabilities that the actions a_0 and a_1 are taken when the test δ_0^c is performed. We use the notations $P_p^c(a_0)$ and $P_p^c(a_1)$ for those probabilities.

LEMMA 2. *The probability $P_p^c(a_0)$ has the bounds*

$$\frac{\exp[\rho_c h(p)] - 1}{\exp[\rho_c h(p)] - \exp[-(\rho_c + 1)h(p)]} \leqq P_p^c(a_0) \leqq \frac{\exp[(\rho_c + 1)h(p)] - 1}{\exp[(\rho_c + 1)h(p)] - \exp[-\rho_c h(p)]} \quad \text{for } p < p_0,$$

$$\frac{1 - \exp[\rho_c h(p)]}{\exp[-(\rho_c + 1)h(p)] - \exp[\rho_c h(p)]} \leqq P_p^c(a_0) \leqq \frac{1 - \exp[(\rho_c + 1)h(p)]}{\exp[\rho_c h(p)] - \exp[(\rho_c + 1)h(p)]} \quad \text{for } p > p_0.$$

LEMMA 3. (Tail estimates). *For $p < p_0$, $1 - P_p^c(a_0) = P_p^c(a_1) \leqq \exp[-\rho_c h(p)]$ and for $p > p_0$, $1 - P_p^c(a_1) = P_p^c(a_0) \leqq \exp[\rho_c h(p)]$.*

The limits in these two lemmas are readily obtained by Wald methods. They are also obtained by different methods in Hall [4] (Corollary 7.4 and Lemma 6.2).

The next two lemmas give estimates of the expectation $E_p^c[N]$ of the sample size N for the test δ_0^c and its derivative with respect to p. Lemma 4 may readily be proved by Wald methods and it is also a special case of Lemma 6.4 in Hall [4]. The estimates given in this lemma are not good for p in a neighbourhood of p_0, and the estimate of $|(d/dp)E_p^c[N]|$ in Lemma 5 is given in order to overcome this inadequacy in later applications.

LEMMA 4.

$$\left| E_p^c[N] - \frac{(1 - 2P_p^c(a_0))\rho_c}{p - p_0} \right| \leqq \frac{1}{|p - p_0|}.$$

LEMMA 5. *The derivative $(d/dp)E_p^c[N]$ exists for $0 < p < 1$ and*

$$\left| \frac{d}{dp} E_p^c[N] \right| \leqq \frac{1}{p(1-p)} \{(\rho_c + 1)E_p^c[N] + |p - p_0| E_p^c[N^2]\}.$$

PROOF. If $N_{k,l}$ denotes the number of possibilities to reach decision with k successful and l not successful experiments in total then

$$E_p^c[N] = \sum_{n=0}^\infty n \cdot P_p^c(N = n) = \sum_{n=0}^\infty n \cdot \sum_{l=0}^n N_{n-l,l} p^{n-l}(1-p)^l,$$

and

$$E_p^c[N^2] = \sum_{n=0}^\infty n^2 \cdot P_p^c(N = n) = \sum_{n=0}^\infty n^2 \sum_{l=0}^n N_{n-l,l} p^{n-l}(1-p)^l,$$

Both series converge uniformly for $0 \leqq p \leqq 1$. In order to see this we first observe that the terms in the first series are dominated by the terms in the second one. Further if we define M by

$$M = \left[\frac{2\rho_c}{\min(p_0; 1 - p_0)} \right] + 1,$$

the probability of making a decision during M successive trials is at least $p^M + (1-p)^M \geqq (\frac{1}{2})^{M-1}$, independent of the state at the beginning of the series of trials, and then

$$\sum_{n=L}^{\infty} n^2 \cdot P_p^c(N=n) \leqq \sum_{l=[L/M]}^{\infty} (lM)^2 \tfrac{1}{2}^{M-1}(1-\tfrac{1}{2}^{M-1})^{l-1}$$
$$\leqq L^2(1-\tfrac{1}{2}^{M-1})^{[L/M]-1} \frac{2-\frac{1}{2}^{M-1}}{\frac{1}{2}^{2M-2}} .$$

This gives the uniform convergence since the estimate is independent of p and tends to 0 when L tends to ∞.

Define $f_n(p)$ for $n = 0, 1, \cdots$ by $f_n(p) = n \sum_{l=0}^{n} N_{n-l,l} p^{n-l}(1-p)^l$. Then

$$|f_n'(p)| \leqq n \sum_{l=0}^{n} N_{n-l,l}|(n-l)p^{n-l-1}(1-p)^l - lp^{n-l}(1-p)^{l-1}|$$
$$\leqq \frac{1}{p(1-p)} n^2 \sum_{l=0}^{n} N_{n-l,l} p^{n-l}(1-p)^l$$

and the series $\sum_{n=0}^{\infty} f_n'(p)$ converges uniformly in $[a; 1-a]$ for any fixed a, $0 < a < \frac{1}{2}$. Thus $(d/dp)E_p^c[N] = (d/dp)[\sum_{n=0}^{\infty} f_n(p)] = \sum_{n=0}^{\infty} f_n'(p)$ according to Rudin [7] Theorem 7.17. Now

$$\left|\frac{d}{dp} E_p^c[N]\right| = |\sum_{n=0}^{\infty} f_n'(p)| \leqq \sum_{n=0}^{\infty} |f_n'(p)|$$

where

$$|f_n'(p)| = n \sum_{l=0}^{n} N_{n-l,l} p^{n-l-1}(1-p)^{l-1}|(n-l)(1-p) - lp|$$

and

$$|(n-l)(1-p) - lp| \leqq |(n-l)(1-p_0) - lp_0| + n|p - p_0| .$$

When decision is reached after $n - l$ successful and l not successful experiments one has

$$\rho_c \leqq |Z_n| = |(n-l)(1-p_0) - lp_0| \leqq \rho_c + 1$$

and finally we get

$$\left|\frac{d}{dp} E_p^c[N]\right| \leqq \sum_{n=0}^{\infty} |f_n'(p)|$$
$$\leqq \frac{1}{p(1-p)} \sum_{n=0}^{\infty} n \sum_{l=0}^{n} N_{n-l,l} p^{n-l}(1-p)^l(\rho_c + 1 + n|p-p_0|)$$
$$= \frac{1}{p(1-p)} ((\rho_c + 1)E_p^c[N] + |p - p_0|E_p^c[N^2]) .$$

3. Asymptotic power, expected sample size and risk of the tests. The limiting behaviour when $c \to 0$ of the power, expected sample size and risk of the tests δ_0^c, $0 < c < \infty$, described in Section 2 is given by the following Theorems 1, 2 and 3. The properly normalized parameter is $m = c^{-\frac{1}{3}} p_0^{-\frac{1}{3}}(1-p_0)^{-\frac{1}{3}} k^{\frac{1}{3}}(p - p_0)$, and it turns out that the probability $P_p^c(a_0)$ converges to a function of m; the same is the case with the normalized expectation $c^{\frac{2}{3}}E_p^c[N]$ and the normalized risk $c^{-\frac{1}{3}}R_c(p;\,\delta_0^c)$.

The limiting formulas given in the three theorems are local at $p = p_0$, albeit uniform in an interval for the normalized parameter m. Since the constant $p_0^{-\frac{1}{3}}(1-p_0)^{-\frac{1}{3}}k^{\frac{1}{3}}$ will appear in many contexts in the following we introduce the notation $\vartheta = p_0^{-\frac{1}{3}}(1-p_0)^{-\frac{1}{3}}k^{\frac{1}{3}}$ and $\Delta_c = c^{\frac{1}{3}}m/\vartheta = p - p_0 \to 0$ as $c \to 0$.

THEOREM 1. *The probability $P_p^c(a_0)$ satisfies*

$$\lim_{c\to 0} P^c_{p_0+\Delta_c}(a_0) = 1/(e^{m\eta}+1)\,.$$

The convergence is uniform for $-m_0 \leqq m \leqq m_0$ if m_0 is any fixed number.

PROOF. The convergence of $\lim_{c\to 0} P^c_{p_0+\Delta_c}(a_0)$ for $m \neq 0$ follows directly from Lemmas 1 and 2. From Lemma 2 it also follows that upper and lower estimates of $P^c_{p_0+\Delta_c}(a_0)$ are obtained by multiplying the terms in the limiting value $(e^{-m\eta}-1)/(e^{-m\eta}-e^{m\eta})$ by factors varying between $\exp[-|\rho_c h(p) + m\eta| - |h(p)|]$ and $\exp[|\rho_c h(p) + m\eta| + |h(p)|]$.

By Lemma 1 there exists for each $\varepsilon_1 > 0$ a $\delta_1 > 0$ such that $|h(p)/(p-p_0) + 2p_0^{-1}(1-p_0)^{-1}| \leqq \varepsilon_1$ for $|p-p_0| \leqq \delta_1$. This implies

$$|\rho_c h(p) + m\eta| \leqq \tfrac{1}{2}|m|p_0(1-p_0)\eta\varepsilon_1$$

for $0 < |\Delta_c| \leqq \delta_1$, i.e. for $0 < |m| \leqq m_0$ when $c^{\frac{1}{3}} \leqq m_0^{-1}\vartheta\delta_1$.

From Lemma 1 it further follows that there exists a δ_2 such that

$$|h(p)| \leqq 3p_0^{-1}(1-p_0)^{-1}|p-p_0| = 3p_0^{-\frac{2}{3}}(1-p_0)^{-\frac{2}{3}}k^{-\frac{1}{3}}c^{\frac{1}{3}}|m|$$

for $|p-p_0| = |\Delta_c| \leqq \delta_2$, i.e. for $|m| \leqq m_0$ when $c^{\frac{1}{3}} \leqq m_0^{-1}\vartheta\delta_2$. It is thus seen that for each $\varepsilon > 0$ there exists a $\delta < 0$ such that

$$|\rho_c h(p) + m\eta| + |h(p)| \leqq \varepsilon\frac{\eta}{3}|m|$$

for $0 < |m| \leqq m_0$ if $c \leqq \delta$. For $0 < m \leqq m_0$ and $c \leqq \delta$ we now have the upper and lower estimates

$$\frac{\left(\exp\left[m\eta\frac{\varepsilon}{3}\right] - \exp\left[-m\eta\left(1+\frac{\varepsilon}{3}\right)\right]\right)}{\left(\exp\left[m\eta\left(1-\frac{\varepsilon}{3}\right)\right] - \exp\left[-m\eta\left(1-\frac{\varepsilon}{3}\right)\right]\right)} \qquad \text{and}$$

$$\frac{\left(\exp\left[-m\eta\frac{\varepsilon}{3}\right] - \exp\left[-m\eta\left(1-\frac{\varepsilon}{3}\right)\right]\right)}{\left(\exp\left[m\eta\left(1+\frac{\varepsilon}{3}\right)\right] - \exp\left[-m\eta\left(1+\frac{\varepsilon}{3}\right)\right]\right)}$$

whose limits for $m \to 0$ are $\frac{1}{2} + \varepsilon/2$ and $\frac{1}{2} - \varepsilon/2$. The case $m_0 \leqq m < 0$ is analogous. Then for each $\varepsilon > 0$ there exist $\delta > 0$ and $\nu > 0$ such that

$$|P^c_{p_0+\Delta_c}(a_0) - 1/(e^{m\eta}+1)| \leqq \varepsilon$$

for $c \leqq \delta$ and $0 < |m| \leqq \nu$. But $P_p{}^c(a_0)$ is a continuous function of p, which is easily seen by using the definition of $P_p{}^c(a_0)$ and Theorem 7.12 in Rudin [7] page 136. Then the point $m = 0$ may also be included, and the uniform convergence is established for $0 \leqq |m| \leqq \nu$. Finally it is trivial that for each $\varepsilon > 0$ there exists a $\delta' > 0$ such that $|P^c_{p_0+\Delta_c}(a_0) - 1/(e^{m\eta} + 1)| \leqq \varepsilon$ for $c \leqq \delta'$ and $\nu \leqq |m| \leqq m_0$, which proves the uniform convergence for $0 \leqq |m| \leqq m_0$.

THEOREM 2. *The expectation* $E_p{}^c[N]$ *satisfies*

$$\lim_{c\to 0} c^{\frac{2}{3}} E^c_{p_0+\Delta_c}(N) = \frac{\eta(e^{m\eta} - 1)}{2m(e^{m\eta} + 1)}\, k\vartheta^{-1}\,,$$

where $\Delta_c = c^{\frac{1}{3}} m/\vartheta$. *The convergence is uniform for* $-m_0 \leqq m \leqq m_0$ *if* m_0 *is any fixed number.*

PROOF. By Theorem 1 there exists for each $\varepsilon_1 > 0$ a $\delta_1 > 0$ such that

$$|1 - 2P_p{}^c(a_0) - (e^{m\eta} - 1)/(e^{m\eta} + 1)| \leqq \varepsilon_1$$

for $c \leqq \delta_1$ and all m in $[-m_0, m_0]$, and then by Lemma 4

$$\begin{aligned} &|c^{\frac{2}{3}} E_p{}^c[N] - (\eta k \vartheta^{-1}(e^{m\eta} - 1))/(2m(e^{m\eta} + 1))| \\ &\qquad \leqq (\varepsilon_1 \rho_c + c^{\frac{2}{3}})|\Delta_c|^{-1} = (k_1\varepsilon_1 + k_2 c^{\frac{1}{3}})|m|^{-1} \end{aligned}$$

where k_1 and k_2 are constants, which gives uniform convergence for $m_1 \leqq |m| \leqq m_0$ with any fixed $m_1 > 0$.

For c small enough we get from Lemma 5

$$\left|\frac{d}{dp} E_p{}^c[N]\right| \leqq \frac{2}{p_0(1 - p_0)} \cdot \{2\rho_c E_p{}^c[N] + |p - p_0| E_p{}^c[N^2]\}$$

for $|m| \leqq m_0$. If $M(c) = [4\rho_c{}^2/(p_0(1 - p_0))]$ then by the Berry–Esséen theorem $P(|Z_{M(c)}| \geqq 2\rho_c) \geqq 2(1 - \Phi(1)) - 0.01 > 0,3$ for c small enough which gives the rough bounds

$$E_p{}^c[N] \leqq M(c)\frac{1}{0,3} \leqq 15\rho_c{}^2 p_0{}^{-1}(1 - p_0)^{-1}$$

and

$$E_p{}^c[N^2] \leqq (M(c))^2 \frac{2 - 0,3}{0,3} \leqq 25\rho_c{}^4 p_0{}^{-2}(1 - p_0)^{-2}$$

by use of geometrical estimates. Then for c small enough we have

$$\left|\frac{d}{dp} E_p{}^c[N]\right| \leqq 60\rho_c{}^3 p_0{}^{-2}(1 - p_0)^{-2} + 50\Delta_c \rho_c{}^4 p_0{}^{-3}(1 - p_0)^{-3} = Ac^{-1}$$

for $|m| \leqq m_0$, where A is a constant, which implies $|E^c_{p'}[N] - E^c_{p''}[N]| \leqq |p' - p''|c^{-1}A = |m' - m''|c^{-\frac{2}{3}}\vartheta^{-1}A$ and $|c^{\frac{2}{3}}E^c_{p'}[N| - c^{\frac{2}{3}}E^c_{p''}[N]| \leqq |m' - m''|\vartheta^{-1}A$ for $|m'|, |m''| \leqq m_0$. This yields

$$\left|c^{\frac{2}{3}} E_p{}^c[N] - \frac{\eta(e^{m\eta} - 1)}{2m(e^{m\eta} + 1)} k\vartheta^{-1}\right| \leqq \varepsilon$$

for $|m| \leqq m_0$ and c small enough, by first choosing m_1 to satisfy

$$|(\eta(e^{m\eta} - 1))/(2m(e^{m\eta} + 1)) - \eta^2/4| \leqq \varepsilon/3$$

for $0 < m < m_1$ and $m_1 \leqq \frac{1}{3}\vartheta^{-1}A\varepsilon$, and then by using the above inequality for $|m| \leqq m_1$ and the earlier mentioned uniform convergence for $m_1 \leqq |m| \leqq m_0$.

Note. Lemmas 2, 3, 4 and 5 and Theorems 1 and 2 may be given in more general settings for tests $\delta^c_{(\alpha,\beta)}$ which choose action a_0 if $Z_n \leqq -\alpha c^{-\frac{1}{3}}$ and choose action a_1 if $Z_n \geqq \beta c^{-\frac{1}{3}}$. The limiting probability $P_p^c(a_0)$ for these tests is

$$\frac{1 - e^{-\beta\kappa m}}{e^{\alpha\kappa m} - e^{-\beta\kappa m}}$$

and the limiting normalized expectation $c^{\frac{2}{3}}E_p^c[N]$ is

$$\vartheta \frac{\beta(e^{\alpha\kappa m} - 1) - \alpha(1 - e^{-\beta\kappa m})}{m(e^{\alpha\kappa m} - e^{-\beta\kappa m})}$$

where

$$\kappa = 2p_0^{-\frac{2}{3}}(1 - p_0)^{-\frac{2}{3}}k^{-\frac{1}{3}}\,.$$

THEOREM 3. *If the loss is locally linear in p then*

$$\lim_{c\to 0} c^{-\frac{1}{3}}R_c(p_0 + \Delta_c; \delta_0^c) = k\vartheta^{-1}\left\{\frac{|m|}{e^{|m|\eta} + 1} + \frac{\eta(e^{m\eta} - 1)}{2m(e^{m\eta} + 1)}\right\}$$

and the convergence is uniform for $-m_0 \leqq m \leqq m_0$ *where* m_0 *is any fixed number.*

PROOF. For $p \geqq p_0$ the risk $R(p; \delta_0^c)$ is given by

$$L(p; a_0) \cdot P_p^c(a_0) + c \cdot E_p^c[N]\,,$$

and the uniform convergence follows directly from condition

$$\text{(C)} \qquad \lim_{p\downarrow p_0} \frac{L(p; a_0)}{p - p_0} = k$$

and Theorems 1 and 2. The case $p \leqq p_0$ is analogous.

4. Asymptotic minimax properties. The asymptotic minimax properties of the studied tests are given in the following theorem.

THEOREM 4. *Let* $\mathscr{P}$ *be the set of probability measures of a sequence* $X_1, X_2, \cdots$ *of independent random variables taking the values* 0 *and* 1 *with the probabilities* $1 - p$ *and* p, *where* $0 \leqq p \leqq 1$, *and suppose the decision problem has local linear loss structure in p. Then the set* $\{\delta_0^c: 0 < c < \infty\}$ *of decision rules* δ_0^c *given in Section* 2 *is asymptotically minimax in the set* $\mathscr{S}$ *of all decision rules and the set* $\mathscr{P}$ *of probability measures.*

PROOF. Let

$$\mathscr{L}(m; \eta) = |m|/(e^{|m|\eta} + 1) + (\eta(e^{m\eta} - 1))/(2m(e^{m\eta} + 1))\,.$$

Then according to De Groot [2] page 1196 there exists a unique $m > 0$ maximizing

$\mathscr{L}(m; \eta)$ which will be denoted by m^*. From Lemma 1 follows the existence of

$$h_0 = \inf_{0 \leqq p \leqq 1; p \neq p_0} -\frac{h(p)}{p - p_0} > 0$$

and Lemmas 3 and 4 now give for the risk $R(p; \delta_0^c)$ the rough upper bound

$$L|\Delta_c| \exp(-\rho_c h_0 |\Delta_c|) + c(1 + \rho_c)|\Delta_c|^{-1} \leqq Lc^{\frac{1}{3}}\vartheta^{-1}|m| \exp(-\tfrac{1}{2}p_0(1 - p_0)h_0|m|) + 4c^{\frac{1}{3}}\vartheta|m|^{-1}$$

for small c. Choosing m_0 big enough it is seen that this estimate is smaller than $\frac{1}{2}c^{\frac{1}{3}}k\vartheta^{-1}\mathscr{L}(m^*; \eta)$ for all m such that $|m| \geqq m_0$. This together with the uniform convergence in Theorem 3 shows that

$$\sup_{0 \leqq p \leqq 1} R(p; \delta_0^c) \leqq k\vartheta^{-1}\mathscr{L}(m^*; \eta)(c^{\frac{1}{3}} + o(c^{\frac{1}{3}})) .$$

Now consider only the alternatives $p_c' = p_0 - c^{\frac{1}{3}}\vartheta^{-1}m^*$ and $p_c'' = p_0 + c^{\frac{1}{3}}\vartheta^{-1}m^*$. The minimax test with respect to these alternatives is a SPR-test such that experimentation is continued as long as

$$\alpha(c) < Y(n) \ln \frac{p_c''(1 - p_c')}{p_c'(1 - p_c'')} + n \ln \frac{1 - p_c''}{1 - p_c'} < \beta(c) ,$$

where $Y(n) = \sum_{l=1}^{n} X_l$. This is an immediate consequence of Theorem 8 in Lehmann [5] and the comments following it.

The steps in the process

$$Y(n) \ln \frac{p_c''(1 - p_c')}{p_c'(1 - p_c'')} + n \ln \frac{1 - p_c''}{1 - p_c'}$$

are of a magnitude smaller than $\delta(c) = 2\vartheta^2k^{-1}m^*c^{\frac{1}{3}}$. If p_1 and p_2 denote the probabilities of accepting the hypothesis $p = p_c'$ when the parameter is p_c' or p_c'' respectively, we get the usual SPR test inequalities

$$p_1 e^{\alpha} \geqq p_2 , \qquad (1 - p_1)e^{\beta} \leqq 1 - p_2$$
$$p_1 e^{\alpha - \delta} \leqq p_2 , \qquad (1 - p_1)e^{\beta + \delta} \geqq 1 - p_2 .$$

Let N denote the sample size of the test and S_N the value of the variable

$$Y(n) \ln \frac{p_c''(1 - p_c')}{p_c'(1 - p_c'')} + n \ln \frac{1 - p_c''}{1 - p_c'}$$

at stopping. In the fundamental identity $E_p[S_N] = E_p[N]E_p[W]$ where

$$W = X_1 \ln \frac{p_c''(1 - p_c')}{p_c'(1 - p_c'')} + \ln \frac{1 - p_c''}{1 - p_c'}$$

we get for $p = p_c'$ and $p = p_c''$ the expectations $E_{p_c'}|[W]$ and $E_{p_c''}[W]$ which are easily seen to satisfy

$$\lim_{c \to 0} c^{-\frac{2}{3}}E_{p_c''}[W] = \lim_{c \to 0} - c^{-\frac{2}{3}}E_{p_c'}[W] = 2\vartheta k^{-1}(m^*)^2 .$$

Further,

$$\lim_{c\to 0} c^{-\frac{1}{3}} L(p_c'; a_1) = \lim_{c\to 0} c^{-\frac{1}{3}} L(p_c''; a_0) = \vartheta^{-1} k m^*$$

and we introduce

$$L(c) = \min\left[L(p_c'; a_1); L(p_c''; a_0); \frac{2c(m^*)^3}{E_{p_c''}[W]}; -\frac{2c(m^*)^3}{E_{p_c'}[W]}\right]$$

which satisfies $\lim_{c\to 0} c^{-\frac{1}{3}} L(c) = \vartheta^{-1}k$. Now we get the risk estimates

$$L(p_c'; a_1)(1-p_1) + cE_{p_c'}[S_N]/E_{p_c'}[W] \geqq L(c)[(1-p_1) - \tfrac{1}{2}(m^*)^{-3}(\beta(c)(1-p_1) + \alpha(c)p_1 + \delta(c))]$$

and

$$L(p_c''; a_0)p_2 + cE_{p_c''}[S_N]/E_{p_c''}[W] \geqq L(c)[p_2 + \tfrac{1}{2}(m^*)^{-3}(\beta(c)(1-p_2) + \alpha(c)p_2 - \delta(c))] .$$

The sum of these lower estimates is

$$L(c)[(1+p_2-p_1) + \tfrac{1}{2}(m^*)^{-3}(\beta(c) - \alpha(c))(p_1 - p_2) - \delta(c)(m^*)^{-3}] .$$

Using the four given inequalities for p_1 and p_2 we get for $p_1 - p_2$ the inequalities

$$(1-e^\alpha)(e^\beta - 1)/(e^\beta - e^\alpha) \leqq p_1 - p_2 \leqq (1 - e^{\alpha-\delta})(e^{\beta+\delta} - 1)/(e^{\beta+\delta} - e^{\alpha-\delta}) .$$

Further if $f(\alpha, \beta) = (1-e^\alpha)(e^\beta - 1)/(e^\beta - e^\alpha)$, then $|f(\alpha, \beta)/\partial\alpha| \leqq 1$ and $|f(\alpha, \beta)/\partial\beta| \leqq 1$ for $\alpha \leqq 0$ and $\beta \geqq 0$, which implies

$$(1-e^{\alpha-\delta})(e^{\beta+\delta} - 1)/(e^{\beta+\delta} - e^{\alpha-\delta}) = f(\alpha - \delta; \beta + \delta) \leqq f(\alpha, \beta) + 2\delta .$$

Thus a lower bound of the above sum is

$$L(c)[1 - f(\alpha, \beta) + \tfrac{1}{2}(m^*)^{-3}(\beta - \alpha)f(\alpha, \beta) - (2 + (m^*)^{-3})\delta(c)] .$$

Elementary calculations show that the minimum of this is attained when $\beta = -\alpha$ is the solution of the equation $\sinh\beta + \beta = (m^*)^3$. This is the second equation (3.10) in De Groot [2] and the solution is $\beta = m^*\eta$.

A lower bound for the risk sum is then

$$L(c)[1 - f(-m^*\eta; m^*\eta) + (m^*)^{-3}(m^*\eta)f(-m^*\eta; m^*\eta) - (2 + (m^*)^{-3})\delta(c)] = L(c)[2(m^*)^{-1}\mathscr{L}(m^*; \eta) - (2 + (m^*)^{-3})\delta(c)]$$

and the minimal maximum risk in the two points can accordingly not fall below $L(c)[(m^*)^{-1}\mathscr{L}(m^*; \eta) - (1 + \frac{1}{2}(m^*)^{-3})\delta(c)]$. Finally we have

$$\lim_{c\to 0} \frac{\sup_{0\leqq p\leqq 1} R(p; \delta_0^c)}{L(c)[(m^*)^{-1}\mathscr{L}(m^*; \eta) - (1 + \frac{1}{2}(m^*)^{-3})\delta(c)]} \leqq \lim_{c\to 0} \frac{k\vartheta^{-1}\mathscr{L}(m^*; \eta)(c^{\frac{1}{3}} + o(c^{\frac{1}{3}}))}{L(c)[(m^*)^{-1}\mathscr{L}(m^*; \eta) - (1 + \frac{1}{2}(m^*)^{-3})\delta(c)]} = 1$$

which proves the theorem.

The result in Theorem 4 may be extended to a more general case where the repeated experiment does not need to be a Bernoulli trial but may be any kind of experiment as soon as the the decision problem has local linear loss structure in the probability of a fixed event A in the experiment. An asymptotic minimax sequence of tests is then obtained by putting X_k equal to the indicator of this event and using the previously defined tests δ_0^c, which then become sequential sign tests.

The risks of these tests are the same as in the Bernoulli case. Let p_c' and p_c'' be the probabilities defined in the proof of Theorem 4, P_c' any probability measure satisfying $P_c'(A) = p_c'$ and P_c'' the probability measure defined by

$$P_c''(B) = \frac{p_c''}{p_c''} P_c'(B \cap A) + \frac{1 - p_c''}{1 - p_c'} P_c'(B \backslash A) .$$

Then P_c'' has the Radon–Nikodym derivative $(p_c''/p_c')T_A + ((1 - p_c'')/(1 - p_c'))(1 - T_A)$ with respect to P_c', and the test that minimizes the maximum risk for these two distributions only then is a SPR-test such that experimentation is continued as long as

$$\alpha(c) < Y(n) \ln \frac{p_c''(1 - p_c')}{p_c'(1 - p_c'')} + n \ln \frac{1 - p_c''}{1 - p_c'} < \beta(c)$$

for suitable $\alpha(c)$ and $\beta(c)$ where $Y(n) = \sum_{k=1}^{n} X_k$. But these tests have risks equal to the risks of the tests studied at the end of Theorem 4, and the asymptotic minimax property in the general casc is proved analogous to Theorem 4.

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